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Some combinatorial problems on the measurability of functions with respect to invariant extensions of the Lebesgue measure

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It is proved that, for every natural number $n \geq 2$, there exist real-valued functions $f_1, f_2, ..., f_n$ such that any $n - 1$ of them can be made measurable with respect to a translation-invariant extension of the Lebesgue measure, but there is no nonzero σ-finite translation-quasi-invariant measure for which all of these functions become measurable. A related result is obtained, under Martin's Axiom, in terms of absolutely nonmeasurable real-valued functions.

Let λ denote the standard Lebesgue measure on the real line **R**. It is known that there are various translation-invariant measures on **R** which strongly extend λ (see, for instance, [1], [2], [5], [6], [10], and [11]). Consequently, there are many subsets of **R** (hence, many real-valued functions on **R**) which are not measurable in the Lebesgue sense but become measurable with respect to certain translation-invariant extensions of λ. Moreover, it was proved that there exists even a nonseparable translation-invariant extension ν of λ (see [1], [2], [6]). Clearly, the domain of such a ν contains in itself a very rich class of subsets of **R** which are not measurable with respect to λ.

In this paper we would like to consider some problems on the measurability of real-valued functions with respect to translation-invariant extensions of λ. These problems are of combinatorial character, because they are concerned with certain combinations of finite families of real-valued functions.

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In the sequel, we will use the following fairly standard notation.

ω = the set of all natural numbers (and, simultaneously, the cardinality of this set).

ω_1 = the least uncountable cardinal number.

Q = the field of all rational numbers.

R = the real line. In the sequel, **R** will also be considered as a vector space over **Q** and some other vector subspaces of **R** will be used below. So, speaking of the linear independence (or of the linear hull) of a family of real numbers, we always mean the linear independence (the linear hull) over **Q**.

c = the cardinality of the continuum.

$\mathbf{R}^m$ = the m-dimensional Euclidean space (so $\mathbf{R} = \mathbf{R}^1$).

λ_m = the m-dimensional Lebesgue measure on $\mathbf{R}^m$ (so $\lambda_1 = \lambda$).

T = the unit circle (equivalently, the one-dimensional torus) in the plane $\mathbf{R}^2$ (so $\mathbf{T} = \mathbf{S}_1$). Naturally, we treat this torus as a commutative compact group $(\mathbf{T}, +)$ endowed with its Haar probability measure θ. In fact, θ coincides with the Lebesgue probability measure on **T** which is invariant under all translations of **T**. In our further considerations, we also need the product probability measure θ_k on the k-dimensional torus $\mathbf{T}^k$, which coincides with the Haar probability measure on $\mathbf{T}^k$.

$dom(\mu)$ = the domain of a given σ-finite measure μ (i.e., the σ-algebra of all μ-measurable sets).

A nonzero measure μ on **R** (on **T**) is called translation-quasi-invariant if every translation of **R** (of **T**) preserves the σ-ideal of all μ-measure zero sets.

A measure μ is called diffused (or continuous) if it vanishes on all singletons.

A subset U of a Polish topological space E is called universal measure zero if, for any σ-finite continuous Borel measure μ on E, the equality $\mu^*(U) = 0$ holds true, where μ^* denotes, as usual, the outer measure associated with μ. It is easy to see that the family of all universal measure zero subsets of an uncountable Polish space forms a σ-ideal and that the topological product of any two universal measure zero sets is also universal measure zero. According to a deep result of descriptive set theory, if E is an uncountable Polish space, then there are universal measure zero subsets of E having cardinality ω_1 (a more general version of this result can be found in [9]).

A subset L of an uncountable Polish topological space E is called a generalized Luzin set if $card(L) = \mathbf{c}$ and $card(L \cap X) < \mathbf{c}$ for every first category subset X of E. Under Martin's Axiom, there exist generalized Luzin sets in E and all of them are universal measure zero. Moreover, under the same assumption, there exists a generalized Luzin set in **R** (in **T**) which simultaneously is a vector space over **Q** (is a divisible subgroup of **T**).

A function $f : \mathbf{R} \to \mathbf{R}$ (respectively, $f : \mathbf{R} \to \mathbf{T}$) is called absolutely nonmeasurable if it is nonmeasurable with respect to every nonzero σ-finite continuous measure on **R**. It is well known that the existence of such functions cannot be established within **ZFC** set theory, but follows, e.g., from Martin's Axiom. In addition, it can be shown that the following two assertions are equivalent:

(i) f is absolutely nonmeasurable;

(ii) the range of f is universal measure zero in $\mathbf{R}$ (respectively, in $\mathbf{T}$) and $card(f^{-1}(t)) \leq \omega$ for each $t \in \mathbf{R}$ (respectively, for each $t \in \mathbf{T}$).

The proof of the equivalence of these two assertions is given in [5].

Let $\{f_i : i \in I\}$ be a family of real-valued functions on $\mathbf{R}$. It is natural to ask whether there exists a translation-invariant extension μ of λ such that all f_i $(i \in I)$ become μ-measurable. At present, no sufficient and necessary conditions are known under which a family $\{f_i : i \in I\}$ has the above-mentioned property. In this context, the example below seems to be relevant.

Example 1. There is a countable family $\{f_i : i \in \omega\}$ of real-valued functions on $\mathbf{R}$ possessing the following properties:

(1) for every finite set $J \subset \omega$, there exists a translation-invariant extension μ of λ such that all functions f_j $(j \in J)$ are measurable with respect to μ;

(2) there exists no nonzero σ-finite translation-quasi-invariant measure on $\mathbf{R}$ for which all functions $\{f_i : i \in \omega\}$ are measurable.

Such a family $\{f_i : i \in \omega\}$ can be presented by using some results from [3] or [5]. Namely, as shown in [3] and [5], there exists a countable covering $\{X_i : i \in \omega\}$ of $\mathbf{R}$ consisting of so-called absolutely negligible sets. For each index $i \in \omega$, let f_i denote the characteristic function of X_i. Then it is not difficult to verify that (1) and (2) are valid for $\{f_i : i \in \omega\}$.

Let $n \geq 2$ be a natural number. Here we are going to construct a family $(f_1, f_2, ..., f_n)$ of real-valued functions on $\mathbf{R}$ such that any $n-1$ of them can be made measurable with respect to a suitable translation-invariant extension of λ, but there is no nonzero σ-finite translation-quasi-invariant measure on $\mathbf{R}$ for which all of these functions become measurable. Also, assuming Martin's Axiom, we will establish below a similar (but much stronger) result in terms of absolutely nonmeasurable functions.

For this purpose, we need several auxiliary propositions.

Lemma 1. *Let $g_1 : \mathbf{R} \to \mathbf{R}$ (respectively, $g_1 : \mathbf{R} \to \mathbf{T}$) be a function satisfying the following conditions:*

(1) g_1 is a homomorphism of the additive group $\mathbf{R}$ into itself (respectively, to the commutative group $(\mathbf{T}, +)$);

(2) the range of g_1 is uncountable and universal measure zero in $\mathbf{R}$ (respectively, in $\mathbf{T}$).

Then, for any nonzero σ-finite translation-quasi-invariant measure μ on $\mathbf{R}$, the function g_1 is nonmeasurable with respect to μ.

The proof of this lemma (and the existence of a homomorphism $g_1 : \mathbf{R} \to \mathbf{R}$ with properties (1) and (2)) is given in [4]. It is easy to see that the same argument works in the case of $g_1 : \mathbf{R} \to \mathbf{T}$.

It should be underlined that Lemma 1 is a statement of **ZFC** set theory. By assuming Martin's Axiom, this lemma can be significantly strengthened. Namely, we have

Lemma 2. *Suppose that Martin's Axiom holds and let* $g_2 : \mathbf{R} \to \mathbf{R}$ *(respectively,* $g_2 : \mathbf{R} \to \mathbf{T}$*) be a function satisfying the following conditions:*

(1) g_2 *is an injective homomorphism of the additive group* $\mathbf{R}$ *into itself (respectively, to the commutative group* $(\mathbf{T}, +)$*);*

(2) the range of g_2 *is a generalized Luzin set in* $\mathbf{R}$ *(respectively, in* $\mathbf{T}$*).*

Then, for any nonzero σ*-finite continuous measure* μ *on* $\mathbf{R}$*, the function* g_2 *is nonmeasurable with respect to* μ *(in our terminology,* g_2 *is absolutely nonmeasurable).*

The proof of this proposition and the existence (under **MA**) of a homomorphism g_2 with properties (1) and (2) can be found in [5].

Lemma 3. *Let* $k \geq 1$ *be a natural number and let*

$$(\phi_1, \phi_2, ..., \phi_k) : \mathbf{R} \to \mathbf{T}^k$$

be a group homomorphism such that its graph is $(\lambda \otimes \theta_k)$*-thick in the product space* $\mathbf{R} \times \mathbf{T}^k$*, i.e., every Borel set* $B \subset \mathbf{R} \times \mathbf{T}^k$ *with* $(\lambda \otimes \theta_k)(B) > 0$ *has nonempty intersection with this graph. Then there exists a translation-invariant extension* μ *of* λ *for which all functions* $\phi_1, \phi_2, ..., \phi_k$ *are* μ*-measurable homomorphisms acting from* $\mathbf{R}$ *to* $\mathbf{T}$.

This proposition is known and, actually, goes back to the classical result of Kodaira and Kakutani [6] stating the existence of nonseparable translation-invariant extensions of λ. It should be noticed that another, substantially different construction of nonseparable translation-invariant extensions of λ was also given by Kakutani and Oxtoby [2].

Lemma 4. *Let* $\Phi : \mathbf{R} \to \mathbf{T}$ *denote the canonical continuous surjective group homomorphism defined by the formula*

$$\Phi(x) = (cos(x), sin(x)) \ (x \in \mathbf{R}).$$

There exists a Borel mapping $\Psi : \mathbf{T} \to \mathbf{R}$ *such that the composition* $\Phi \circ \Psi$ *coincides with the identity transformation of* $\mathbf{T}$.

This proposition is almost trivial from the geometrical view-point and, in fact, is a straightforward consequence of the widely known theorem of Kuratowski and Ryll-Nardzewski on measurable selectors (see [8]). Notice also that, for any $t \in \mathbf{T}$, the set $\Phi^{-1}(t)$ is countable. Consequently, if a set $X \subset \mathbf{R}$ is uncountable, then the set $\Phi(X)$ is uncountable, too.

Lemma 5. *Let* Φ *be as in Lemma 4, let* $k \geq 1$ *be a natural number and let*

$$(h_1, h_2, ..., h_k) : \mathbf{R} \to \mathbf{R}^k$$

be a mapping whose graph is λ_{k+1}*-thick in the Euclidean space* $\mathbf{R}^{k+1}$*. Then the graph of the mapping*

$$(\Phi \circ h_1, \Phi \circ h_2, ..., \Phi \circ h_k) : \mathbf{R} \to \mathbf{T}^k$$

is $(\lambda \otimes \theta_k)$*-thick in the product group* $\mathbf{R} \times \mathbf{T}^k$.

We omit an easy proof of this lemma based on the Fubini theorem.

Lemma 6. *Let E_1 and E_2 be two Polish spaces and let $g : E_1 \to E_2$ be a Borel mapping such that $card(g^{-1}(y)) \leq \omega$ for each point $y \in E_2$. If U is a universal measure zero subset of E_1, then $g(U)$ is a universal measure zero subset of E_2.*

Proof. According to a well-known theorem of descriptive set theory (see, e.g., [7]), there exists a countable partition $\{B_i : i \in \omega\}$ of E_1 into Borel subsets such that all restrictions $g|B_i$ ($i \in \omega$) are injective. For each index $i \in \omega$, the set $U \cap B_i$ is universal measure zero in E_1. The set $g(U \cap B_i)$ being an injective Borel image of $U \cap B_i$ is universal measure zero in E_2. It remains to apply the simple fact that the family of all universal measure zero subsets of E_2 is countably additive (as was already mentioned, if E_2 is uncountable, then this family forms a σ-ideal of subsets of E_2).

Remark 1. In Lemma 6, the restriction on g is very essential. For example, under Martin's Axiom, there exists a generalized Luzin set $L \subset \mathbf{R}$ such that $L + L = \mathbf{R}$. Considering the continuous mapping

$$\phi : \mathbf{R}^2 \to \mathbf{R}$$

which is defined by the simple formula

$$\phi(x, y) = x + y \ (x \in \mathbf{R}, y \in \mathbf{R}),$$

we see that the ϕ-image of the universal measure zero set $L \times L$ coincides with the whole real line $\mathbf{R}$.

The next proposition is crucial for obtaining the main result of this paper.

Lemma 7. *Let $n \geq 2$ be a natural number and let $g_1 : \mathbf{R} \to \mathbf{R}$ be as in Lemma 1. There exist functions $h_1, h_2, ..., h_n$ acting from $\mathbf{R}$ into itself and satisfying the following relations:*

(1) all h_i ($i = 1, ..., n$) are group homomorphisms;

(2) for any $i \in \{1, ..., n\}$, the graph of the mapping

$$(h_1, ..., h_{i-1}, h_{i+1}, ..., h_n) : \mathbf{R} \to \mathbf{R}^{n-1}$$

is λ_n-thick in $\mathbf{R}^n$;

(3) $h_1 + h_2 + ... + h_n = g_1$.

Proof. Denote by α the least ordinal number of cardinality $\mathbf{c}$ and let $\{B_\xi : \xi < \alpha\}$ be an enumeration of all Borel subsets of $\mathbf{R}^n$ having strictly positive λ_n-measure. Without loss of generality, we may suppose that every Borel subset of $\mathbf{R}^n$ with strictly positive λ_n-measure occurs continuumly many times in $\{B_\xi : \xi < \alpha\}$.

Now, we will need some fixed subsets Ξ_i ($i = 0, 1, ..., n$) of the interval $[0, \alpha[$. We may choose all the above-mentioned sets Ξ_i so that the following relations would be satisfied:

(a) all these sets form a partition of $[0, \alpha[$;

(b) $card(\Xi_0) = \mathbf{c}$;

(c) for any index $i \in \{1, 2, ..., n\}$, the corresponding partial transfinite sequence $\{B_\xi : \xi \in \Xi_i\}$ contains all Borel subsets of $\mathbf{R}^n$ having strictly positive λ_n-measure.

Let $\preceq$ be a fixed well-ordering of $\mathbf{R}$ isomorphic to α.

We are going to construct (by means of the method of transfinite recursion) an α-sequence $(x_\xi)_{\xi<\alpha}$ of points of $\mathbf{R}$ and the corresponding α-sequence

$$(h_1(x_\xi), h_2(x_\xi), ..., h_n(x_\xi))_{\xi<\alpha}$$

of elements of $\mathbf{R}^n$. Suppose that our construction has already been done for all ordinals $\zeta < \xi$, where ξ is an arbitrary ordinal strictly less than α. Only two cases are possible.

1. $\xi \in \Xi_0$. In this case, let x be the least element of $\mathbf{R}$ (with respect to $\preceq$) which does not belong to the $\mathbf{Q}$-linear hull of $\{x_\zeta : \zeta < \xi\}$. Denote $x_\xi = x$ and choose the values

$$h_1(x_\xi) \in \mathbf{R},\ h_2(x_\xi) \in \mathbf{R},\ ...,\ h_n(x_\xi) \in \mathbf{R}$$

arbitrarily but taking into account the restriction:

$$h_1(x_\xi) + h_2(x_\xi) + ... + h_n(x_\xi) = g_1(x_\xi).$$

Clearly, there are many possibilities for such a choice.

2. $\xi \in \Xi_i$, where $i \in \{1, ..., n\}$. In this case, we take the set B_ξ and an element $x \in pr_1(B_\xi)$ which does not belong to the $\mathbf{Q}$-linear hull of $\{x_\zeta : \zeta < \xi\}$ and for which the inequality $\lambda_{n-1}(B_\xi(x)) > 0$ holds true, where

$$B_\xi(x) = \{y \in \mathbf{R}^{n-1} : (x, y) \in B_\xi\}.$$

Notice that the existence of x follows directly from the Fubini theorem. Since $B_\xi(x) \neq \emptyset$, we may choose a point

$$(y_1, ..., y_{i-1}, y_{i+1}, ..., y_n) \in B_\xi(x).$$

Further, we put $x_\xi = x$ and

$$h_1(x_\xi) = y_1,\ \ldots, h_{i-1}(x_\xi) = y_{i-1},\ h_{i+1}(x_\xi) = y_{i+1},\ \ldots,\ h_n(x_\xi) = y_n,$$

$$h_i(x_\xi) = g_1(x_\xi) - h_1(x_\xi) - ... - h_{i-1}(x_\xi) - h_{i+1}(x_\xi) - ... - h_n(x_\xi).$$

Proceeding in this manner, we get the required two α-sequences

$$(x_\xi)_{\xi<\alpha},\ (h_1(x_\xi), h_2(x_\xi), ..., h_n(x_\xi))_{\xi<\alpha}.$$

By virtue of our construction, it is not difficult to derive that the family of points $\{x_\xi : \xi < \alpha\}$ is a Hamel basis for $\mathbf{R}$ (because the well-ordering $\preceq$ is isomorphic to α and $card(\Xi_0) = \mathbf{c}$). Consequently, all partial functions h_i ($i = 1, ..., n$) can uniquely be extended to group homomorphisms h_i acting from $\mathbf{R}$ into itself. Also, it is clear that, for any index $i \in \{1, 2, ..., n\}$, the graph of the homomorphism

$$(h_1, ..., h_{i-1}, h_{i+1}, ..., h_n) : \mathbf{R} \to \mathbf{R}^{n-1}$$

is λ_n-thick in the product space $\mathbf{R} \times \mathbf{R}^{n-1} = \mathbf{R}^n$. Lemma 7 has thus been proved.

Now, by using the presented lemmas, we can establish the main statement of this paper.

Theorem 1. *Let $n \geq 2$ be a natural number. There exist functions*

$$f_i : \mathbf{R} \to \mathbf{R} \ (i = 1, 2, ..., n)$$

possessing the following properties:

(1) any $n - 1$ of these functions can be made measurable with respect to a translation-invariant extension of λ;

(2) there is no nonzero σ-finite translation-quasi-invariant measure on **R** *for which all of these functions are measurable.*

Proof. Let Φ be as in Lemma 4 and let h_i $(i = 1, 2, ..., n)$ be as in Lemma 7. We denote

$$\phi_i = \Phi \circ h_i \ (i = 1, 2, ..., n).$$

Then, in view of Lemmas 3 and 5, any $n - 1$ of the obtained functions $\phi_1, \phi_2, ..., \phi_n$ can be made measurable with respect to a translation-invariant extension of λ. On the other hand, we have the equality

$$h_1 + h_2 + ... + h_n = g_1,$$

where $g_1 : \mathbf{R} \to \mathbf{R}$ is nonmeasurable with respect to any nonzero σ-finite translation-quasi-invariant measure on **R**. This equality implies

$$\Phi \circ g_1 = \phi_1 + \phi_2 + ... + \phi_n,$$

where $\Phi \circ g_1$ has the same non-measurability property (because of Lemmas 1 and 6). We thus see that the functions ϕ_1, ϕ_2,..., ϕ_n cannot simultaneously be measurable with respect to a nonzero σ-finite translation-quasi-invariant measure on **R**. Finally, taking Ψ as in Lemma 4 and putting

$$f_1 = \Psi \circ \phi_1, \ f_2 = \Psi \circ \phi_2, \ \ldots, \ f_n = \Psi \circ \phi_n,$$

we get the required functions f_1, f_2, ..., f_n with properties (1) and (2). This completes the proof of Theorem 1.

The next statement can be established by applying a completely analogous argument.

Theorem 2. *Assume Martin's Axiom and let $n \geq 2$ be a natural number. There exist functions f_1, f_2, ..., f_n acting from* **R** *into itself such that:*

(1) any $n - 1$ of these functions can be made measurable with respect to a translation-invariant extension of λ;

(2) there is no nonzero σ-finite continuous measure on **R** *for which all of these functions are measurable.*

The proof of Theorem 2 is carried out by the same scheme as for Theorem 1. Indeed, in the corresponding argument, we only should replace the function g_1 of Lemma 1 by the function g_2 of Lemma 2.

Remark 2. Two direct analogues of Theorems 1 and 2 are valid for the m-dimensional Euclidean space $\mathbf{R}^m$ and for the m-dimensional Lebesgue measure λ_m

on this space. Actually, the proof for $(\mathbf{R}^m, \lambda_m)$ is almost identical with the argument presented above.

Remark 3. It would be interesting to extend Theorems 1 and 2 to the more general case of an uncountable σ-compact locally compact topological group equipped with its Haar measure.

In connection with Theorem 1, it is natural to pose the following combinatorial problem.

Problem 1. Let $n \geq 2$ and $0 < k < n$ be natural numbers. Prove (or disprove) that there is a family $\{f_1, f_2, ..., f_n\}$ of real-valued functions on $\mathbf{R}$ satisfying the following conditions:

(a) for every k-element subfamily of $\{f_1, f_2, ..., f_n\}$, there exists a translation-invariant extension of λ such that all members from the subfamily are measurable with respect to this extension;

(b) for every $(k + 1)$-element subfamily of $\{f_1, f_2, ..., f_n\}$, there exists no nonzero σ-finite translation-quasi-invariant measure on $\mathbf{R}$ for which all functions from this subfamily become measurable.

So far, we were concerned with real-valued functions on $\mathbf{R}$ and the results presented above were formulated in terms of the measurability of those functions. But similar questions can be envisaged for subsets of $\mathbf{R}$. In this context, the following problem seems to be of interest.

Problem 2. Let $n \geq 2$ and $0 < k < n$ be natural numbers. Prove (or disprove) that there is a family $\{Z_1, Z_2, ..., Z_n\}$ of subsets of $\mathbf{R}$ satisfying the following conditions:

(a) for any k-element subfamily of $\{Z_1, Z_2, ..., Z_n\}$, there exists a translation-invariant extension of λ such that all members from the subfamily are measurable with respect to this extension;

(b) for any $(k + 1)$-element subfamily of $\{Z_1, Z_2, ..., Z_n\}$, there exists no nonzero σ-finite translation-quasi-invariant measure on $\mathbf{R}$ whose domain contains this subfamily.

Notice that the positive solution of Problem 2 automatically implies the positive solution of Problem 1.

Example 2. Assuming the Continuum Hypothesis, Sierpiński was able to construct two subsets Z_1 and Z_2 of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ satisfying the following conditions:

(1) $card(Z_1 \cap (\{x\} \times \mathbf{R})) \leq 1$ for every $x \in \mathbf{R}$;

(2) $card(Z_2 \cap (\mathbf{R} \times \{y\})) \leq 1$ for every $y \in \mathbf{R}$;

(3) there exists a countable family $\{s_i : i \in \omega\}$ of translations of $\mathbf{R}^2$ such that $\cup\{s_i(Z_1 \cup Z_2) : i \in \omega\} = \mathbf{R}^2$.

The above-mentioned conditions imply that Z_1 and Z_2 have also the following properties:

(a) there exists a translation-invariant measure μ_1 on $\mathbf{R}^2$ extending λ_2 and such that $\mu_1(Z_1) = 0$;

(b) there exists a translation-invariant measure μ_2 on $\mathbf{R}^2$ extending λ_2 and such that $\mu_2(Z_2) = 0$;

(c) there exists no nonzero σ-finite translation-quasi-invariant measure μ on $\mathbf{R}^2$ such that $\{Z_1, Z_2\} \in dom(\mu)$.

In [3] the existence of sets Z_1 and Z_2 with properties (a), (b) and (c) was shown without appealing to any additional set-theoretical assumptions. Notice that the conditions (1)-(3) are significantly stronger than (a)-(c), because they imply the Continuum Hypothesis.

Remark 4. It would be interesting to investigate analogues of Problems 1 and 2 for the Euclidean space $\mathbf{R}^m$ which is equipped with its Lebesgue measure λ_m and with its group of all isometric transformations (as known, the latter group is much more complicated than the group of all translations of the same space).

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