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QUASITRIVIAL SEMIMODULES VI

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The paper continues the investigation of quasitrivial semimodules and related problems. In particular, endomorphisms of semilattices are investigated.

This part is a continuation of [1], [2], [3], [4] and [5] with main emphasis on endomorphisms of semilattices. The notation introduced in the preceding parts is used. All the results collected here are fairly basic and we will not attribute them to any particular source.

1. Introduction

Throughout the paper, let $M = M(+)$ be a non-trivial semilattice (i.e., a commutative idempotent semigroup). As usual, a relation of order is defined on M by $a \leq b$ if and only if $a + b = b$. The ordered set $M(\leq)$ has the smallest element if and only if the semilattice M has the neutral element (usually denoted as 0_M). Then 0_M is the smallest element and *minimal elements* (or *atoms*) are the elements covering the neutral element. If $0_M \notin M$ then *minimal elements* (or *atoms*) are just the minimal elements of the ordered set $M(\leq)$. This set has the greatest element if and only if the semilattice M has the absorbing element (denoted by o_M throughout this paper). Then o_M is

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the greatest element and *maximal elements* (or *coatoms, dual atoms*) are the elements that are covered by the absorbing element o_M. If $o_M \notin M$ then the ordered set $M(\leq)$ has no maximal elements at all. An element $a \in M$ is *irreducible* if $a \neq x + y$ for all $x, y \in M \setminus \{a\}$.

1.1 Proposition. (i) *An element w is the smallest element of $M(\leq)$ if and only if $w = 0_M$ is the neutral (or zero) element of the semilattice M.*
(ii) *An element w is the greatest element of $M(\leq)$ if and only if $w = o_M$ is the absorbing element of the semilattice M.*
(iii) *If $0_M \in M$ then a is minimal if and only if $a \neq 0_M$ and $a \notin (M \setminus \{0_M, a\}) + M$ (or $a \notin (M \setminus \{0_M, a\}) + a$).*
(iv) *If $0_M \notin M$ then a is minimal if and only if $a \notin (M \setminus \{a\}) + M$ (or $a \notin M \setminus \{a\}) + a$).*
(v) *0_M is irreducible.*
(vi) *Every minimal element is irreducible.*
(vii) *If $o_M \in M$ then a is maximal if and only if $a \neq o_M$ and $M + a \subseteq \{a, o_M\}$.*
(viii) *If $o_M \notin M$ then $M(\leq)$ has no maximal element.*

Proof. It is easy. □

1.2 Example. (i) $M(+, *)$ is a semiring, where $a * b = a$ for all $a, b \in M$.
(ii) $M(+, \circ)$ is a semiring, where $a \circ b = b$.
(iii) $M(+, +)$ is a semiring.
(iv) Let $w \in M$ and $a \cdot b = w$ for all $a, b \in M$. Then $M(+, \cdot)$ is a semiring.

1.3 Remark. A non-empty subset I of M is an ideal if $M + I \subseteq I$.
(i) If I, J are ideals then the sets $I+J$, $I\cap J$ and $I\cup J$ are ideals and $I+J \subseteq I\cap J \subseteq I\cup J$.
(ii) A one-element set $\{w\}$ is an ideal iff $w = o_M$.
(iii) If $o_M \notin M$ then no ideal is minimal.
(iv) If $o_M \in M$ then an ideal I is minimal iff $I = \{a, o_M\}$, where a is maximal.
(v) If $0_M \in M$ then the set $M \setminus \{0_M\}$ is the only maximal ideal of M.
(vi) If $0_M \notin M$ then I is a maximal ideal of M iff $I = M \setminus \{a\}$, where a is minimal.
(vii) For every $a \in M$, the set $M + a = \{\, x \,|\, a \leq x \,\}$ is just the ideal generated by the one-element set $\{a\}$. If $a \neq o_M$ then the set $(M + a) \setminus \{a\} = \{\, y \,|\, a < y \,\}$ is an ideal, too.
(viii) Consider the following conditions

- (1) M is finite;
- (2) Every strictly decreasing sequence $I_1 \supset I_2 \supset I_3 \supset \dots$ of ideals of M is finite;
- (3) Every strictly decreasing sequence $J_1 \supset J_2 \supset J_3 \supset \dots$ of one-generated ideals of M (see (vii)) is finite;
- (4) Every strictly increasing sequence $a_1 < a_2 < a_3 < \dots$ of elements from M is finite;
- (5) $o_M \in M$.

One sees easily that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).
(ix) Any infinite strictly decreasing chain $a_1 > a_2 > a_3 > \dots$ of elements from M satisfies (2) (but not (1)).

(x) Consider the following semilattice: $M = \{a_1, a_2, a_3, b_1, b_2, b_3, \dots, o_M\}$, where $a_i < b_i$ and $a_i < b_{i+1}$ for every i. Then M satisfies (3) and (4), but not (2).

1.4 Remark. An ideal I of M is said to be *prime* if the set $M \setminus I$ is a subsemilattice of M (i.e., $I \neq M$ and $a + b \in M \setminus I$ for all $a, b \in M \setminus I$).
(i) If I, J are prime ideals such that $I \cup J \neq M$ then $I \cup J$ is a prime ideal.
(ii) For every $a \in M$, $a \neq o_M$, the set $P_a = \{\, x \in M \mid x \nleq a \,\}$ $(= \{\, x \mid a < x + a \,\})$ is a prime ideal of M. (These prime ideals are called *principal*.)
(iii) Let P be a prime ideal of M. Then $P \subseteq P_a$ for every $a \in M \setminus P$ and we have $P = \bigcup P_a$.
(iv) Let P be a prime ideal of M. Put $N = M \setminus P$. Then P is principal iff $o_N \in N$.
(v) Let $a_1 < a_2 < a_3 < \dots$ be an infinite strictly increasing sequence and let P be the set of $x \in M$ such that $x \nleq a_i$ for every i. If $P \neq \emptyset$ then P is a non-principal prime ideal.
(vi) The following conditions are equivalent:
  (1) Every prime ideal is principal.
  (2) Every infinite strictly increasing sequence $a_1 < a_2 < a_3 < \dots$ is upwards cofinal in M.
  (3) $o_N \in N$ for every (proper) subsemilattice N of M.

2. Endomorphisms (a)

We denote by $\underline{E}$ the full endomorphism semiring of the semilattice M $(= M(+))$. That is, $\underline{E}$ is the set of transformations f of M such that $f(a + b) = f(a) + f(b)$ for all $a, b \in M$. The basic operations of addition and multiplication are defined by $(f + g)(a) = f(a) + g(a)$ and $(fg)(a) = f(g(a))$. The identity automorphism id_M is the (unique) multiplicatively neutral element of the semiring $\underline{E}$, i.e., $\mathrm{id}_M = 1_{\underline{E}}$. The additive semigroup $\underline{E}(+)$ is a semilattice and, for all $f, g \in \underline{E}$, we have $f \leq g$ iff $f + g = g$ (or $f(a) \leq g(a)$ for every $a \in M$).

2.1 For every $a \in M$, the constant transformation $\sigma_a : M \to \{a\}$ belongs to $\underline{E}$. We put $\underline{E}^{(1)} = \{\, \sigma_a \mid a \in M \,\}$.

2.1.1 Proposition. (i) $\sigma_a + \sigma_b = \sigma_{a+b}$.
(ii) $\sigma_a f = \sigma_a$ *for every* $f \in \underline{E}$.
(iii) $f\sigma_a = \sigma_{f(a)}$.
(iv) $\sigma_a \sigma_b = \sigma_a$.

Proof. It is easy. □

2.1.2 Proposition. (i) $\underline{E}^{(1)}$ *is an ideal of the semiring* $\underline{E}$.
(ii) $\underline{E}^{(1)}$ *is the smallest (left) ideal of* $\underline{E}$.
(iii) $\underline{E}^{(1)}$ *is the set of left multiplicatively absorbing elements of the semiring* $\underline{E}$.

Proof. It is easy (use 2.1.1). □

2.1.3 Proposition. (i) $|\underline{E}^{(1)}| = |M|$.
(ii) *The semiring* $\underline{E}^{(1)}$ *is left-ideal-free.*
(iii) *The semiring* $\underline{E}^{(1)}$ *is bi-idempotent.*
(iv) *Every subsemilattice of* $\underline{E}^{(1)}(+)$ *is a right ideal of the semiring* $\underline{E}^{(1)}$.
(v) *Every element from* $\underline{E}^{(1)}$ *is left multiplicatively absorbing and right multiplicatively neutral in* $\underline{E}^{(1)}$.
(vi) *The semiring* $\underline{E}^{(1)}$ *has no left multiplicatively neutral element.*
(vii) *The semiring* $\underline{E}^{(1)}$ *has no right multiplicatively absorbing element.*
(viii) *The semiring* $\underline{E}^{(1)}$ *has an additively neutral element iff* $0_M \in M$*; then* σ_{0_M} *is the additively neutral element.*
(ix) *The semiring* $\underline{E}^{(1)}$ *has an additively absorbing element iff* $o_M \in M$*; then* σ_{o_M} *is the additively absorbing element.*
(x) *The semiring* $\underline{E}^{(1)}$ *is congruence-simple iff* $|M| = 2$ *(or* $|\underline{E}^{(1)}| = 2$*).*

Proof. It is easy (use 2.1.1). □

2.1.4 Proposition. $\mathrm{id}_M \notin \underline{E}^{(1)}$ *and* $\underline{E}^{(1)} \neq \underline{E}$.

Proof. It is obvious. □

2.2 Proposition. (i) *The semiring* $\underline{E}$ *is not ideal-simple.*
(ii) $\underline{E}$ *has an additively neutral element iff* $0_M \in M$*; then* σ_{0_M} *is the additively neutral element and* σ_{0_M} *is left multiplicatively absorbing.*
(iii) $\underline{E}$ *has an additively absorbing element iff* $o_M \in M$*; then* σ_{o_M} *is the additively absorbing element and* σ_{o_M} *is left multiplicatively absorbing.*
(iv) $\underline{E}$ *has no right multiplicatively absorbing element.*
(v) $\underline{E}$ *is bi-idempotent iff* $|M| = 2$ *(or* $|\underline{E}| \leq 3$*).*

Proof. It is easy. □

2.3 Proposition. (i) *For every* $a \in M$*, the one-element set* $\{\sigma_a\}$ *is a right ideal of* $\underline{E}$.
(ii) *A subset* I *of* $\underline{E}$ *is a minimal right ideal of* $\underline{E}$ *iff* $I = \{\sigma_a, \sigma_b\}$ *for some* $a, b \in M$, $a < b$.

Proof. (i) This is obvious.
(ii) First, let I be a minimal right ideal of $\underline{E}$ and let $f \in I$. For every $a \in M$, we have $\sigma_{f(a)} = f\sigma_a \in I$, and hence $K = I \cap \underline{E}^{(1)} \neq \emptyset$. Of course, K is a right ideal. If $|K| = 1$ then $\sigma_{f(a)} = \sigma_{f(b)}$, and hence $f(a) = f(b)$ for all $a, b \in M$. Thus $f \in \underline{E}^{(1)}$, $I \subseteq \underline{E}^{(1)}$ and $I = K$, a contradiction with $|I| \geq 2$. Thus $|K| \geq 2$, and hence $K = I$, since I is a minimal right ideal. Thus $I \subseteq \underline{E}^{(1)}$ and our result easily follows.
Conversely, if $I = \{\sigma_a, \sigma_b\}$, $a < b$, then the result is clear. □

Let $\underline{E}^{(\alpha)} = E + \underline{E}^{(1)} = \{ f \in \underline{E} \,|\, \sigma_a \leq f$ for some $a \in M \}$.

2.4 Proposition. (i) $\underline{E}^{(\alpha)}$ *is the smallest bi-ideal of the semiring* $\underline{E}$.
(ii) $\underline{E}^{(1)} \subseteq \underline{E}^{(\alpha)}$.

Proof. Let I be a bi-ideal of $\underline{E}$. Since I is an ideal, we have $\underline{E}^{(1)} \subseteq I$ by 2.1.2(ii). Since I is a bi-ideal, we have $\underline{E}^{(\alpha)} = \underline{E} + \underline{E}^{(1)} \subseteq I$. □

2.5 Proposition. *The following conditions are equivalent:*

(i) $\underline{E}$ *is bi-ideal-simple.*
(ii) $\underline{E}$ *is bi-ideal-free.*
(iii) $\underline{E} = \underline{E}^{(\alpha)}$.
(iv) $\mathrm{id}_M \in \underline{E}^{(\alpha)}$.
(v) $\underline{E}^{(\alpha)}$ *has a mulitplicatively neutral element.*
(vi) $\underline{E}^{(\alpha)}$ *has an additively neutral element.*
(vii) $0_M \in M$.

Proof. It is easy (use 2.4). □

2.6 Proposition. *The semiring* $\underline{E}^{(\alpha)}$ *is bi-ideal-free.*

Proof. Let I be a bi-ideal of $\underline{E}^{(\alpha)}$. Then $\underline{E}^{(1)} \subseteq I$, and hence $\underline{E}^{(\alpha)} \subseteq I$. □

Put $\underline{E}^{(\beta)} = \underline{E}^{(1)} \cup (\underline{E}^{(1)} + \mathrm{id}_M) \cup \{\mathrm{id}_M\}$ and $\underline{E}^{(\beta 1)} = \underline{E}^{(1)} \cup (\underline{E}^{(1)} + \mathrm{id}_M)$

2.7 Proposition. (i) $\underline{E}^{(\beta)}$ *is a bi-idempotent subsemiring of* $\underline{E}$.
(ii) $\underline{E}^{(\beta)}$ *is the subsemiring generated by* $\underline{E}^{(1)} \cup \{\mathrm{id}_M\}$.
(iii) $\underline{E}^{(\beta)} \subseteq \underline{E}^{(\alpha)}$ *iff* $0_M \in M$.
(iv) $\underline{E}^{(\beta)} = \underline{E}$ *iff* $|M| = 2$.
(v) $\underline{E}^{(\beta 1)} \subseteq \underline{E}^{(\alpha)}$.
(vi) *If* $0_M \in M$ *then* $\underline{E}^{(\beta)} = \underline{E}^{(\beta 1)}$.
(vii) *If* $0_M \notin M$ *then* $\underline{E}^{(\beta 1)}$ *is a proper bi-ideal of* $\underline{E}^{(\beta)}$.

Proof. It is easy. □

2.8 For every $a \in M$, the translation λ_a, where $\lambda_a(x) = a + x$, is an endomorphism of M. We put $\underline{E}^{(\gamma)} = \{ \lambda_a \,|\, a \in M \}$.

2.8.1 Proposition. (i) $\lambda_a + \lambda_b = \lambda_{a+b} = \lambda_a \lambda_b$.
(ii) $f\lambda_a = \lambda_{f(a)}$ *for every* $f \in \underline{E}$.

Proof. It is easy. □

2.8.2 Proposition. (i) $\underline{E}^{(\gamma)}$ *is a subsemiring of* $\underline{E}$.
(ii) $\underline{E}^{(\gamma)}$ *is bi-idempotent.*
(iii) $\underline{E}^{(\gamma)}$ *is ideal-simple iff it is congruence-simple and iff* $|M| = 2$.

Proof. It is easy. □

2.8.3 Lemma. $\lambda_a = \sigma_a + \mathrm{id}_M$.

Proof. It is obvious. □

2.8.4 Corollary.. (i) $\underline{E}^{(\gamma)} = \underline{E}^{(1)} + \mathrm{id}_M \subseteq \underline{E}^{(\alpha)} \cap \underline{E}^{(\beta)}$.
(ii) $\underline{E}^{(\beta)} = \underline{E}^{(1)} \cup \underline{E}^{(\gamma)} \cup \{\mathrm{id}_M\}$. □

Let $\underline{E}^{(\delta)} = \{ f \in \underline{E} \,|\, f \leq \sigma_a \text{ for some } a \in M \} = \{ f \,|\, f + \sigma_a = \sigma_a \text{ for some } a \in M \}$.

2.9 Proposition. (i) *$\underline{E}^{(\delta)}$ is an ideal of the semiring $\underline{E}$.*
(ii) $\underline{E}^{(1)} \subseteq \underline{E}^{(\delta)}$.
(iii) *$\underline{E}^{(\delta)} = \underline{E}$ iff $o_M \in M$ (and iff $\mathrm{id}_M \in \underline{E}^{(\delta)}$).*

Proof. It is easy. □

Put $\underline{E}^{(\epsilon)} = \underline{E}^{(\alpha)} \cap \underline{E}^{(\delta)} = \{ f \in \underline{E} \,|\, a \leq f(M) \leq b \text{ for some } a, b \in M \}$.

2.10 Proposition. (i) *$\underline{E}^{(\epsilon)}$ is an ideal of the semiring $\underline{E}$.*
(ii) $\underline{E}^{(1)} \subseteq \underline{E}^{(\epsilon)}$.
(iii) *$\underline{E}^{(\epsilon)} = \underline{E}$ iff $0_m, o_m \in M$ (and iff $\mathrm{id}_M \in \underline{E}^{(\epsilon)}$).*

Proof. It is easy. □

2.11 Proposition. *Let S be a subsemiring of $\underline{E}$. Define a relation $\varrho_{\alpha,S}$ on S by $(f,g) \in \varrho_{\alpha,S}$ iff there is an element $a \in M$ such that $f(x) + a = g(x) + a$ for every $x \in M$. Then $\varrho_{\alpha,S}$ is a congruence of the semiring S. Moreover:*
(i) *$(f,g) \in \varrho_{\alpha,S}$ iff $f + \sigma_a = g + \sigma_a$ for at least one $a \in M$.*
(ii) *$T \times T \subseteq \varrho_{\alpha,S}$, where $T = S \cap \underline{E}^{(1)}$.*
(iii) *If $\underline{E}^{(1)} \subseteq S$ then $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho_{\alpha,S}$.*
(iv) *If $o_M \in M$ then $\varrho_{\alpha,S} = S \times S$.*
(v) *If $f, \sigma_a \in S$ and $(f, \sigma_a) \in \varrho_{\alpha,S}$ then $f \in \underline{E}^{(\delta)}$.*

Proof. Clearly, $\varrho = \varrho_{\alpha,S}$ is reflexive and symmetric and (i) is true. Now, if $(f,g) \in$ $\in \varrho$ and $(g,h) \in \varrho$ then $f + \sigma_a = g + \sigma_a$, $g + \sigma_b = h + \sigma_b$ for some $a, b \in M$, and so $f + \sigma_{a+b} = f + \sigma_a + \sigma_b = g + \sigma_a + \sigma_b = h + \sigma_a + \sigma_b = h + \sigma_{a+b}$ and $(f,h) \in \varrho$. It follows that ϱ is an equivalence defined on S. If $(f,g) \in \varrho$, $f + \sigma_a = g + \sigma_b$, then $f+h+\sigma_a = f+h+\sigma_a$, $(f+h, g+h) \in \varrho$, $hf+\sigma_{h(a)} = h(f+\sigma_a) = h(g+\sigma_a) = hg+\sigma_{h(a)}$, $(hf, hg) \in \varrho$, $fh + \sigma_a = (f + \sigma_a)h = (g + \sigma_a)h = gh + \sigma_a$, $(fh, gh) \in \varrho$. It follows that ϱ is a congruence of the semiring S.

If $\sigma_a, \sigma_b \in T$ then $(\sigma_a, \sigma_b) \in \varrho$, since $\sigma_a + \sigma_{a+b} = \sigma_{a+b} = \sigma_b + \sigma_{a+b}$. That is, $T \times T \subseteq \varrho$ and (ii),(iii) are clear. If $o_M \in M$ then $f + \sigma_{o_M} = \sigma_{o_M} = g + \sigma_{o_M}$ for all $f, g \in S$, and so $\varrho = S \times S$. Finally, if $(f, \sigma_a) \in \varrho$ then $\sigma_{a+b} = \sigma_a + \sigma_b = f + \sigma_b$ for some $b \in M$, and hence $\sigma_{a+b} = \sigma_{a+b} + \sigma_a = f + \sigma_b + \sigma_a = f + \sigma_{a+b}$, $f \leq \sigma_{a+b}$ and $f \in \underline{E}^{(\delta)}$. □

2.12 Proposition. *Let S be a subsemiring of $\underline{E}$ and $T = S \cap \underline{E}^{(\alpha)}$. Then:*
(i) $\varrho_{\beta,S} = (T \times T) \cup \mathrm{id}_S$ *is a congruence of S.*
(ii) $\varrho_{\gamma,S} = \varrho_{\alpha,S} \cap \varrho_{\beta,S}$ *is a congruence of S.*
(iii) *If $\underline{E}^{(1)} \subseteq S$ then $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho_{\gamma,S}$.*

Proof. If $T = \emptyset$ then $\varrho_{\beta,S} = \mathrm{id}_S$. If $T \neq \emptyset$ then T is a bi-ideal of S and $\varrho_{\beta,S}$ is a congruence again. The rest is clear. □

2.13 Lemma. *Let $a \in M$. Define a relation ξ_a on M by $(u, v) \in \xi_a$ iff either $u = v$ or $u + a = u$, $v + a = v$. Then ξ_a is a congruence of the semilattice M (namely the congruence corresponding to the ideal $\{\, x \mid a \leq x \,\}$).*

Proof. It is easy. □

2.14 Lemma. *Let I be an ideal of M. Denote by S the set of endomorphisms $f \in \underline{E}$ such that either $f(I) \subseteq I$ or $|f(I)| = 1$. Then:*
(i) *S is a subsemiring of $\underline{E}$.*
(ii) $\underline{E}^{(1)} \subseteq S$.
(iii) $\underline{E}^{(\gamma)} \subseteq S$.
(iv) $\mathrm{id}_M \in S$.
(v) $\underline{E}^{(\beta)} \subseteq S$.
(vi) $(I \times I) \cup \mathrm{id}_M$ *is a congruence of the S-semimodule ${}_S M$.*

Proof. It is easy. □

2.15 Lemma. *Let I and S be as in 2.14. Define a relation ϱ on S by $(f, g) \in \varrho$ iff $f(x) = g(x)$ for every $x \in M \setminus I$ such that $\{(f(x), g(x)\} \nsubseteq I$. Then:*
(i) *ϱ is a congruence of the semiring S.*
(ii) *$(\sigma_a, \sigma_b) \in \varrho$ for all $a, b \in I$.*
(iii) *$(\sigma_a, \sigma_c) \notin \varrho$ for all $a \in I$ and $c \in M \setminus I$.*
(iv) *$(\sigma_c, \sigma_d) \notin \varrho$ for all $c, d \in M \setminus I$, $c \neq d$.*
(v) *$(\lambda_a, \lambda_b) \in \varrho$ for all $a, b \in I$.*
(vi) *$(\lambda_a, \lambda_c) \notin \varrho$ for all $a \in I$ and $c \in M \setminus I$.*
(vii) *$(\lambda_c, \lambda_d) \notin \varrho$ for all $c, d \in M \setminus I$, $c \neq d$.*

Proof. It is easy. □

2.16 Corollary. *Let I and S be as in 2.14. If $|I| \geq 2$ and $I \neq M$ then the semiring S is not congruence-simple.* □

3. Endomorphisms (b)

Let S be a subsemiring of $\underline{E}$ such that $\underline{E}^{(1)} \subseteq S$.

3.1 Proposition. *S is bi-ideal-simple (bi-ideal-free) if and only if $S \subseteq \underline{E}^{(\alpha)}$.*

Proof. Put $I = S \cap \underline{E}^{(\alpha)}$. Then $\underline{E}^{(1)} \subseteq I$ and I is a non-trivial bi-ideal of the semiring S. If S is bi-ideal-simple then $I = S$ and $S \subseteq \underline{E}^{(\alpha)}$. Conversely, if $S \subseteq \underline{E}^{(\alpha)}$ then $S = S + \underline{E}^{(1)}$. Now, if K is a bi-ideal of S then $\underline{E}^{(1)} \subseteq K$ and $S = S + \underline{E}^{(1)} \subseteq K$. Thus $K = S$ and S is bi-ideal-free. □

3.2 Proposition.. (i) $0_S \in S$ *iff* $0_M \in M$ *(then* $0_S = \sigma_{0_M}$*).*
(ii) $o_S \in S$ *iff* $o_M \in M$ *(then* $o_S = \sigma_{o_M}$*).*
(iii) $\underline{E}^{(1)}$ *is the smallest (left) ideal of S and it is the set of left multiplicatively absorbing element of S.*
(iv) *S has no right multiplicatively absorbing elements.*

Proof. (i) If $0_S \in S$ then $a = \sigma_a(x) = (\sigma_a + 0_S)(x) = \sigma_a(x) + 0_S(x) = a + 0_S(x)$ for all $a, x \in M$. Thus $0_S(x) = 0_M \in M$ and $0_S = \sigma_{0_M}$. Conversely, if $0_M \in M$ then $\sigma_{0_M} = 0_S$ is clear.
(ii) If $o_S \in S$ then $o_S(x) = (o_S + \sigma_a)(x) = o_S(x) + \sigma_a(x) = o_S(x) + a$ for all $a, x \in M$. Thus $o_S(x) = o_M \in M$ and $o_S = \sigma_{o_M}$. Conversely, if $o_M \in M$ then $\sigma_{o_M} = o_S$ is clear.
(iii) and (iv) Easy to see. □

3.3 Proposition. *The following conditions are equivalent:*

(i) *The semiring S has at least one left multiplicatively neutral element.*
(ii) $1_S \in S$.
(iii) $\mathrm{id}_M \in S$.
(iv) $\underline{E}^{(\beta)} \subseteq S$.

Proof. If $e \in S$ is left multiplicatively neutral then $a = \sigma_a(x) = (e\sigma_a)(x) = e(\sigma_a(x)) = e(a)$ for all $a, x \in M$. Thus $e = \mathrm{id}_M \in S$. The rest is clear. □

3.4 Proposition. *Let S be bi-ideal-simple and let $1_S \in S$. Then $0_M \in M$, $0_S \in S$ and $\underline{E}^{(\beta)} \subseteq S \subseteq \underline{E}^{(\alpha)}$.*

Proof. By 3.1, $S \subseteq \underline{E}^{(\alpha)}$. By 3.3, $\underline{E}^{(\beta)} \subseteq S$. Consequently, $\underline{E}^{(\beta)} \subseteq \underline{E}^{(\alpha)}$, $0_M \in M$ by 2.7(iii) and $0_S \in S$ by 3.2(i). □

3.5 Lemma. *Let $\varrho \neq \mathrm{id}_S$ be a congruence of the semiring S. Then there are $a, b \in M$ such that $a < b$ and $(\sigma_a, \sigma_b) \in \varrho$.*

Proof. There are $f, g \in S$ such that $f \neq g$ and $(f, g) \in \varrho$. Then $f(u) \neq g(u)$ for at least one $u \in M$ and we can assume that $f(u) \nleq g(u) = a$. If $b = a + f(u)$ then $\sigma_a = g\sigma_u$, $\sigma_{f(a)} = f\sigma_u$ and $(\sigma_a, \sigma_{f(u)} \in \varrho$. We have $f(u) \nleq a$, $a < b$ and $(\sigma_a, \sigma_b) = (\sigma_a + \sigma_a, \sigma_a + \sigma_{f(u)}) \in \varrho$. □

3.6 Lemma. *Let ϱ be a congruence of S and let $a, b, c \in M$ be such that $a < b$ and $(\sigma_a, \sigma_b) \in \varrho$. Let $f \in S$ be such that $f(a) = c$ and $f(x) > c$ whenever $a < x$. Then $c < f(b)$ and $(\sigma_c, \sigma_{f(b)}) \in \varrho$.*

Proof. Since $a < b$, we have $c < f(b)$. Since $(\sigma_a, \sigma_b) \in \varrho$, we have $(\sigma_c, \sigma_{f(b)}) = = (\sigma_{f(a)}, \sigma_{f(b)}) = (f\sigma_a, f\sigma_b) \in \varrho$. □

Consider the following condition:

(A) For all $a, b \in M \setminus \{o_M\}$ there is at least one endomorphism $f \in S$ such that $f(a) = b$ and $f(x) > b$ for every $x > a$.

3.7 Lemma. *Assume that* (A) *is true. Let $\varrho \neq \mathrm{id}_S$ be a congruence of S. Then for every $a \in M$, $a \neq o_M$, there is at least one $b \in M$ such that $a < b$ and $(\sigma_a, \sigma_b) \in \varrho$.*

Proof. By 3.5, $(\sigma_c, \sigma_d) \in \varrho$ for some $c, d \in M$, $c < d$. Then $c \neq o_M$ and, using (A), we find $f \in S$ with $f(c) = a$ and $f(x) > a$ whenever $x > c$. By 3.6, $(\sigma_a, \sigma_b) \in \varrho$, where $b = f(d) > a$. □

3.8 Lemma. *Let ϱ be a congruence of S and let $a, b, c \in M$ be such that $a \leq c \leq b$. If $(\sigma_a, \sigma_b) \in \varrho$ then $(\sigma_a, \sigma_c) \in \varrho$ and $(\sigma_c, \sigma_b) \in \varrho$.*

Proof. We have $\sigma_a + \sigma_c = \sigma_c$ and $\sigma_b + \sigma_c = \sigma_b$. □

3.9 Lemma. *Let ϱ be a congruence of S such that $(\sigma_a, \sigma_b) \in \varrho$ whenever $a < b$. Then $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$.*

Proof. If $a, b \in M$ then $a \leq a + b$, $(\sigma_a, \sigma_{a+b}) \in \varrho$, $b \leq a + b$, $(\sigma_b, \sigma_{a+b}) \in \varrho$ and, finally, $(\sigma_a, \sigma_b) \in \varrho$. □

3.10 Lemma. *Assume that $S \subseteq \underline{E}^{(\epsilon)}$. Let ϱ be a congruence of S such that $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \subseteq \varrho$. Then $\varrho = S \times S$.*

Proof. First, let $f, g \in S$, $f \leq g$. Since $S \subseteq \underline{E}^{(\epsilon)}$, we have $\sigma_a \leq f \leq g \leq \sigma_b$ for some $a, b \in M$, $a \leq b$. Now, $(\sigma_a, \sigma_b) \in \varrho$, and hence $(f, \sigma_b) = (f + \sigma_a, f + \sigma_b) \in \varrho$ and $(g, \sigma_b) = (g + \sigma_a, g + \sigma_b) \in \varrho$. Thus $(f, g) \in \varrho$. In the general case, we have $f \leq f + g$, $g \leq f + g$, $(f, f + g) \in \varrho$, $(g, f + g) \in \varrho$ and $(f, g) \in \varrho$. □

Consider the following condition:

(B) Every infinite strictly increasing chain $a_1 < a_2 < a_3 < \ldots$ of elements from M is upwards cofinal in M.

3.11 Lemma. *Assume that both* (A) *and* (B) *are satisfied. Then* $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$ *for every congruence* $\varrho \neq \mathrm{id}_S$ *of* S.

Proof. In view of 3.9, we have to show that $(\sigma_a, \sigma_b) \in \varrho$ whenever $a < b$. If $o_M \in M$ and $(\sigma_a, \sigma_{o_M}) \in \varrho$ then $(\sigma_a, \sigma_b) \in \varrho$ by 3.8. Consequently, assume that either $o_M \notin M$ or $(\sigma_a, \sigma_{o_M}) \notin \varrho$. Using 3.7 repeatedly, we get an infinite strictly increasing chain $a_1 < a_2 < a_3 < \dots$ such that $a_1 = a$ and $(\sigma_a, \sigma_{a_i}) \in \varrho$ for every $i = 1, 2, 3, \dots$. With respect to (B), we have $b \leq a_n$ for some n. By 3.8, $(\sigma_a, \sigma_b) \in \varrho$. □

3.12 Proposition. *Assume that* $S \subseteq \underline{E}^{(\epsilon)}$ *and that the conditions* (A) *and* (B) *are satisfied. Then the semiring* S *is congruence-simple.*

Proof. Let ϱ be a congruence of S. Then $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$ is proved in 3.11. It remains to use 3.10. □

4. Endomorphisms (c)

Let S be a subsemiring of $\underline{E}$ such that $\underline{E}^{(1)} \subseteq S$.

4.1 Let ϱ be a congruence of the semiring S. Define a relation τ on M by $(a, b) \in \tau$ iff $(\sigma_a, \sigma_b) \in \varrho$.

4.1.1 Lemma. τ *is a congruence of the left* S*-semimodule* ${}_SM$.

Proof. It is easy. □

4.1.2 Lemma. $\varrho \neq \mathrm{id}_S$ *iff* $\tau \neq \mathrm{id}_M$.

Proof. If $\varrho \neq \mathrm{id}_S$ then $\tau \neq \mathrm{id}_M$ by 3.5. If $\tau \neq \mathrm{id}_M$ then $\varrho \neq \mathrm{id}_S$ trivially. □

4.1.3 Lemma. *Assume that* ${}_SM$ *is congruence-simple and that* $\varrho \neq \mathrm{id}_S$. *Then* $\underline{E}^{(1)} \times \times \underline{E}^{(1)} \subseteq \varrho$.

Proof. We have $\tau \neq \mathrm{id}_M$ by 4.1.2. Since the semimodule ${}_SM$ is congruence-simple, we get $\tau = M \times M$. Thus $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$. □

4.2 Proposition. *The (left* S*-)semimodule* ${}_SM$ *is faithful and strictly minimal.*

Proof. It is easy. □

4.3 Let α be a congruence of the semimodule ${}_SM$. Define a relation ϱ on S by $(f, g) \in \varrho$ iff $(f(x), g(x)) \in \alpha$ for every $x \in M$.

4.3.1 Lemma. ϱ *is a congruence of the semiring* S.

Proof. It is easy. □

4.3.2 Lemma. $\varrho \neq \mathrm{id}_S$ *iff* $\alpha \neq \mathrm{id}_M$.

Proof. If $(f, g) \in \varrho$, $f \neq g$, then $f(u) \neq g(u)$ for at least one $u \in M$ and we have $(f(u), g(u)) \in \alpha$. Thus $\alpha \neq \mathrm{id}_M$. Conversely, if $(a, b) \in \alpha$, $a \neq b$, then $\sigma_a \neq \sigma_b$ and $(\sigma_a, \sigma_b) \in \varrho$. Thus $\varrho \neq \mathrm{id}_S$. □

4.3.3 Lemma. $\varrho = S \times S$ *iff* $\alpha = M \times M$.

Proof. We can proceed similarly as in the proof of 4.3.2. □

4.3.4 Lemma. *If* $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$ *then* $\varrho = S \times S$.

Proof. Obvious. □

4.4 Proposition. *The following conditions are equivalent:*
(i) $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$ *for every congruence* $\varrho \neq \mathrm{id}_S$ *of the semiring* S.
(ii) *The semimodule* ${}_SM$ *is congruence-simple.*

Proof. (i) implies (ii). Let $\alpha \neq \mathrm{id}_M$ be a congruence of ${}_SM$. Consider the congruence ϱ defined in 4.3. By 4.3.2, we have $\varrho \neq \mathrm{id}_S$, and hence $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$. By 4.3.4, $\varrho = S \times S$ and, finally, $\alpha = M \times M$ by 4.3.3. Thus ${}_SM$ is congruence-simple.
(ii) implies (i). Let $\varrho \neq \mathrm{id}_S$ be a congruence of S. Consider the congruence τ defined in 4.1. Then $\tau \neq \mathrm{id}_M$ by 4.1.2, and hence $\tau = M \times M$. Now, $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$ follows from 4.1.3. □

4.5 Corollary. *If the semiring* S *is congruence-simple then the semimodule* ${}_SM$ *is congruence-simple.* □

4.6 Proposition. *Assune that both* (A) *and* (B) *are satisfied. Then* ${}_SM$ *is congruence-simple.*

Proof. Combine 3.11 and 4.4. □

4.7 Proposition. *Assume that* $S \subseteq \underline{E}^{(\epsilon)}$. *Then the semiring* S *is congruence-simple if and only if the semimodule* ${}_SM$ *is congruence-simple.*

Proof. First, let S be congruence-simple. Then ${}_SM$ is congruence-simple by 4.5. Conversely, if ${}_SM$ is congruence-simple and if $\varrho \neq \mathrm{id}_S$ is a congrunece of S then $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$ by 4.1.3, and hence $\varrho = S \times S$ by 3.10. Thus S is congruence-simple. □

4.8 Remark. Notice that the mapping $a \mapsto \sigma_a$ is an isomorphism of the seminodule ${}_SM$ onto the semimodule ${}_S\underline{E}^{(1)}$.

4.9 Lemma. *Let* ϱ *be a congruence of* S *such that* $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$. *Then* $\varrho_{\gamma,S} \subseteq \varrho$.

Proof. Let $(f,g) \in \varrho_{\gamma,S}$, $f \neq g$. Then $(f,g) \in \varrho_{\beta,S}$, and hence $f,g \in \underline{E}^{(\alpha)} \cap S$ (see 2.12) and there are $a,b \in M$ such that $f + \sigma_a = f$ and $g + \sigma_b = g$. Further, $(f,g) \in \varrho_{\alpha,S}$ (see 2.11) and there is $c \in M$ with $f + \sigma_c = g + \sigma_c$. Now, $(\sigma_a, \sigma_c) \in \varrho$, and so $(f, \sigma_c+f) = (f+\sigma_a, f+\sigma_c) \in \varrho$. Similarly, $(g, \sigma_c+g) \in \varrho$. Since $\sigma_c+f = \sigma_c+g$, we have $(f,g) \in \varrho$. □

4.10 Proposition. $\varrho_{\gamma,S}$ *is just the congruence of* S *generated by the set* $\underline{E}^{(1)} \times \underline{E}^{(1)}$.

Proof. We have $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho_{\gamma,S}$ by 2.12(iii) and the rest follows from 4.9. □

4.11 Proposition. *If* $S \subseteq \underline{E}^{(\epsilon)}$ *then* $\varrho_{\gamma,S} = S \times S$.

Proof. Combine 4.10 and 3.10. □

4.12 Proposition. *Assume that the conditions* (A) *and* (B) *are satisfied. Then the semiring* S *is subdirectly irreducible and the congruence* $\varrho_{\gamma,S}$ *is just the monolith congruence of* S.

Proof. Let $\varrho \neq \mathrm{id}_S$ be a congruence of S. By 3.11, $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho$. By 4.9, $\varrho_{\gamma,S} \subseteq \varrho$. Of course, $\varrho_{\gamma,S} \neq \mathrm{id}_S$, and hence $\varrho_{\gamma,S}$ is the monolith congruence of S. □

4.13 Remark. Combining 4.11 and 4.12, we get another proof of 4.6.

Consider the following condition:

(C) For all $a,b,c,d \in M$ such that $a < b$ and $c < d$ there is at least one $f \in S$ such that $f(a) = c$ and $f(b) = d$.

4.14 Lemma. *Assume that* (C) *is true. If* $\varrho \neq \mathrm{id}_S$ *is a congruence of* S *then* $\underline{E}^{(1)} \times \times \underline{E}^{(1)} \subseteq \varrho$.

Proof. By 3.5, $(\sigma_a, \sigma_b) \in \varrho$ for some $a,b \in M$, $a < b$. Since (C) is true, we get $(\sigma_c, \sigma_d) \in \varrho$ for all $c,d \in M$, $c < d$. Now, if $x,y \in M$ then $x \leq x+y$, $y \leq x+y$, $(\sigma_x, \sigma_{x+y}) \in \varrho$, $(\sigma_y, \sigma_{x+y}) \in \varrho$ and, finally, $(\sigma_x, \sigma_y) \in \varrho$. □

4.15 Proposition. (cf. 4.12) *Assume that the condition* (C) *is satisfied. Then the semiring* S *is subdirectly irreducible and the congruence* $\varrho_{\gamma,S}$ *is just the monolith congruence of* S.

Proof. Combine 4.10 and 4.14. □

4.16 Proposition. *Assume that either the conditions* (A), (B) *are true or that the condition* (C) *is true. The following conditions are equivalent:*

(i) *The semiring* S *is congruence-simple.*

(ii) $S \subseteq \underline{E}^{(\epsilon)}$.

(iii) *For every* $f \in S$ *there are* $a,b \in M$ *such that* $\sigma_a + f = f$ *and* $\sigma_b + f = f$ *(i.e.,* $\sigma_a \leq f \leq \sigma_b$*).*

Moreover, if these equivalent conditions are satisfied then the semimodule ${}_SM$ is congruence-simple.

Proof. (i) implies (ii). We have $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \varrho_{\gamma,S}$ (see 2.12(iii)), and hence $\varrho_{\gamma,S} \neq$ $\neq \mathrm{id}_S$. Since S is congruence-simple, we get $\varrho_{\gamma,S} = S \times S$. Of course, $\varrho_{\gamma,S} =$ $= \varrho_{\alpha,S} \cap \varrho_{\beta,S}$, and therefore $\varrho_{\alpha,S} = S \times S = \varrho_{\beta,S}$. Since $\varrho_{\beta,S} = S \times S$, the inclusion $S \subseteq \underline{E}^{(\alpha)}$ follows from 2.12(i). Since $\varrho_{\alpha,S} = S \times S$, the inclusion $S \subseteq \underline{E}^{(\delta)}$ follows from 2.11(v). Thus $S \subseteq \underline{E}^{(\alpha)} \cap \underline{E}^{(\delta)} = \underline{E}^{(\epsilon)}$.
(ii) is equivalent to (iii). This is clear from the definitions.
(ii) implies (i). If (A),(B) are true then S is congruence-simple by 3.12. If (C) is true then S is congruence-simple by 4.15 and 4.11. □

4.17 Proposition. *Assume that either the conditions* (A),(B) *are true or that the condition* (C) *is true. Then the semimodule ${}_SM$ is congruence-simple.*

Proof. Use 4.6, 4.14 and 4.4. □

4.18 Remark. Let (C) be true and let $a, b \in M \setminus \{o_M\}$. Choose $c > b$. For every $d > a$ there is $f_d \in S$ such that $f_d(a) = b$ and $f_d(d) = c$. Now, assume that the set $\{d\} d > a\}$ is finite and put $f = \sum f_d$. Then $f(a) = b$ and $f(d) \geq c > b$ for every $d > a$. In particular, if M is finite then (C) implies (A) (and, of course, (B) is true).

4.19 Proposition. *Assume that the semimodule ${}_SM$ is congruence-simple. Then the semiring S is subdirectly irreducible and the congruence $\varrho_{\gamma,S}$ is the monolith congruence of S.*

Proof. See 4.1.3. □

4.20 Proposition. *The semiring S is congruence-simple if and only if $S \subseteq \underline{E}^{(\epsilon)}$ and the semimodule ${}_SM$ is congruence-simple.*

Proof. If S is congruence-simple then $S \subseteq \underline{E}^{(\alpha)}$ by 3.1. Furthermore, $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq$ $\subseteq \varrho_{\gamma,S}$, $\varrho_{\gamma,S} = S \times S$, and hence $\varrho_{\alpha,S} = S \times S = \varrho_{\beta,S}$. Using 2.11(v), we get $S \subseteq \underline{E}^{(\delta)}$. Thus $S \subseteq \underline{E}^{(\alpha)} \cap \underline{E}^{(\delta)} = \underline{E}^{(\epsilon)}$. By 4.5, ${}_SM$ is congruence-simple. Conversely, if $S \subseteq \underline{E}^{(\epsilon)}$ and ${}_SM$ is congruence-simple then S is subdirectly irreducible and $\varrho_{\gamma,S}$ is the monolith congruence. By 4.1, $\varrho_{\gamma,S} = S \times S$, and so S is congruence-simple. □

5. Endomorphisms (d)

Let S be a subsemiring of $\underline{E}$ with $\underline{E}^{(1)} \subseteq S$.

5.1 Lemma. *Let $u, v \in M$, $u \neq v$. Define a relation $\alpha_{u,v}$ on M by $(a, b) \in \alpha_{u,v}$ iff $\{u, v\} \nsubseteq \{f(a), f(b)\}$ for every $f \in S$. Then $\alpha_{u,v}$ is reflexive, symmetric and stable.*

Proof. First, $(a, a) \in \alpha_{u,v}$, since $|\{u, v\}| = 2$ and $|\{f(a), f(a)\}| = 1$. The symmetry is clear. If $(a, b) \in \alpha_{u,v}$ then $(f(a), f(b)) \in \alpha_{u,v}$ for every $f \in S$. Finally, $\{f(a + c), f(b + c)\} = \{f(a) + f(c), f(b) + f(c)\} = \{(f + \sigma_{f(c)})(a), (f + \sigma_{f(c)})(b)\}$, and hence $\{u, v\} \nsubseteq \{f(a + c), f(b + c)\}$ and $(a + c, b + c) \in \alpha_{u,v}$. □

5.2 Lemma. *Let $u, v \in M$, $u \neq v$, and let $\beta_{u,v}$ be the transitive closure of the relation $\alpha_{u,v}$. Then $\beta_{u,v}$ is a congruence of the semimodule ${}_S M$.*

Proof. It follows easily from 5.1. □

5.3 Lemma. *Let $u, v \in M$, $u \neq v$. The following conditions are equivalent:*

(i) $\alpha_{u,v} = \mathrm{id}_M$.

(ii) $\beta_{u,v} = \mathrm{id}_M$.

(iii) *For all $a, b \in M$, $a \neq b$, there is at least one $f \in S$ such that* $\{u, v\} = \{f(u), f(v)\}$.

Moreover, if these equivalent conditions are satisfied then either $u < v$ or $v < u$.

Proof. It is easy (choose $a < b$). □

5.4 Remark. Using 5.3, one sees easily that the condition (C) is true if and only if $\alpha_{u,v} = \mathrm{id}_M$ for all $u, v \in M$, $u < v$.

5.5 Lemma. *Let $u, v, w \in M$ be such that $u \neq v$ and $(u, w) \in \alpha_{u,v}$.*

(i) *If $f \in S$ is such that $f(u) = u$ then $f(w) \neq v$.*

(ii) *If $f \in S$ is such that $f(u) = v$ then $f(w) \neq u$.*

Proof. It is easy. □

Consider the following condition:

(C1) For all $a, b, c \in M$ such that $a < b$ and $a < c$ there is at least one $f \in S$ such that $f(a) = a$ and $f(b) = c$.

Clearly, (C) implies (C1).

5.6 Lemma. *The following conditions are equivalent:*

(i) (C1) *is true.*

(ii) $(u, w) \notin \alpha_{u,v}$ *whenever $u < v$ and $u < w$.*

(iii) $(u, w \notin \beta_{u,v}$ *whenever $u < v$ and $u < w$.*

Proof. Clearly, (i) is equivalent to (ii) and (iii) implies (ii). It remains to show that (ii) implies (iii). For, suppose that (ii) is true and $(u, w) \in \beta_{u,v}$. Then there are $u_0, u_1, \dots, u_n$, $n \geq 1$, such that $u_0 = u$, $u_n = w$ and $(u_i, u_{i+1}) \in \alpha_{u,v}$ for $i = 0, 1, \dots, n - 1$. Since $\alpha_{u,v}$ is stable, we get $(z_i, z_{i+1}) \in \alpha_{u,v}$ for $i = 0, 1, \dots, n - 1$, where $z_0 = u_0 = u$ and $z_i = \sum_{j \leq i} u_j$ for $i \geq 1$. Of course, $z_0 \leq z_1 \leq z_2 \leq \cdots \leq z_n$, $z_0 = u$ and $w = u_n \leq z_n$, and there is k such that $0 \leq k < n$ and $z_0 = z_1 = \cdots = z_k = u$ and $z_{k+1} \neq u$. Now, $(u, z_{k+1}) \in \alpha_{u,v}$ and $u < z_{k+1}$, a contradiction with (ii). □

5.7 Proposition. *Assume that the semimodule* ${}_SM$ *is congruence-simple. Then* (C) *is true if and only if* (C1) *is true.*

Proof. The direct implication is trivial. Now, let (C1) be true and let $a, b, c, d \in M$ be such that $a < b$ and $c < d$. By 5.6, $(c, d) \notin \beta_{c,d}$ and so $\beta_{c,d} \neq M \times M$. But $\beta_{c,d}$ is a congruence of ${}_SM$ and, since ${}_SM$ is congruence-simple, we have $\beta_{c,d} = \mathrm{id}_M$. In particular, $(a, b) \notin \beta_{c,d}$, and hence $(a, b) \notin \alpha_{c,d}$ either. That is, $\{c, d\} \subseteq \{f(a), f(b)\}$ for some $f \in S$. Since $c < d$ and $a < b$, we conclude that $f(a) = c$ and $f(b) = d$. We have proved that (C) is true. □

5.8 Proposition. *The following conditions are equivalent:*

(i) ${}_SM$ *is congruence-simple and* (C1) *is true.*
(ii) (C) *is true.*

Proof. Combine 5.7 and 4.17. □

5.9 Proposition. *If the semiring* S *is congruence-simple and* (C1) *is true then* (C) *is true.*

Proof. By 4.5, the semimodule ${}_SM$ is congruence-simple and 5.8 applies. □

5.10 Lemma. *Assume that the condition*(C1) *is satisfied. Let* ϱ *be a congruence of* ${}_SM$, $\varrho \neq \mathrm{id}_M, M \times M$. *Then:*
(i) *There is an ideal* A *of* M *such that* $\varrho = (A \times A) \cup \mathrm{id}_M$ *(*$M + A = A$*).*
(ii) $A \neq M$ *and* $|A| \geq 2$.

Proof. First, let A be a block of ϱ such that $|A| \geq 2$. Of course, A is a non-trivial subsemilattice of M and we claim that $M + A = A$. For, take $a \in A$, $a \neq o_A$. Then $a < b$ for some $b \in A$ and we have $(a, b) \in \varrho$. If $c \in M$ is such that $a < c$ then there is $f \in S$ with $f(a) = a$ and $f(b) = c$. Now, $(a, c) = (f(a), f(b) \in \varrho$, and therefore $c \in A$. Thus $M + a \subseteq A$. If $o_A \in A$ then $a < o_A$ for some $a \in A$ (since $|A| \geq 2$) and $M + o_A = M + a + o_A \subseteq A + o_A = \{o_A\}$. We have proved that $M + A \subseteq A$ (then $M + A = A$ and $o_A = o_M$). Now, if B is a block of ϱ with $|B| \geq 2$ then B is an ideal as well, and hence $\emptyset \neq A + B \subseteq A \cap B$ and $A = B$. Consequently, $\varrho = (A\times) \cup \mathrm{id}_M$. Since $\varrho \neq M \times M$, we have $A \neq M$ and the proof is finished. □

5.11 Remark. Consider the situation from 5.10. If $f \in S$ then either $f(A) \subseteq A$ or $|f(A)| = 1$. In the latter case, $f(A) = \{u\}$ and if $a \in A$ and $b \in M$ then $a + b \in A$ and $u = f(a + b) = f(a) + f(b) = u + f(b)$. Thus $f \leq \sigma_u$ and $f \in \underline{E}^{(\delta)}$. Put $S_\varrho = \{ f \in S \mid f(A) \subseteq A \}$. Clearly, S_ϱ is a subsemiring of S and $\sigma_a \in S_\varrho$ for every $a \in A$. Now, A becomes a left S_ϱ-semimodule. If $a, b, c \in A$ are such that $a < b$ and $a < c$ then $f(a) = a$, $f(b) = c$ for some $f \in S_\varrho$.
(i) Assume that the S_ϱ-semimodule A is congrneunce-simple. Then, for all $a, b, c, d \in$ $\in A$ such that $a < b$ and $c < d$, there is $f \in S_\varrho$ such that $f(a) = c$ and $f(b) = d$ (see the proof of 5.7).

Let α be a non-identical congruence of the semiring S_ϱ. Proceeding similarly as in 4.1, we get $(\sigma_a, \sigma_b) \in \alpha$ for all $a, b \in A$.

Define a relation α_1 on S_ϱ by $(f, g) \in \alpha_1$ iff $f|A = g|A$. Then α_1 is a congruence of S_ϱ and $(\sigma_a, \sigma_b) \notin \alpha_1$ whenever $a, b \in A$ are such that $a \neq b$. Consequently, $\alpha_1 = \mathrm{id}_A$ and A is a faithful S_ϱ-semimodule.

Now, consider again the congruence α. Let $f, g \in S_\varrho$ and $a, b \in A$ be such that $a \leq \leq f(x) \leq g(x) \leq b$ for every $x \in A$. We have $(\sigma_a, \sigma_b) \in \alpha$, and hence $(f+\sigma_a, f+\sigma_b) \in \in \alpha$ and $(g + \sigma_a, g + \sigma_b) \in \alpha$. Furthermore, $(f + \sigma_a)|A = f|A$, $(f + \sigma_b)|A = \sigma_b|A$, $(g + \sigma_a)|A = g|A$ and $(g + \sigma_b)|A = \sigma_b|A$. Since A is a faithful S_ϱ-semimodule, we conclude that $f + \sigma_a = f$, $f + \sigma_b = \sigma_b$, $g + \sigma_a = g$ and $g + \sigma_b = \sigma_b$. Thus $(f, \sigma_b) \in \alpha$, $(g, \sigma_b) \in \alpha$ and $(f, g) \in \alpha$. Now, more generally, let $f, g \in S_\varrho$ and $a_1, a_2, b_1, b_2 \in A$ be such that $a_1 \leq f(x) \leq b_1$ and $a_2 \leq g(x) \leq b_2$ for every $x \in A$. Then $a_1 \leq f(x) \leq (f+g)(x) \leq b_1+b_2$, and hence $(f, f+g) \in \alpha$. Similarly, $(g, f+g) \in \alpha$ and we get $(f, g) \in \alpha$.
(ii) Assume that the S_ϱ-semimodule A is congruence-simple and that for every $f \in S_\varrho$ there are elements $a \in A$ and $b \in M$ such that $a \leq f(x) \leq b$ for every $x \in A$ (then, in fact, $b \in A$). Then the semiring S_ϱ is congruence-simple (use (i)).
(iii) Put $T = \underline{E}^{(1)} \cup S_\varrho \cup (S_\varrho + \underline{E}^{(1)})$. It is immediately clear that T is a subsemiring of $\underline{E}$ and that $S_\varrho \subseteq T \subseteq S$. Moreover, $T_1 = \underline{E}^{(1)} \cup (S_\varrho + \underline{E}^{(1)})$ is a bi-ideal of the semiring T. Clearly, $T_1 = T$ iff $S_\varrho \subseteq \underline{E}^{(\alpha)}$.
(a) Now, let τ be a congruence of the semiring T such that $S_\varrho \times S_\varrho \subseteq \tau$. Take $a \in M$, $b \in A$ and $f \in S_\varrho$. Then $a \leq a + b \in A$ and, using (C1), we find $g \in S_\varrho$ such that $g(a) = a$ and $g(a + b) = a + b$ (see 5.11; $g = \sigma_a \in S_\varrho$ if $a = a + b$). Furthermore, $(g, \sigma_{a+b}) \in \tau$, and hence $(\sigma_a, \sigma_{a+b}) = (g\sigma_a, \sigma_{a+b}\sigma_a) \in \tau$. Since $(f, \sigma_{a+b}) \in \tau$, we get $(\sigma_a, f) \in \tau$. Now, it is easy to conclude that $\underline{E}^{(1)} \times \underline{E}^{(1)} \subseteq \tau$ and, in fact, $\tau = T \times T$.
(b) Define a relation θ on T by $(f, g) \in \theta$ iff, for every $x \in M$, we have either $f(x) = = g(x)$ or $f(x), g(x) \in A$. Then θ is a congruence of the semiring T (cf. 4.3). We have $(\sigma_a, \sigma_b) \in \theta$ for all $a, b \in A$, and hence $\theta \cap (S_\varrho \times S_\varrho) \neq \mathrm{id}_{S_\varrho}$. On the other hand, if $c \in A$ and $d \in M \setminus A$ then $(\sigma_c, \sigma_d) \notin \theta$. Consequently, $\mathrm{id}_T \neq \theta \neq T \times T$ and it follows that the semiring T is not congruence-simple.
(iv) Define a relation μ of S_ϱ by $(f, g) \in \mu$ iff, for every $x \in M$, we have either $f(x) = g(x)$ or $f(x), g(x) \in A$ (of course, $\mu = \theta \cap (S_\varrho \times S_\varrho)$). Then μ is a congruence of the semiring S_ϱ. We have $(\sigma_a, \sigma_b) \in \mu$ for all $a, b \in A$, and hence $\mu \neq \mathrm{id}_{S_\varrho}$.

Now, let $a \in M$ and $b \in A$. Then $a \leq a + b \in A$ and, using (C1), we find $f \in S$ such that $f(a) = a$ and $f(a + b) = a + b$ ($f = \sigma_a$ if $a = a + b$). Of course, $f \in S_\varrho$ and if $a \in M \setminus A$ then $f(a) \notin A$ and $f(a) \neq \sigma_{a+b}(a)$. Consequently, $(f, \sigma_{a+b} \notin \mu$ and $\mu \neq S_\varrho \times S_\varrho$. Thus the semiring S_ϱ is not congruence-simple (cf. (ii)).

5.12 Remark. 5.11 continued. Now, denote by $S_{\varrho,1}$ the set of all endomorphisms $f \in S_\varrho$ such that there is an element $a \in A$ with $a \leq f(x)$ for every $x \in A$. It is easy to see that $S_{\varrho,1}$ is a bi-ideal (and hence a subsemiring) of the semiring S_ϱ and that $\sigma_A \in S_{\varrho,1}$ for every $a \in A$.
(i) $S_{\varrho,1} = S_\varrho$, provided that the semiring S_ϱ is bi-ideal-simple.

(ii) Let $a, b, c \in A$ be such that $a < b$ and $a < c$. Since the condition (C1) is true, there is $f \in S_\varrho$ such that $f(a) = a$ and $f(b) = c$ (see 5.11). If $g = f + \sigma_a \in S_\varrho$ then $g(a) = a$ and $g(b) = c + a = c$. Clearly, $a \leq g(x)$ for every $x \in M$ and we get $g \in S_{\varrho,1}$.
(iii) Assume that the $S_{\varrho,1}$-semimodule A is congruence-simple. Then, for all $a, b, c, d \in A$ such that $a < b$ and $c < d$, there is $f \in S_{\varrho,1}$ with $f(a) = c$ and $f(b) = d$ (see the proof of 5.7). Furthermore, if α is a non-identical congruence of the semiring $S_{\varrho,1}$ then $(\sigma_a, \sigma_b) \in \alpha$ for all $a, b \in A$. The $S_{\varrho,1}$-semimodule A is faithful (see 5.11(i)). Assume, moreover, that for every $f \in S_{\varrho,1}$ there is an element $a \in M$ such that $f(x) \leq a$ for every $x \in A$ (this condition is satisfied, provided that $o_M \in M$). Then the semiring $S_{\varrho,1}$ is congruence-simple (see 5.11).

5.13 Remark. 5.11 and 5.12 continued. Let $S_{\varrho,2}$ be the set of all endomorphisms $f \in S_\varrho$ such that there are elements $a \in A$ and $b \in M$ with $a \leq f(x) \leq b$ for every $x \in A$ (then $b \in A$). It is easy to see that $S_{\varrho,2}$ is a subsemiring of the semiring $S_{\varrho,1}$ and that $\sigma_a \in S_{\varrho,2}$ for every $a \in A$. Besides, $S_{\varrho,2}$ is an ideal of the semiring S_ϱ (and $S_{\varrho,1}$, too).
(i) $S_{\varrho,2} = S_{\varrho,1}$, provided that $o_M \in M$.
(ii) $S_{\varrho,2} = S_{\varrho,1} = S_\varrho$, provided that the semiring S_ϱ is ideal-simple.
(iii) $S_{\varrho,2} = S_{\varrho,1}$, provided that the semiring $S_{\varrho,1}$ is ideal-simple.
(iv) $S_{\varrho,2} = S_{\varrho,1} = S_\varrho$, provided that $o_M \in M$ and the semiring S_ϱ is bi-ideal-simple.
(v) $S_{\varrho,2} = S_{\varrho,1}$, provided that $S \subseteq \underline{E}^{(\delta)}$.
(vi) If $S_{\varrho,2} = S_{\varrho,1}$ and the $S_{\varrho,1}$-semimodule A is congruence-simple then the semiring $S_{\varrho,1}$ is congruence-simple (see 5.12(iii)).

5.14 Remark. 5.11 continued. Let ϱ_1 be a non-identical congruence of the S_ϱ-semimodule A. Proceeding similarly as in 5.10, we find that $\varrho_1 = (A_1 \times A_1) \cup \mathrm{id}_A$, where A_1 is a non-trivial ideal of A. Then A_1 is an ideal of M and $A_1 \neq A$ iff $\varrho_1 \neq A \times A$.

Let $f \in S$. If $f \in S_\varrho$ then $f(A) \subseteq A$, and hence either $f(A_1) \subseteq A_1$ or $f(A_1) = \{v\}$ for some $v \in A \setminus A_1$. If $f \in S \setminus S_\varrho$ then $f(A_1) = f(A) = \{u\}$ for some $u \in M \setminus A$. The set $S_{\varrho_1} = \{ f \in S \mid f(A_1) \subseteq A_1 \}$ is a subsemiring of S_ϱ.

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