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TRANSITIVE CLOSURES OF BINARY RELATIONS IV

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Transitive closures of binary relations α are investigated.

This note is a continuation of [1]. We therefore refer to [1] for terminology, notation, etc.

1. Confluent relations (a)

Throughout this work, let α be a binary relation defined on a set S. Let $\gamma = \mathbf{t}(\alpha)$, $\delta = \mathbf{rt}(\alpha)$ $(= \mathbf{tr}(\alpha))$ and $\epsilon = \mathbf{t}(\delta \cup \delta^{-1})$ $(= \mathbf{rt}(\alpha \cup \alpha^{-1}))$. Note that ϵ is the equivalence relation generated by α.

Now let A denote the set of elements $a \in S$ such that for all $b, c \in S$ satisfying $(a, b), (a, c) \in \alpha$ there exists at least one element $d \in S$ such that $(b, d), (c, d) \in \delta$. Likewise, let B denote the set of all elements $a \in S$ such that for every $b, c \in S$ satisfying $(a, b), (a, c) \in \delta$ (and $a \neq b \neq c \neq a$ with $(b, c) \notin \delta$ and $(c, b) \notin \delta$) there exists at least one element $d \in S$ such that $(b, d), (c, d) \in \delta$.

An element $a \in A$ will be called *critical* if there doesn't exist an element $b \in S \setminus B$ with $b \neq a$ and $(a, b) \in \delta$.

Lemma 1 *Let $a \in S$ be critical. Then $a \in B$.*

Proof. Suppose $b, c \in S$ such that $(a, b), (a, c) \in \delta$. Since δ is reflexive, we may assume that $a \neq b \neq c \neq a$. Since $(a, b) \in \gamma$ and $a \neq b$, there exists an element

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$b_1 \in S$ such that $(a, b_1) \in \alpha$, $(b_1, b) \in \delta$, and $b_1 \neq a$. Similarly, there exists an element $c_1 \in S$ such that $(a, c_1) \in \alpha$, $(c_1, c) \in \delta$, and $c_1 \neq a$. Since $a \in A$, there exists a $d_1 \in S$ such that $(b_1, d_1), (c_1, d_1) \in \delta$. Now let $d_2 = d_1$ if $d_1 \neq a$ and let $d_2 = b_1$ if $d_1 = a$. In both cases, $(a, d_2) \in \delta$ and $d_2 \neq a$. Since a is critical, the elements b_1, c_1, and d_2 are contained in B. Consequently, there are elements $d_3, d_4 \in S$ such that $(b, d_3), (d_2, d_3), (d_2, d_4), (c, d_4) \in \delta$. Finally, there is a $d \in S$ such that $(d_3, d), (d_4, d) \in \delta$. Clearly, $(b, d), (c, d) \in \delta$ and hence $a \in B$. □

Lemma 2 *Suppose $a, b \in S$ with $(a, b) \in \delta$. If $a \in B$ then $b \in B$.*

Let A_1 denote the set of elements $a \in S$ such that for all $b, c, d \in S$ satisfying $(a, b) \in \delta$ and $(b, c), (b, d) \in \alpha$ (and $b \neq c \neq d \neq b$ with $(c, d) \notin \delta$ and $(d, c) \notin \delta$), there exists at least one element $e \in S$ such that $(c, e), (d, e) \in \delta$.

Lemma 3 *(i) $B \subseteq A_1 \subseteq A$.*
(ii) If $(a, b) \in \delta$ with $a \in A_1$ then $b \in A_1$.

Lemma 4 *For every $a \in A_1 \setminus B$ there exists an infinite α-sequence, say $(a_0, a_1, a_2, ...)$, such that $a_0 = a$, $a_i \in A_1 \setminus B$, and $a_i \neq a_{i+1}$ for all $i \geq 0$.*

Proof. By Lemma 1, the element $a_0 = a$ is not critical. Thus $(a_0, b) \in \delta$ for some $b \in S \setminus B$ with $b \neq a_0$. Consequently $(a_0, b) \in \gamma$ and there are elements $a_1, ..., a_m \in S$, $m \geq 1$, such that $(a_0, ..., a_m)$ is an α-sequence, $a_m = b$ and $a_i \neq a_{i+1}$ for all $i \in$ $\in \{0, .., m-1\}$. By Lemmas 2 and 3, all of the elements $a_0, ..., a_m$ are contained in A_1 and none of them are in B since $b \notin B$. Since $a_m = b \in A_1$ is not critical, one can proceed by induction to form an infinite α-sequence. □

Corollary 5 *Assume that there exists no weakly pseudoirreducible infinite α-sequence containing only elements from $A_1 \setminus B$. Then $A_1 = B$.*

Corollary 6 *Assume that there exists no weakly pseudoirreducible infinite α-sequence. Then the reflexive and transitive closure $\delta = \mathbf{rt}(\alpha)$ of the relation α is right (strictly) confluent if and only if the following condition is satisfied:*
(A) For all $a, b, c \in S$ such that $(a, b), (a, c) \in \alpha$ (and $a \neq b \neq c \neq a$ with $(b, c) \notin \delta$ and $(c, b) \notin \delta$) there exists at least one element $d \in S$ such that $(b, d), (c, d) \in \delta$.

Example 7 *Let $S = \{a, b, c, d\}$ be a four-element set and let $\alpha = \{(a, b), (b, a), (a, c), (b, d)\}$. Then $\gamma = \alpha \cup \{(a, a), (b, b), (a, d), (b, c)\}$ and thus α is irreflexive but not superirreflexive. The elements c and d are right strictly isolated. Here $B = \{c, d\}$ and the relation δ is not right confluent. On the other hand, $A = S$, namely, condition (A) is satisfied.*

Example 8 *Let $S = \{a_i, b_i, c, d \mid i \in \mathbb{Z}\}$ and $\alpha = \{(a_i, c), (b_i, d), (a_i, b_i), (b_i, a_{i+1}) \mid i \in$ $\in \mathbb{Z}\}$. Then $\gamma = \alpha \cup \{(a_i, a_j), (b_i, b_j), (a_i, d), (b_i, c) \mid i, j \in \mathbb{Z}, i < j\}$ and the relation α is superirreflexive. Here the elements c and d are right strictly isolated. We have $B = \{c, d\}$ and the relation δ is not right confluent. On the other hand, the condition (A) is satisfied. Notice that $\mathbb{Z}$ can be replaced by $\mathbb{N}$.*

Remark 9 *Suppose there exists no weakly pseudoirreducible infinite α-sequence. Then α is antisymmetric and $\mathrm{i}(\alpha)$ is superirreflexive. For every $a \in S$ there exists at least one right α-isolated element $b \in S$ with $(a, b) \in \delta$. If $(a_0, a_1, a_2, ...)$ is an infinite α-sequence then $a_m = a_{m+1} = a_{m+2} = \cdots$ for some $m \geq 0$. Note that if α is irreflexive then there exists no infinite α-sequence at all.*

Remark 10 *Suppose that δ is right confluent. Let $(a, b) \in \epsilon$. Then either $(a, b) \in \delta$ or $(b, a) \in \delta$ or there exists an element $c \in S$ such that $(a, c), (b, c) \in \delta$. Hence $\epsilon = \delta \cup \delta^{-1} \cup (\delta^{-1} \circ \delta)$.*

Remark 11 *If A and B are two distinct blocks of the equivalence ϵ then $(a, b) \notin \delta$ for all $a \in A$ and $b \in B$. Thus, for practical purposes, one can always assume that $\epsilon = S \times S$. In particular, δ is confluent if the restriction $\delta|_A$ is confluent.*

2. Confluent relations (b)

Lemma 12 *For every $a \in S$,*

$$\mathbf{R}(a, \gamma) = \mathbf{R}(a, \alpha) \cup \bigcup_{\substack{b \in \mathbf{R}(a,\alpha) \\ b \neq a}} \mathbf{R}(b, \gamma).$$

Lemma 13 *Let $a \in S$ such that the set $\mathbf{R}(a, \alpha)$ is finite and the set $\mathbf{R}(a, \gamma)$ is infinite. Then:*
(i) $\mathbf{R}(a, \alpha)$ contains at least two elements, and
(ii) $\mathbf{R}(b, \gamma)$ is infinite for at least one element $b \in \mathbf{R}(a, \alpha)$ with $b \neq a$.

Proof. This follows from Lemma 12. □

Lemma 14 *Suppose that $\mathbf{R}(a, \alpha)$ is finite for every $a \in S$. If there exists an element $b \in S$ such that $\mathbf{R}(b, \gamma)$ is infinite then there exists a weakly pseudoirreducible infinite α-sequence $(b_0, b_1, b_2, ...)$ with $b_0 = b$.*

Proof. This follows inductively using Lemma 14 (ii). □

Corollary 15 *Suppose there exists no weakly pseudoirreducible infinite α-sequence and that $\mathbf{R}(a, \alpha)$ is finite for any $a \in S$. Then the set $\mathbf{R}(a, \gamma)$ is finite for any $a \in S$.*

Proposition 16 *The following are equivalent:*
(i) The irreflexive core $\mathbf{i}(\alpha)$ is superirreflexive and $\mathbf{R}(a, \gamma)$ is finite fore every $a \in S$.
(ii) There exists no weakly pseudoirreducible infinite α-sequence and $\mathbf{R}(a, \alpha)$ is finite for every $a \in S$.

Proof. Suppose $\mathbf{i}(\alpha)$ is superirreflexive and that $\mathbf{R}(a, \gamma)$ is finite fore every $a \in \in S$. If $(a_0, a_1, a_2, ...)$ is a weakly pseudoirreducible infinite α-sequence then it is a

$\mathbf{i}(\alpha)$-sequence and is therefore pseudoirreducible. But then $\mathbf{R}(a_0, \gamma)$ would be infinite. Thus, there doesn't exist a weakly pseudoirreducible infinite α-sequence and $\mathbf{R}(a, \alpha) \subseteq \mathbf{R}(a, \gamma)$ is finite for every $a \in S$.

Now suppose that there are no weakly pseudoirreducible infinite α-sequences and that $\mathbf{R}(a, \alpha)$ is finite for every $a \in S$. Hence, by Corollary 15, all of the sets $\mathbf{R}(a, \gamma)$ are finite and thus $\mathbf{i}(\alpha)$ is superirreflexive. □

Proposition 17 *Suppose that* $\mathbf{i}(\alpha)$ *is superirreflexive and that* $\mathbf{R}(a, \gamma)$ *is finite fore every* $a \in S$. *Then the relation* δ *is right confluent if and only if condition (A) is satisfied.*

Proof. This follows from Corollary 6 and Proposition 16. □

3. Confluent relations (c)

Proposition 18 *The transitive and reflexive closure* δ *is right confluent if and only if the following condition is satisfied:*
(B) For all $a, b \in S$ *such that* $(a, b) \in \epsilon$ *there exists at least one* $c \in S$ *such that* $(a, c), (b, c) \in \delta$.

Proof. Suppose that δ is right confluent and that $(a, b) \in \epsilon = \mathbf{t}(\delta \cup \delta^{-1})$. Then there exists a sequence $(d_0, ..., d_n)$ with $n \geq 1$ such that $d_0 = a$, $d_n = b$ and $(d_i, d_{i+1}) \in \delta \cup \delta^{-1}$ for every $0 \leq i \leq n-1$. Now we proceed by inducting on n.

For $n = 1$ either $(a, b) \in \delta$ or $(a, b) \in \delta^{-1}$. If $(a, b) \in \delta$ then let $c = b$ otherwise let $c = a$.

Now assume that $n \geq 2$ and that there exists an element $c_1 \in S$ with (a, c_1), $(d_{n-1}, c_1) \in \delta$. Also, $(d_{n-1}, b) = (d_{n-1}, d_n) \in \delta \cup \delta^{-1}$. If $(d_{n-1}, b) \in \delta$ then, since δ is right confluent, there exists a $c \in S$ such that $(c_1, c), (b, c) \in \delta$. Thus $(a, c) \in \delta$ and $(b, c) \in \delta$. Otherwise, if $(d_{n-1}, b) \in \delta^{-1}$ then $(b, d_{n-1}) \in \delta$ and thus $(b, c_1) \in \delta$. Now just let $c = c_1$. Hence, by induction on n, condition (B) is satisfied.

Now suppose that (B) is true and let $a, b, c \in S$ such that $(a, b), (a, c) \in \delta$. Then $(b, c) \in \epsilon$ and by (B), there exists an element $d \in S$ such that $(b, d), (c, d) \in \delta$. Hence δ is confluent. □

Proposition 19 *Suppose that the relation* δ *is right confluent. Then for any* $a \in S$ *there exists at most one right* α*-isolated element* b *such that* $(a, b) \in \epsilon$.

Proof. Let b_1 and b_2 be right α-isolated elements such that $(a, b_1), (a, b_2) \in \epsilon$. Then $(b_1, b_2) \in \epsilon$ and by Proposition 18 there exists a $c \in S$ with $(b_1, c), (b_2, c) \in \delta$. Since b_1 and b_2 are right α-isolated, and therefore right δ-isolated, $b_1 = c = b_2$. □

Corollary 20 *Suppose that* δ *is right confluent. If* $a, b \in S$ *are right* α*-isolated with* $(a, b) \in \epsilon$ *then* $a = b$.

Proposition 21 *Suppose that there are no weakly pseudoirreducible infinite* α*-sequences. Then the following are equivalent:*

(i) δ is right confluent.
(ii) For any $a \in S$ there exists exactly one right α-isolated element $b \in S$ such that $(a, b) \in \epsilon$.
(iii) For every $a \in S$ there exists exactly one right α-isolated element $b \in S$ such that $(a, b) \in \delta$.

Proof. First note that, by [1, 5.4(iii)], for any $a \in S$ there exists at least one right α-isolated element $b \in S$ such that $(a, b) \in \delta$. Now, by Proposition 19, (i) implies (ii). Also, (iii) clearly follows from (ii). So suppose (iii) holds and let $(a, b), (a, c) \in \delta$. Then there are right α-isolated elements $e, f \in S$ such that $(b, e), (c, f) \in \delta$. Thus $(a, e), (a, f) \in \delta$ and from (iii), $e = f$. □

Remark 22 *(i) Every infinite α-sequence is weakly pseudoirreducible if and only if α is irreflexive.*
(ii) If there are no infinite weakly pseudoirreducible α-sequences then α is antisymmetric and $\mathbf{i}(\alpha)$ is superirreflexive.
(iii) The relation δ is confluent provided that for any $a \in S$ there exists exactly one right α-isolated element $b \in S$ with $(a, b) \in \delta$ (see the proof of Proposition 21).

Example 23 *Let $S = \{a_i \,|\, i \geq 0\} \cup \{b\}$ with $b \neq a_i$ and $\alpha = \{(a_0, b), (a_i, a_{i+1}) \,|\, i \geq 0\}$. Then the relation δ is not confluent. On the other hand, for every $u \in S$ there exists exactly one $v \in S$ (namely, $v = b$) such that $(u, v) \in \epsilon$.*

4. Technical results (a)

Let α, γ, δ, and ϵ be as usual. The set of right α-isolated elements will be denoted by $\mathrm{Ir}(\alpha)$. From this point on we will assume that for every $a \in S$ there exists at least one element $b \in \mathrm{Ir}(\alpha)$ such that $(a, b) \in \delta$.

Let σ be a symmetric relation defined on S and let $\eta = \mathbf{rt}(\sigma)$ so that η is the equivalence relation generated by σ. Now define a relation ρ on S by $(a, b) \in \rho$ if and only if $(a, b) \in \eta$ and the following condition is true:
For every $c, d \in S$ such that $(a, c), (b, d) \in \delta$ there exist elements $e, f \in S$ such that $(c, e), (d, f) \in \delta$ and $(e, f) \in \eta$.

Lemma 24 *ρ is a symmetric relation and $\rho \subseteq \eta$.*

Lemma 25 *The following are equivalent:*
(i) $(a, b) \in \rho$,
(ii) $(a, b) \in \eta$ and if $(a, c), (b, d) \in \delta$ with $c, d \in Ir(\alpha)$ then $(c, d) \in \eta$.

Proof. Suppose $(a, b) \in \rho$. Then there exist elements $c_1, d_1 \in S$ such that (c, c_1), $(d, d_1) \in \delta$ and $(c, d_1) \in \eta$. Since $c, d \in \mathrm{Ir}(\alpha)$, $c_1 = c$ and $d_1 = d$. Hence, $(c, d) =$
$= (c_1, d_1) \in \eta$.

Now let $(a,c),(b,d) \in \delta$. There exist elements $c_1, d_1 \in \mathrm{Ir}(\alpha)$ such that (c,c_1), $(d,d_1) \in \delta$. Thus $(a,c_1),(b,d_1) \in \delta$. From (ii), $(c_1,d_1) \in \eta$. Hence, by definition of ρ, $(a,b) \in \rho$. □

Lemma 26 *If $(a,b),(b,b),(b,c) \in \rho$ then $(a,c) \in \rho$.*

Proof. Since η is transitive, $(a,c) \in \eta$. Now it must be shown that Lemma 25 (ii) is true for the pair (a,c). Let $(a,d),(c,e) \in \delta$ where $d,e \in \mathrm{Ir}(\alpha)$. Since $(a,b) \in \rho$ there exist $d_1, f \in S$ such that $(d,d_1),(b,f) \in \delta$ and $(d,f) \in \eta$. Since $d \in \mathrm{Ir}(\alpha)$, $d_1 = d$ and hence $(d,f) \in \eta$. Furthermore, $(f,h) \in \delta$ for some $h \in \mathrm{Ir}(\alpha)$ and thus $(b,h) \in \delta$. Since $(a,b) \in \rho$, there are $d_2, h_1 \in S$ such that $(d,d_2),(h,h_1) \in \delta$ and $(d_2,h_1) \in \eta$. But $d,h \in$ $\in \mathrm{Ir}(\alpha)$ and hence $d_2 = d$, $h_1 = h$ and $(d,h) \in \eta$. Similarly, there exists an element $k \in$ $\in \mathrm{Ir}(\alpha)$ such that $(b,k) \in \delta$ and $(e,k) \in \eta$. Now $(b,h) \in \delta$, $(b,k) \in \delta$, $(b,b) \in \rho$ and $h,k \in \mathrm{Ir}(\alpha)$. Therefore, $(h,k) \in \eta$. Since η is transitive, $(d,e) \in \eta$. Hence, by Lemma 25, $(a,c) \in \rho$. □

Lemma 27 *The relation ρ is an equivalence relation if and only if ρ is reflexive.*

Proof. The relation ρ is symmetric. Thus, if ρ is reflexive then, by Lemma 26, ρ would be transitive and therefore an equivalence relation. □

An ordered pair $(a,b) \in \eta \setminus \rho$ will be called *critical* if there doesn't exist a pair $(c,d) \in \eta \setminus \rho$ such that:

(i) $(c,d) \neq (a,b)$,

(ii) $(a,c) \in \rho$, and

(iii) $(b,d) \in \rho$.

Clearly, (a,b) is critical if and only if (b,a) is critical.

Lemma 28 *Suppose that there doesn't exist any infinite weakly pseudoirreducible α-sequence. Then for every pair $(a,b) \in \eta \setminus \rho$ there exists a critical pair $(c,d) \in \eta \setminus \rho$ such that $(a,c) \in \delta$ and $(b,d) \in \delta$.*

Proof. Assume this is not true and let $(a,b) \in \eta \setminus \rho$ be a counter example. Then (a,b) is not critical. Let $a_0 = a$ and $b_0 = b$. Since (a,b) is not critical, there exists a pair $(a_1,b_1) \in \eta \setminus \rho$ such that $(a_0,a_1),(b_0,b_1) \in \delta$ and $(a_1,b_1) \neq (a_0,b_0)$. Now, by induction, for $i > 0$ there exists a pair $(a_i,b_i) \in \eta \setminus \rho$ such that $(a_{i-1},a_i),(b_{i-1},b_i) \in \delta$ and $(a_i,b_i) \neq (a_{i-1},b_{i-1})$. Let $I = \{i \geq 0 \mid a_i \neq a_{i+1}\}$ and $J = \{j \geq 0 \mid b_j \neq b_{j+1}\}$. Then $I \cup J = \mathbb{N}_o$, and hence either I or J is infinite. Thus there exists an infinite weakly pseudoirreducible α-sequence forming a contradiction. □

Lemma 29 *If $a,b \in Ir(\alpha)$ and $(a,b) \in \eta$ then $(a,b) \in \rho$.*

Observation 30 *Let $(a,b) \in \eta$. First note that if $(a,c) \in \delta$ with $c \in Ir(\alpha)$ and $c \neq a$ then there exists a $c_1 \in S \setminus \{a\}$ such that $(a,c_1) \in \alpha$ and $(c_1,c) \in \delta$. Likewise, if $(b,d) \in \delta$ with $d \in Ir(\alpha)$ and $d \neq b$ then there exists a $d_1 \in S \setminus \{b\}$ such that $(b,d_1) \in \alpha$ and $(d_1,d) \in \delta$.*

Now assume that $(c_1, e), (b, f) \in \delta$ *and that* $(e, f) \in \eta$. *Assume also that* $(c_1, c_1) \in \rho$. *Let* $g \in Ir(\alpha)$ *be such that* $(e, g) \in \delta$. *Since* $(c_1, c_1) \in \rho$, *by definition of* ρ, $(c, g) \in \eta$. *Now let* $h \in Ir(\alpha)$ *be such that* $(f, h) \in \delta$. *Assume that* $(e, f) \in \rho$. *Then by definition of* ρ, $(g, h) \in \eta$. *Since* η *is transitive,* $(c, h) \in \eta$.

(i) Let $b \neq f$. *Then there is an* $f_1 \in S\smallsetminus\{b\}$ *such that* $(b, f_1) \in \alpha$ *and* $(f_1, f) \in \delta$. *Now assume that* $(f_1, p), (d_1, q) \in \delta$ *and that* $(p, q) \in \eta$. *Choose* $r, s \in Ir(\alpha)$ *such that* $(p, r), (q, s) \in \delta$. *Assume that* $(f_1, f_1), (d_1, d_1) \in \rho$. *Then* $(h, r) \in \eta$ *and* $(s, d) \in \eta$. *Thus, since* η *is transitive,* $(c, r) \in \eta$. *Assume also that* $(p, q) \in \rho$. *Then* $(r, s) \in \eta$ *and hence* $(c, d) \in \eta$.

(ii) Let $b = f$. *Thus* $(e, f) \in \rho$. *Assume that* $(e, p), (d_1, q) \in \delta$ *and that* $(p, q) \in \eta$. *Choose* $r, s \in Ir(\alpha)$ *such that* $(p, r), (q, s) \in \delta$. *Let* $(d_1, d_1), (e, e), (p, q) \in \rho$. *Then* $(d, s) \in \eta$, $(r, s) \in \eta$ *and* $(g, r) \in \eta$. *Thus, since* η *is transitive,* $(c, d) \in \eta$.

Observation 31 *Let* $(a, b) \in \eta$ *where* $b \in Ir(\alpha)$ *and let* $(a, c) \in \delta$ *where* $c \in Ir(\alpha)$ *with* $c \neq a$. *Then there exists a* $c_1 \in S\smallsetminus\{a\}$ *such that* $(a, c_1) \in \alpha$ *and* $(c_1, c) \in \delta$. *Let* $d \in S$ *such that* $(c_1, d) \in \delta$ *and* $(d, b) \in \eta$. *Now choose* $e \in Ir(\alpha)$ *such that* $(d, e) \in \delta$. *If* $(c_1, c_1), (d, b) \in \rho$ *then* $(c, e), (b, e) \in \eta$. *Hence* $(c, b) \in \eta$.

5. Technical results (b)

The preceding section is immediately continued.

Lemma 32 $(a, a) \in \rho$ *for every* $a \in Ir(\alpha)$.

An element $a \in S$ will be called *1-critical* if $(a, a) \notin \rho$ and for every $b \in S\smallsetminus\{a\}$ with $(a, b) \in \delta$, $(b, b) \in \rho$.

Lemma 33 *Assume that there is no infinite weakly pseudoirreducible* α*-sequence. Then for every* $a \in S$ *such that* $(a, a) \notin \rho$ *there is at least one 1-critical element* $b \in S$ *such that* $(a, b) \in \delta$.

Proof. If a is 1-critical then, since $(a, a) \in \delta$, we are done. Suppose a is not 1-critical and let $a_0 = a$. Then there exists an element $a_1 \in S\smallsetminus\{a_0\}$ such that $(a_0, a_1) \in$ $\in \delta$ and $(a_1, a_1) \notin \rho$. Then $(a_0, a_1) \in \gamma$ and proceeding in this way, by induction, there exists an infinite weakly pseudoirreducible γ-sequence forming a contradiction. Hence, there exists an $n \geq 0$, such that $(a_0, a_n) \in \delta$ and a_n is 1-critical with $(a_n, a_n) \notin$ $\notin \rho$. □

Lemma 34 *Suppose there is no infinite weakly pseudoirreducible* α*-sequence and that there are no 1-critical elements. Then the relation* ρ *is an equivalence relation.*

Proof. From Lemma 33, $(a, a) \in \rho$ for every $a \in S$. Hence, by Lemma 27, ρ is an equivalence relation. □

Observation 35 *Suppose $(a,b),(a,c) \in \delta$ with $b,c \in Ir(\alpha)$ and $a \notin Ir(\alpha)$. Then $b \neq a \neq c$ and thus there exist elements $b_1, c_1 \in S \setminus \{a\}$ such that $(a,b_1),(a,c_1) \in \alpha$ and $(b_1,b),(c_1,c) \in \delta$. Now suppose that there are elements $d,e \in S$ such that $(b_1,d),(c_1,e) \in \delta$ and $(d,e) \in \eta$. Choose $f,g \in Ir(\alpha)$ such that $(d,f),(e,g) \in \delta$. Then $(b_1,f),(c_1,g) \in \delta$.*

(i) Assume that $(b_1,b_1),(c_1,c_1) \in \rho$. This is true provided that a is 1-critical. Then $(b,f) \in \eta$ and $(c,g) \in \eta$. Thus, if $(d,e) \in \rho$ then $(f,g) \in \eta$ and hence $(b,c) \in \eta$.

(ii) Let a be 1-critical. Then, by Lemma 25, the elements b and c can be chosen so that $(b,c) \notin \eta$. But then $(f,g) \notin \eta$ and $(d,e) \in \eta \setminus \rho$. So if the pair (a,a) is critical then $d = a = e$ and hence the relation $\mathbf{i}(\alpha)$ is not superirreflexive.

Lemma 36 *Suppose that the following condition is satisfied:*
(α) If a is 1-critical and $(a,b),(a,c) \in \alpha$ with $b \neq a \neq c$ then there exist $d,e \in S$ such that $(b,d),(c,e) \in \delta$ and $(d,e) \in \eta$.
(i) Then there exist elements $a_1, a_2 \in S$ and a 1-critical element $a \in S \setminus Ir(\alpha)$ such that $(a,a_1),(a,a_2) \in \gamma$ and $(a_1,a_2) \in \eta \setminus \rho$.
(ii) If $\mathbf{i}(\alpha)$ is superirreflexive then the pair (a,a) is not critical.

Proof. This follows from 35. □

Proposition 37 *Suppose that there exists no weakly pseudoirreducible infinite α-sequence. Then the following are equivalent:*
(i) The relation ρ is an equivalence relation.
(ii) The relation ρ is reflexive.
(iii) If a is 1-critical then the pair (a,a) is critical and if $(a,b),(a,c) \in \alpha$ with $b \neq a \neq c$ then $(b,d),(c,e) \in \delta$ and $(d,e) \in \eta$ for some $d,e \in S$.

Proof. Suppose that ρ is reflexive. Then, by definition, there are no 1-critical elements at all and condition (iii) is true.

Suppose condition (*iii*) is true. It then follows from Lemma 36 that there are no 1-critical elements. Hence, by Lemma 34, ρ is an equivalence relation. □

6. Technical results (c)

Again, we continue with the preceding two sections.

Observation 38 *Let $a \in S$ be 1-critical and let $(a,b),(a,c) \in \delta$ be such that $b,c \in$ $\in Ir(\alpha)$ and $a \notin Ir(\alpha)$. Then $b \neq a \neq c$ and there exist elements $b_1, c_1 \in S \setminus \{a\}$ such that $(a,b_1),(a,c_1) \in \alpha$ and $(b_1,b),(c_1,c) \in \delta$.*

Now assume that $(b_1,d),(c_1,e) \in \delta$ and $(d,e) \in \eta$ for some $d,e \in S$ (property (α) of Lemma 36).

(i) Suppose $(d,e) \in \rho$. *Since* a *is 1-critical,* $(b_1,b_1),(c_1,c_1) \in \rho$ *and hence* $(d,f) \in \delta$, $(b,f) \in \eta$, $(e,g) \in \delta$, *and* $(c,g) \in \eta$ *for some* $f,g \in S$. *Furthermore,* $(f,h),(g,k) \in \delta$ *for some* $h,k \in Ir(\alpha)$. *Since* $(d,e) \in \rho$, $(h,k) \in \eta$. *Similarly, since* $(b_1,h) \in \delta$ *and* $(b_1,b_1) \in \rho$, $(b,h) \in \eta$. *Likewise* $(c,k) \in \eta$. *Hence, since* η *is an equivalence relation,* $(b,c) \in \eta$.

(ii) Now suppose $(d,e) \notin \rho$. *Let* $(f,g) \in \eta \setminus \rho$ *be a critical pair such that* $(d,f),(e,g) \in \delta$ *(see Lemma 28). Then* $(b_1,f),(e,g) \in \delta$ *and since* $(b_1,b_1),(c_1,c_1) \in \rho$, *there exist elements* $h,k \in S$ *such that* $(f,h),(g,k) \in \delta$ *and* $(b,h),(c,k) \in \eta$.

If $f = h$ *and* $g = k$ *then* $(b,f),(c,g) \in \eta$ *and so* $(b,c) \in \eta$. *Now assume that either* $f \neq h$ *or* $g \neq k$. *Without loss we may assume that* $f \neq h$. *Thus there exists an element* $f_1 \in S \setminus \{f\}$ *such that* $(f,f_1) \in \alpha$ *and* $(f_1,h) \in \delta$. *Further assume that there exist elements* $p,q \in S$ *such that* $(f_1,p),(g,q) \in \delta$ *and* $(p,q) \in \eta$. *Thus there exist* $r,s \in Ir(\alpha)$ *such that* $(h,r) \in \delta$ *and* $(k,s) \in \delta$. *Since* $(b_1,b_1),(c_1,c_1) \in \rho$, $(b,r) \in \eta$ *and* $(c,s) \in \eta$.

Now choose $u,v \in Ir(\alpha)$ *such that* $(p,u),(q,v) \in \delta$.

(ii1) If either $p = f$ *or* $f_1 = a$ *or* $g = a$ *then the relation* $\mathbf{i}(\alpha)$ *is not superirreflexive*

(ii2) Now assume that $p \neq f$, $f_1 \neq a$ *and* $g \neq a$. *Since the pair* (f,g) *is critical we have* $(p,q) \in \rho$ *and it follows that* $(u,v) \in \eta$. *Since* a *is 1-critical, by definition,* $(f_1,f_1) \in \rho$ *and hence* $(r,u) \in \eta$. *Likewise,* $(g,g) \in \rho$ *and* $(v,s) \in \eta$. *Thus* $(b,r),(r,u),(u,v),(v,s),(s,c) \in \eta$ *and hence* $(b,c) \in \eta$.

Lemma 39 *Suppose that there exists no infinite weakly pseudoirreducible* α*-sequence such that condition* (α) *in Lemma 36 is satisfied and the following condition is also satisfied:*

(β) If $(a,b) \in \eta \setminus \rho$ *is critical and* $(a,c) \in \alpha$ *with* $c \neq a$ *then there exist elements* $d,e \in S$ *such that* $(c,d),(b,e) \in \delta$ *and* $(d,e) \in \eta$.

Then the relation ρ *is an equivalence relation.*

Proof. Since there are no infinite weakly pseudoirreducible α-sequences, the relation $\mathbf{i}(\alpha)$ is superirreflexive. Now by using Lemma 25, Lemma 28, condition (β) and condition (α) from Lemma 36, it follows from Observation 38 that there are no 1-critical elements in S. Hence, by Lemma 34, ρ is an equivalence relation. □

7. Technical results (d)

Here an element $a \in S$ will be called *2-critical* if $(a,b) \in \eta \setminus \rho$ for some $b \in S$ and there are no elements $c \in S \setminus \{a\}$ and $d \in S$ such that $(a,c) \in \delta$ and $(c,d) \in \eta \setminus \rho$.

Lemma 40 *Suppose that there exists no infinite weakly pseudoirreducible* α*-sequence. If* $(a,b) \in \eta \setminus \rho$ *then* $(a,c) \in \delta$ *for at least one 2-critical element* $c \in S$.

Proof. Let $a_0 = a$ and $b_0 = b$. If a_0 is not 2-critical then there exists an element $a_1 \in S \setminus \{a_0\}$ such that $(a_0,a_1) \in \delta$ and $(a_1,b_1) \in \eta \setminus \rho$ for some $b_1 \in S$. By induction, there exists a weakly pseudoirreducible infinite α-sequence $(a_0, a_1, a_2, ...)$. But, since there is no weakly pseudoirreducible infinite α-sequence, a_n is 2-critical for some $n \geq 0$. □

Lemma 41 *Suppose there is no weakly pseudoirreducible infinite α-sequence. Then $\rho = \eta$ provided that there are no 2-critical elements in S.*

Proof. This immediately follows from Lemma 40. □

Observation 42 *Let a be 2-critical with $(a, b) \in \eta\setminus\rho$, $(a, c) \in \delta$, and $c \in Ir(\alpha)\setminus\{a\}$. Thus there exists an element $c_1 \in S\setminus\{a\}$ such that $(a, c_1) \in \alpha$ and $(c_1, c) \in \delta$.*

Now suppose that $(c_1, d_1), (b, e) \in \delta$ and $(d, e) \in \eta$ (condition (β) in Lemma 39). Since a is 2-critical, $(c_1, c_1) \in \rho$. Since $(c_1, c_1) \in \rho$ and $(c_1, c), (c_1, d_1) \in \delta$, by definition of ρ, there exist elements $c_2, f \in S$ such that $(c, c_2), (d_1, f) \in \delta$ and $(c_2, f) \in \in \eta$. Since $c \in Ir(\alpha)$, $c = c_2$ and hence $(c, f) \in \eta$.

Let $g \in Ir(\alpha)$ so that $(f, g) \in \delta$. Since a is 2-critical with $(a, c) \in \delta$ and $c \neq a$, $(c, f) \in \rho$ and hence $(c, g) \in \eta$. If $d_1 = a$ then $\mathbf{i}(\alpha)$ would not be superirreflexive. So assume that $d_1 \neq a$. Then $(d_1, e) \in \rho$ and thus $(e, h) \in \delta$ with $h \in Ir(\alpha)$. Hence $(g, h) \in \eta$ and therefore $(c, h) \in \eta$.

Now let $d \in Ir(\alpha)$ such that $(h, d) \in \delta$. Then, since $(b, e), (e, h) \in \delta$, $(b, d) \in \delta$. Furthermore, $(c, d) \in \eta$ provided that $(g, d) \in \eta$. But this can shown using the fact that $g \in Ir(\alpha)$. Since $g \in Ir(\alpha)$ and $a \notin Ir(\alpha)$, $g \neq a$ with $(a, g) \in \delta$. Thus $(g, h) \in \rho$ and, by definition of ρ, $(g, d) \in \eta$. Hence, $(c, d) \in \eta$.

Lemma 43 *Suppose that there does not exist an infinite weakly pseudoirreducible α-sequence and that the following condition is satisfied:*
(γ) If a is 2-critical with $(a, b) \in \eta\setminus\rho$ and $(a, c) \in \alpha$ for some $c \neq a$ then $(c, d), (b, e) \in \in \delta$ and $(d, e) \in \eta$ for some $d, e \in S$.
Then the following is true:
(i) If $a \in S$ is 2-critical with $a \notin Ir(\alpha)$, $(a, b_1) \in \eta\setminus\rho$ and $(a, c_1) \in \delta$ for some $c \in Ir(\alpha)$ then there exists an element $h \in Ir(\alpha)$ such that $(b_1, h) \in \delta$ and $(c, h) \in \eta$.

Proof. This follows from Observation 42. □

Lemma 44 *Suppose that there does not exist a weakly pseudoirreducible infinite α-sequence and that the condition (γ) in Lemma 43 is satisfied. If $(a, b) \in \eta$ is a pair such that a is 2-critical with $a \notin Ir(\alpha)$ and $(b, b) \in \rho$ then $(a, b) \in \rho$.*

Proof. From Lemma 25, it is enough to show that for $c, d \in \mathrm{Ir}(\alpha)$ with $(a, c), (b, d) \in \in \delta$, $(c, d) \in \eta$. By Lemma 43 (i), there exists an element $e \in \mathrm{Ir}(\alpha)$ such that $(b, e) \in \delta$ and $(c, e) \in \eta$. Since $(b, b) \in \rho$ and $d, e \in \mathrm{Ir}(\alpha)$, $(e, d) \in \eta$. Hence $(c, d) \in \eta$. □

Lemma 45 *Suppose that there exists no weakly pseudoirreducible infinite α-sequence and that the condition (γ) in Lemma 43 is satisfied. If ρ is reflexive (Proposition 37 (ii)) then every 2-critical element is contained in $Ir(\alpha)$.*

Proof. Let $a \in S$ be 2-critical. Then there exists an element $b \in S$ such that $(a, b) \in \eta \setminus \rho$. Assume that $a \notin \mathrm{Ir}(\alpha)$. Since $(b, b) \in \rho$, by Lemma 45, $(a, b) \in \rho$. Hence, by contradiction, $a \in \mathrm{Ir}(\alpha)$. □

Lemma 46 *Let $a \in S$ be 2-critical. Then a is 1-critical if and only if $(a, a) \notin \rho$. Moreover, in such a case, the pair (a, a) is critical.*

Proof. If a is 1-critical then, by definition, $(a, a) \notin \rho$. Now suppose that $(a, a) \notin \rho$ and thus $a \notin \mathrm{Ir}(\alpha)$. If $(a, b) \in \delta$ with $b \neq a$ then, since $(b, b) \in \eta$ and a is 2-critical, $(b, b) \in \rho$. Hence a is 1-critical. Moreover, if $(a, c), (a, d) \in \delta$ with $(c, d) \in \eta$ and $(c, d) \neq (a, a)$ then, since a is 2-critical with either c or d not equal to a, $(c, d) \in \rho$. Therefore, (a, a) is critical. □

8. Technical results (e)

Let $a \in \mathrm{Ir}(\alpha)$ be such that $(a, b) \in \eta \setminus \rho$ for at least one $b \in S$. Note that $b \notin \mathrm{Ir}(\alpha)$. An element $c \in S$ will be called *a-critical* if $(a, c) \in \eta \setminus \rho$ and there doesn't exist an element $d \in S \setminus \{c\}$ such that $(c, d) \in \delta$ and $(a, d) \in \eta \setminus \rho$.

Lemma 47 *Suppose that there does not exist a weakly pseudoirreducible infinite α-sequence. If $(a, b) \in \eta \setminus \rho$ with $a \in Ir(\alpha)$ then there exists an a-critical element $c \in S$ such that $(b, c) \in \delta$ and thus $(a, c) \in \eta \setminus \rho$.*

Proof. Assume that there does not exist an a-critical element $c \in S$ such that $(b, c) \in \delta$. Let $b_0 = b$. Since $(a, b_0) \in \eta \setminus \rho$ and b_0 is not a-critical, there exists an element $b_1 \in S \setminus \{b_0\}$ such that $(b_0, b_1) \in \delta$ and $(a, b_1) \in \eta \setminus \rho$. Proceeding by induction, there exists a weakly pseudoirreducible infinite α-sequence $(b_0, b_1, b_2, ...)$ forming a contradiction. □

Observation 48 *Let $a \in Ir(\alpha)$ and let b be a-critical with $(a, b) \in \eta \setminus \rho$. Then $b \notin$ $\notin Ir(\alpha)$. Now let $c \in Ir(\alpha)$ such that $(b, c) \in \delta$. Since $b \neq c$, there exists an element $c_1 \in S \setminus \{b\}$ such that $(b, c_1) \in \alpha$ and $(c_1, c) \in \delta$. Now suppose that $(c_1, d) \in \delta$ where $(a, d) \in \eta$. If $d = b$ then $\mathbf{i}(\alpha)$ is not superirreflexive. So suppose $d \neq b$. Then, since $(b, d) \in \delta$ and b is a-critical, $(a, d) \in \rho$. Assume that (c_1, c_1) then, by definition of ρ, there exists an $e \in S$ such that $(d, e) \in \delta$ and $(c, e) \in \eta$. Since $(a, d) \in \rho$ and $(d, e) \in \delta$ there exists an $f \in S$ such that $(e, f) \in \delta$ and $(a, f) \in \eta$. Choose $g \in Ir(\alpha)$ so that $(f, g) \in \delta$ and $g \in Ir(\alpha)$. Since $(c_1, c_1) \in \rho$ with $(c_1, g) \in \delta$ and $(c_1, c) \in \delta$, $(c, g) \in \eta$. Now if $f = b$ then $\mathbf{i}(\alpha)$ is not superirreflexive. So suppose $f \neq b$. Then, since $(b, f) \in \delta$ and b is a-critical, $(a, f) \in \rho$ and hence $(a, g) \in \eta$. Therefore, $(a, c) \in \eta$.*

Lemma 49 *Suppose that there is no weakly pseudoirreducible α-sequence. Suppose further that the relation ρ is reflexive and that the following condition is satisfied:*
(δ) If $a \in Ir(\alpha)$, b be a-critical with $(a, b) \in \eta \setminus \rho$ and $c \in S \setminus \{b\}$ with $(b, c) \in \alpha$ then there exists an element $d \in S$ such that $(c, d) \in \delta$ and $(a, d) \in \eta$.
Then $(a_1, b_1) \in \rho$ for any $(a_1, b_1) \in \eta$ with $a_1 \in Ir(\alpha)$.

Proof. Assume that $(a_1, b_1) \in \eta \setminus \rho$ with $a_1 \in \mathrm{Ir}(\alpha)$. Then $b_1 \in \mathrm{Ir}(\alpha)$ and, by Lemma 47, b_1 can be chosen to be a_1-critical. Thus, from Observation 48, the pair (a_1, b_1) satisfies condition (ii) in Lemma 25. Thus $(a_1, b_1) \in \rho$ forming a contradiction. □

9. Local summary

Proposition 50 *Let α be a relation such that there are no weakly pseudo-irreducible α-sequences and let η be an equivalence relation defined on S. Consider the following condition:*
(C) For any $(a, b) \in \eta$ and $(a, c), (b, d) \in \delta$ there exist elements $e, f \in S$ such that $(c, e), (d, f) \in \delta$ and $(e, f) \in \eta$.
Then condition (C) is satisfied if and only if the following two (formally weaker) conditions are satisfied:
(D) For any $(a, b), (a, c) \in \alpha$ with $b \neq a \neq c$ there exist elements $d.e \in S$ such that $(b, d), (c, e) \in \delta$ and $(d, e) \in \eta$;
(E) For any $(a, b) \in \eta$ with $a \neq b$ and $(a, c) \in \alpha$ with $c \neq a$ there exist elements $d.e \in S$ such that $(b, d), (c, e) \in \delta$ and $(d, e) \in \eta$.

Proof. Suppose that (D) and (E) are true. Note that in order for (C) to be true ρ must equal η.

Let $(a, b) \in \eta$. From Lemma 39, ρ is reflexive. Hence, if $a = b$ then $(a, b) \in \rho$. Furthermore, by Lemma 49, if either $a \in \mathrm{Ir}(\alpha)$ or $b \in \mathrm{Ir}(\alpha)$ then $(a, b) \in \rho$. Thus, from Lemma 46, there are no 2-critical elements in S. Hence, by Lemma 40, $(a, b) \in \rho$. □

Example 51 *Let*

$$S = \{a, b, c, d\},$$

$$\alpha = \{(a, c), (b, d)\}, \text{ and}$$

$$\eta = \{(a, b), (b, a)\} \cup \mathrm{id}_S.$$

Then $\alpha = \gamma$, $\delta = \alpha \cup id_S$ and there is no infinite α-sequence. Moreover, η and $\rho = = id_S$ are equivalence relations and condition (D) in Proposition 50 is satisfied while condition (E) in Proposition 50 is not satisfied.

Example 52 *Let*

$$S = \{a, b, c, d, e, f, g, h, k\},$$

$$\alpha = \{(a, c), (a, c), (b, d), (b, f), (c, e), (c, g), (f, h), (g, k)\}, \text{ and}$$

$$\eta = \{(d, h), (h, d), (e, k), (k, e), (f, g), (g, f)\} \cup \mathrm{id}_S.$$

There is no infinite α-sequence and $Ir(\alpha) = \{d, e, h, k\}$. Moreover, η is an equivalence relation and $(d, e) \notin \eta$. Thus condition (C) in Proposition 50 is not satisfied while condition (D) in Proposition 50 is satisfied with $\rho = \eta \setminus \{(a, a), (f, g), (g, f)\}$. Thus ρ is not reflexive and a is the only 1-critical element. Also, (f, g) and (g, f) are the only critical pairs and the only 2-critical elements are f and g.

An ordered pair $(a,b) \in \eta \searrow \rho$ will be called *ultracritical* if

(i) (a,b) is critical,

(ii) there doesn't exist a pair $(c,d) \neq (a,a)$ such that $(a,c),(a,d) \in \delta$ and $(c,d) \in \eta \searrow \rho$, and

(iii) there doesn't exist a pair $(e,f) \neq (b,b)$ such that $(b,e),(b,f) \in \delta$ and $(e,f) \in \eta \searrow \rho$.

Note that if (a,b) is ultracritical then (b,a) is also ultracritical and either $(a,a) \in \rho$ ($(b,b) \in \rho$ respectfully) or (a,a) ((b,b) respectfully) is critical. If the latter is true then a (b respectfully) is 1-critical. Also note that if $(a,b) \in \eta \searrow \rho$ where both a and b are 2-critical then (a,b) is ultracritical.

Remark 53 *Let $(a,b) \in \eta \searrow \rho$ be a critical pair that is not ultracritical. Then there exists a pair $(c,d) \in \eta \searrow \rho$ such that at least one of the following four cases is true:*
(1) $(a,c),(a,d) \in \delta$ with $a \neq c$;
(2) $(a,c),(a,d) \in \delta$ with $a = c$ and $a \neq d$;
(3) $(b,c),(b,d) \in \delta$ with $b \neq c$;
(4) $(b,c),(b,d) \in \delta$ with $b = c$ and $b \neq c$.

Remark 54 *Suppose $(a,b) \in \eta \searrow \rho$ such that $(a,b) \in \delta$ with $a \neq b$ (thus $(a,b) \in \gamma$). Then the pair (a,b) is not ultracritical.*

Example 55 *Let $S = \{a,b,c\}$, $\alpha = \{(a,b),(a,c)\}$ and $\eta = \{(a,b),(b,a)\} \cup id_S$. Then $\gamma = \alpha$, $\delta = \alpha \cup id_S$, $\rho = \{(b,b),(c,c)\}$ and there is no infinite α-sequence. Thus $\eta \searrow \rho = \{(a,a),(a,b),(b,a)\}$ and both (a,b) and (b,a) are critical pairs. On the other hand, there are no ultracritical pairs in $S \times S$.*

An ordered pair $(a,b) \in \eta \searrow \rho$ is called *semi-ultracritical* if:

(i) (a,b) is critical,

(ii) $(c,d) \in \rho$ whenever $(a,c),(a,d) \in \delta$ with $c \neq a \neq d$, and

(iii) $(e,f) \in \rho$ whenever $(b,e),(b,f) \in \delta$ with $e \neq b \neq f$.

Note that if (a,b) is semi-ultracritical then (b,a) is also semi-ultracritical and either $(a,a) \in \rho$ ($(b,b) \in \rho$ respectfully) or a (b respectfully) is 1-critical. Also note that if (a,b) is ultracritical then (a,b) is also semi-ultracritical.

Remark 56 *Let $(a,b) \in \eta \searrow \rho$ be a critical pair that is not semi-ultracritical. Then there exists a pair $(c,d) \in \eta \searrow \rho$ such that either $(a,c),(a,d) \in \delta$ with $c \neq a \neq d$ or $(b,c),(b,d) \in \delta$ with $c \neq b \neq d$.*

Proposition 57 *Suppose that there exists no weakly pseudoirreducible infinite α-sequence. Then for every pair $(a,b) \in \eta \searrow \rho$ there is a semi-ultracritical pair $(c,d) \in \eta \searrow \rho$ such that $(a,c),(b,d) \in \delta$.*

Proof. By Lemma 28, for every pair $(a,b) \in \eta \searrow \rho$ there is a critical pair $(c,d) \in$ $\in \eta \searrow \rho$ such that $(a,c),(b,d) \in \delta$.

Let $(a_0, b_0) \in \eta \smallsetminus \rho$ be critical but not semi-ultracritical. Then there exists a pair $(a_1, b_1) \in \eta \smallsetminus \rho$ such that either $(a_0, a_1), (a_0, b_1) \in \delta$ with $a_1 \neq a_0 \neq b_1$ (in which case we let $\mu(0) = 0$) or $(b_0, a_1), (b_0, b_1) \in \delta$ with $a_1 \neq b_0 \neq b_1$ (in which case we let $\mu(0) = 1$). Furthermore, there is a critical pair $(a_2, b_2) \in \eta \smallsetminus \rho$ such that $(a_1, a_2), (b_1, b_2) \in \delta$.

Continuing in this way, by induction, there exists a sequence $\alpha_0 = (a_0, b_0)$, $\alpha_1 = (a_1, b_1)$, $\alpha_2 = (a_2, b_2)$, ... of pairs in $\eta \smallsetminus \rho$ and a mapping $\mu : \{2i\} \to \{0, 1\}$ such that:

(1) (a_{2i}, b_{2i}) is a critical pair for all $i \geq 0$;

(2) $(a_{2i+1}, a_{2i+2}), (b_{2i+1}, b_{2i+2}) \in \delta$ for all $i \geq 0$;

(3) if $i \geq 0$ is such that $\mu(2i) = 0$ then $(a_{2i}, a_{2i+1}), (a_{2i}, b_{2i+1}) \in \delta$ with $a_{2i+1} \neq a_{2i} \neq b_{2i+1}$;

(4) if $i \geq 0$ is such that $\mu(2i) = 1$ then $(b_{2i}, a_{2i+1}), (b_{2i}, b_{2i+1}) \in \delta$ with $a_{2i+1} \neq b_{2i} \neq b_{2i+1}$.

Now let $c_{2i} = a_{2i}$ if $\mu(2i) = 0$ an let $c_{2i} = b_{2i}$ if $\mu(2i) = 1$. Furthermore, let $c_{2i+1} = a_{2i+1}$ if $\mu(2i+2) = 0$ an let $c_{2i+1} = b_{2i+1}$ if $\mu(2i+2) = 1$. Thus, using induction, one can show that $(c_0, c_1, c_2, ...)$ is an infinite δ-sequence. In fact, $(c_{2i}, c_{2i+1}) \in \delta$ for any $i \geq 0$. It remains to show that $(c_{2i+1}, c_{2i+2}) \in \delta$ for any $i \geq 0$.

If $\mu(2i + 2) = 0$ then $c_{2i+1} = a_{2i+1}$ and $c_{2i+2} = a_{2i+2}$ and thus $(c_{2i+1}, c_{2i+2}) = (a_{2i+1}, a_{2i+2}) \in \delta$. Likewise, if $\mu(2i + 2) = 1$ then $c_{2i+1} = b_{2i+1}$ and $c_{2i+2} = b_{2i+2}$ and hence $(c_{2i+1}, c_{2i+2}) = (b_{2i+1}, b_{2i+2}) \in \delta$. Therefore, the sequence $(c_0, c_1, c_2, ...)$ is an infinite δ-sequence with $c_{2i} \neq c_{2i+1}$ for all $i \geq 0$. But this contradicts the fact that there is no weakly pseudoirreducible infinite α-sequence. □

Remark 58 *Suppose that there exists no weakly pseudoirreducible infinite α-sequence. Suppose also that conditions (D) and (E) in Proposition 50 are satisfied.*

Let $(a_0, b_0) \in \eta \smallsetminus \rho$. By Proposition 57, there exists a semi-ultracritical pair $(a, b) \in \eta \smallsetminus \rho$ such that $(a_0, a), (b_0, b) \in \delta$. Now, taking into account Observation 30, either $a \in Ir(\alpha)$ with $b \notin Ir(\alpha)$ or $b \in Ir(\alpha)$ with $a \notin Ir(\alpha)$.

Without loss, assume that $a \in Ir(\alpha)$ and $b \notin Ir(\alpha)$. By Lemma 40 there exists an a-critical element $b_1 \in S$ such that $(b, b_1) \in \delta$ and $(a, b_1) \in \eta \smallsetminus \rho$. Since the pair (a, b) is critical, $b_1 = b$ and hence b is a-critical. By Observation 48, $(a, b) \in \rho$ which forms a contradiction. This here is another proof of Proposition 50.

References

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