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ABOUT THE CLASS OF ORDERED LIMITED OPERATORS

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We give a brief survey of recent results of order limited operators related to some properties on Banach lattices.

1. Introduction

In [10], we have defined a new class of operators that we are called order limited operators. This class of operators is essentially based on the concept of limited sets introduced in [7]. It is bigger than the class of AM-compact operators, but smaller than the class of order Dunford-Pettis operators introduced in [4].

In this paper, we continue on this path by characterizing, in the first time, Banach lattice on which each order interval is limited (Proposition 3.1). As an important consequence, we obtain the same result of Witold Wnuk [14, Proposition 1.1]. Afterwards, we study the class of order limited operators. Also, we derive the following consequences: a characterization of this class of operators (Theorem 3.3, Corollary 3.4), the coincidence of this class of operators and the class of AM-compact operators (Corollary 3.5), the equivalence between a discrete Banach lattice with order continuous norm and a Banach lattice which admits the Gelfand-Phillips property and whose the lattice operations of its topological dual are weak* sequentially.

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2. Preliminaries

A norm bounded subset A of a Banach space X is said limited set if every weak* null sequence (f_n) of X' converges uniformly on A, that is, $\lim\limits_{n\to\infty} \sup_{x\in A} |\langle f_n, x\rangle| = 0$. It is easy to check that every relatively norm compact set is limited but the converse is not true in general. In fact, the set $\{e_n : n \in N\}$ of unit coordinate vectors is a limited set in ℓ^∞ which is not relatively compact. The space X is said to have the Gelfand-Phillips property if this may be reversed, i.e. if every limited subset of X is relatively norm compact (abb. GP-property). Alternatively, the Banach space X has the GP-property if, and only if, every limited and weakly null sequence (x_n) in X is norm null [8]. As example, the classical Banach spaces c_0 and ℓ^1 have the GP-property but the Banach space ℓ^∞ does not have the GP-property.

Let us recall a Banach space X has the Dunford-Pettis* property if every relatively weakly compact subset of X is limited. It turns out that a Banach space X has the DP* property if and only if $\lim f_n(x_n) = 0$ for every weakly null sequence (x_n) in X and every weak* convergent sequence (f_n) in X'. But if X is a Grothendieck space (i.e., weak and weak* convergence of sequences in X are coincide), then these properties are the same on X. As an example the Banach lattice ℓ^1 has the Dunford-Pettis* property.

Recall from [9] that a operator T from a Banach lattice E into a Banach space X is said to be order limited if carries each order bounded subset of E into a limited set of X. i.e. For each $x \in E^+$, the subset $T([-x; x])$ is limited in X. Let us recall that an operator T from a Banach lattice E into a Banach space X is said to be AM-compact if it carries each order-bounded subset of E onto a relatively compact set of X. An operator T from a Banach lattice E into a Banach space X is said to be order Dunford-Pettis if carries each order bounded subset of E into a Dunford-Pettis set of X. i.e. for each $x \in E^+$, the subset $T([-x; x])$ is Dunford-Pettis in X.

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|.\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|\cdot\|$ where the notation $x_\alpha \downarrow 0$ means that (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

The lattice operations in E' are called weak* sequentially continuous if the sequence $(|f_n|)$ converges to 0 in the weak* topology, whenever the sequence (f_n) converges weak* to 0 in E. Note that if E is a Banach lattice, its topological dual E', endowed with the dual norm and the dual order, is also a Banach lattice. A nonzero element x of a vector lattice E is discrete if the order ideal generated by x equals the lattice subspace generated by x. The vector lattice E is discrete, if it admits a complete disjoint system of discrete elements. Also, a vector lattice E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum. A subset A of a vector lattice E is called order bounded, if it includes in an order interval in E. A linear mapping T from a vector lattice E into another F is order bounded if it carries order bounded set of E into order bounded set of F. We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. The operator T is regular if $T = T_1 - T_2$ where T_1 and T_2 are positive operators from E into F. Note that each positive linear mapping on a Banach lattice is continuous. If an operator $T : E \longrightarrow F$ between two Banach lattices is positive, then its adjoint $T' : F' \longrightarrow E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. For terminologies concerning Banach lattice theory and positive operators we refer the reader to the excellent book of Aliprantis-Burkinshaw [2].

3. Mains results

In the following result, we give a characterization of Banach lattice whose its topological dual has weak* sequentially continuous lattice operations,

Proposition 3.1 *Let E be a Banach lattice. Then the following assertions are equivalent*

(1) *E' has weak* sequentially continuous lattice operations,*

(2) *for each $x \in E^+$, $[-x, x]$ is limited,*

(3) *each operator $S : E \rightarrow c_0$ is AM-compact.*

Proof. (2) $\Longleftrightarrow$ (1) $[-x, x]$ is limited if, and only if, $\sup\{|f_n(z)|; z \in [-x, x]\} \longrightarrow 0$. As $|f_n|(x) = \sup\{|f_n(z)|; z \in [-x, x]\}$, we conclude that $|f_n|$ converge weak* to 0. i.e., E' has weak* sequentially continuous lattice operations.

(2) $\Longleftrightarrow$ (3) Follows from Theorem 2.3 [12]. □

As an immediate consequence of Proposition 3.1, we obtain the Proposition 1.1 of [14],

Corollary 3.2 *Let E be a Banach lattice. If E is discrete with order continuous norm then E' has weak* sequentially continuous lattice operations.*

Proof. Let $x \in E^+$. Since E is discrete with order continuous norm then, it follows from Corollary 21.13 [1] that the order interval $[-x, x]$ is norm relatively compact and hence it is a limited subset of E.

Now it follows from Proposition 3.1 that E' has weak* sequentially continuous lattice operations. □

Our first major result gives necessary and sufficient conditions for which each operator is order limited.

Theorem 3.3 *Let T be an operator from a Banach lattice E into a Banach space X. Then the following assertions are equivalent:*

(1) *T is order limited operator,*

(2) *for each lcc operator S from X into an arbitrary Banach space Z, the composed operator $S \circ T$ is AM-compact,*
(3) *for every weak* null sequence (f_n) of X', we have $|T'(f_n)| \to 0$ for $\sigma(E', E)$.*

Proof. (1) $\Longrightarrow$ (2) Let S be an order limited operator from X into an arbitrary Banach space Z and let $x \in E^+$. Then $T([-x, x])$ is a limited subset of X and since S is *lcc*, it follows from Theorem 2.1 [12] that $S \circ T([-x, x])$ is relatively compact of Z. This show that $S \circ T$ is AM-compact.

(2) $\Longrightarrow$ (3) Let (f_n) be a weak* null sequence of X', we consider the operator $S : X \to c_0$ defined by $S(x) = (f_n(x))_{n=1}^{\infty}$ for all $x \in X$.

It is clear that S is well defined and it is a *lcc* operator (because c_0 has the GP-property). Now, it follows from the assumption that $S(T([-x, x]))$ is a relatively compact subset of c_0 for all $x \in E^+$. Then

$$\begin{aligned} |T'(f_n)|(y) &= \sup\{|T'(f_n)(y)|; y \in [-x, x]\} \\ &= \sup\{|f_n(z)|; z \in T([-x, x])\} \to 0 \end{aligned}$$

(see Exercise 14 of Section 3.2 [2])

(3) $\Longrightarrow$ (1) For each $x \in E^+$, we have

$$\sup\{|f_n(y)|; y \in [-x, x]\} = |T'(f_n)|(y) = \sup\{|f_n(z)|; z \in T([-x, x])\} \to 0$$

for each weak* null sequence of X', which prove that $T([-x, x])$ is a limited set of X for each $x \in E^+$ and hence T is an order limited operator. □

As consequence of Theorem 3.3 and Proposition 3.1, we obtain the following characterizations.

Corollary 3.4 *Let E be a Banach lattice. Then the following statements are equivalent*

(1) *each positive operator from E into E is order limited,*
(2) *the identity operator of E is order limited,*
(3) *E' has weak* sequentially continuous lattice operations.*

Proof. (1) $\Longrightarrow$ (2) Obvious.

(2) $\Longrightarrow$ (3) Let $x \in E^+$. Since the identity operator of E is order limited then $[-x, x]$ is a limited set in E and hence, it follows from Proposition 3.1 that E' has weak* sequentially continuous lattice operations.

(3) $\Longrightarrow$ (1) Let $T : E \to E$ be a positive operator and $S : E \to c_0$ an *lcc* operator. Since E' has weak* sequentially continuous lattice operations, it follows from Proposition 3.1 that S is AM-compact and hence $S \circ T$ is AM-compact. Finally, it follows from Theorem 3.3 that T is order limited. □

Another consequence of Theorem 3.3, is giving by the following result:

Corollary 3.5 *Let X and Y be two Banach spaces such that Y has the GP-property. Then each order limited operator $T : X \to Y$ is AM-compact.*

Proof. Let $T : X \to Y$ be an *lcc* operator. Since Y has the GP-property then its identity operator $Id_Y : Y \to Y$ is *lcc*. Now, it follows from Theorem 3.3 that $Id_Y \circ T = T$ is AM-compact. □

There exist operators that are not order limited. In fact, the identity operator of the Banach lattice $L^2[0;1]$ is not order limited (because the lattice operations of $L^2[0;1]$ are not weak* sequentially continuous).

Proposition 3.6 *Let E be a Banach lattice and X a Banach space. If the norm of E is order continuous and either X has the DP* property or E has the DP* property then, each operator T from E into X is order limited.*

Proof. Let $T : E \to X$ be an operator. Since the norm of E is order continuous, then it follows from Theorem 2.4.3 of [11] that for each $x \in E^+$, the order interval $[-x;x]$ is weakly compact then $T([-x;x])$ is weakly compact in X.

- Since X has the DP* property, then $T([-x;x])$ is a limited set in X and hence T is order limited.
- Since E has the DP* property, then the order interval $[-x;x]$ is limited and hence it follows from [5] that $T([-x;x])$ is limited. So, T is order limited.

□

Lemma 3.7 *Let E be a Dedekind σ-complete Banach lattice. Then, for every order bounded disjoint sequence (x_n) in E, the subset $\{x_n, n \in \mathbb{N}\}$ is limited.*

Proof. Let (x_n) be an order bounded disjoint sequence in E and pick some $x \geq 0$ such that $|x_n| \leq x$ for all n. Let (f_n) be a weak* null sequence of E' and let $\varepsilon > 0$. By Theorem 4.42 [2] there exists $f \in (E')^+$ such that $(|f_n| - f)^+(x) \leq \frac{\varepsilon}{2}$ for each n.

As $(|x_n|)$ is an order bounded disjoint sequence in E then $|x_n| \to 0$ weakly. So there exists some m such that $f(|x_n|) \leq \frac{\varepsilon}{2}$ for each $n \geq m$.

Then for every $n \geq m$ we have

$$\begin{aligned} |f_n(x_n)| &\leq |f_n|(|x_n|) \\ &\leq (|f_n| - f)^+(|x_n|) + f(|x_n|) \\ &\leq (|f_n| - f)^+(x) + \frac{\varepsilon}{2} \\ &\leq \varepsilon \end{aligned}$$

This implies that $f_n(x_n) \to 0$. Thus the set $\{x_n, n \in \mathbb{N}\}$ is limited by Corollary 3.2 [10]. □

Proposition 3.8 *Let E be a Dedekind σ-complete Banach lattice and X be a Banach space. Then each lcc operator $T : E \to X$ is order weakly compact.*

Proof. Let (x_n) be an order bounded disjoint sequence of E. It follows from a Remark of ([3], page 185) and Lemma 3.7 that (x_n) is a limited weakly null sequence. Since T is *lcc* then, $\|T((x_n)\| \to 0$. Hence Dodds's Theorem ([2], Theorem 5.57) implies that T is order weakly compact. □

As an immediate consequence of Proposition 3.8, we have the following result,

Corollary 3.9 *Let E be a Dedekind σ-complete Banach lattice. If E has the GP-property then the norm of E is order continuous.*

Proposition 3.10 *Let E be a Banach lattice and X be a Banach space. If E' has weak* sequentially continuous lattice operations and E has the GP-property then, each operator $T : E \to X$ is AM-compact.*

Proof. Let $x \in E^+$, it follows from Theorem 2.4.3 of [11] that the order interval $[-x, x]$ is relatively weakly compact. Since E' has weak* sequentially continuous lattice operations then, it follows from proposition 3.1 that $[-x, x]$ is a limited set of X and since E has the GP-property then $[-x, x]$ is relatively compact and hence $T([-x, x])$ is relatively compact. This show that T is AM-compact. □

As a consequence of Proposition 3.10, we obtain the following result,

Corollary 3.11 *Let E be a Banach lattice. If E' has weak* sequentially continuous lattice operations and E has the GP-property then, E is discrete with an order continuous norm.*

As consequence of Theorem 4.5 [13] and Corollary 3.11, we obtain the following result,

Corollary 3.12 *Let E be a Dedekind σ-complete Banach lattice. Then, the following assertions are equivalent:*

(1) *E is discrete with order continuous norm,*
(2) *E' has weak* sequentially continuous lattice operations and E has the GP-property.*

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