# IDEAL-SIMPLE SEMIRINGS II

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Left- and right-ideal simple semirings are characterized.

This note is an immediate continuation of [1]. Any prospective reader is fully referred to the first part as concerns various prerequisities, terminology, references, etc. (e.g., Lemma 1.14 from [1] is referred to as I.1.14).

## 1. Elementary observations (a)

Let $S$ be a non-trivial semiring such that the multiplicative semigroup $S(\cdot)$ is a group. Then, of course, the semiring $S$ is both left- and right-ideal-free.

**1.1 Proposition.** *$S$ is infinite and the group $S(\cdot)$ is torsionfree.*

*Proof.* First, let $a \in S$ and $m \geq 1$ be such that $a^m = 1\ (= 1_S)$. Put $b = a + a^2 + \cdots + a^m$. Then $ab = b$ (see I.1.14), and hence $a = 1$. It follows that $S(\cdot)$ is torsionfree and $S$ is infinite. □

**1.2 Proposition.** *Either $S$ is additively idempotent or $S$ has no additively idempotent element.*

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*Proof.* It is easy. □

**1.3** (The dual semiring.) Put $a*b = (a^{-1}+b^{-1})^{-1}$ for all $a, b \in S$. Clearly, $a*b = b*a$, $a + b = (a^{-1} * b^{-1})^{-1}$ and $a * a = \frac{a}{2}$. Furthermore, $a * (b * c) = (a^{-1} + (b * c)^{-1})^{-1} = (a^{-1} + b^{-1} + c^{-1})^{-1} = ((a * b)^{-1} + c^{-1})^{-1} = (a * b) * c$. Consequently, $S(*)$ is a commutative semigroup. We have $(a(b*c))^{-1} = (a(b^{-1}+c^{-1})^{-1})^{-1} = (b^{-1}+c^{-1})a^{-1} = b^{-1}a^{-1} + c^{-1}a^{-1} = (ab)^{-1} + (ac)^{-1} = ((ab)^{-1} + (ac)^{-1})^{-1})^{-1} = (ab * ac)^{-1}$ so that $a(b * c) = ab * ac$. Symmetrically, $(b * c)a = ba * ca$. We see that the algebraic structure $S(*, \cdot)$ is again a semiring – the dual of or the conjugate to $S(+, \cdot)$. The mapping $a \mapsto a^{-1}$ is an antiisomorphism of the semirings ($(a + b)^{-1} = a^{-1} * b^{-1}$ and $(ab)^{-1} = b^{-1}a^{-1}$). Notice that the semiring $S(*, \cdot)$ is additively idempotent if and only if the semiring $S(+, \cdot)$ is such.

The dual (or conjugate) semiring $S(*, \cdot)$ is a parastrophe of the original semiring $S(+, \cdot)$. Of course, we also have the usual parastrophe, namely the opposite semiring $S(+, \circ)$, where $a \circ b = ba$. Notice that the mapping $a \mapsto a^{-1}$ is an isomorphism of $S(+, \circ)$ onto $S(*, \cdot)$ and conversely.

**1.4** Assume that $S$ $(= S(+, \cdot))$ is additively idempotent. Then $S(*, \cdot)$ is so and $(a^{-1} + b^{-1})(a+(a*b)) = 1+b^{-1}a+1 = 1+b^{-1}a = (a^{-1}+b^{-1})a$. Thus $a+(a*b) = a$ and, symmetrically, $a * (a + b) = a$ for all $a, b \in S$. It means that the algebraic structure $S(+, *)$ is a lattice and $S(\cdot, +, *)$ is a lattice ordered group.

Conversely, let $G = G(\cdot, \vee, \wedge)$ be a lattice ordered group, i.e., $G(\cdot)$ is a group, $G(\vee, \wedge)$ is a lattice and $a(b \vee c) = ab \vee ac$, $(b \vee c)a = ba \vee ca$, $a(b \wedge c) = ab \wedge ac$, $(b \wedge c)a = ba \wedge ca$ for all $a, b, c \in G$.

We have $(a\vee(a^{-1}\vee b^{-1})^{-1})(a^{-1}\vee b^{-1}) = a(a^{-1}\vee b^{-1})^{-1}$, and so $a\vee(a^{-1}\vee b^{-1})^{-1} = a$. Similarly, $b \vee (a^{-1} \vee b^{-1})^{-1} = b$, $(a \wedge (a^{-1} \vee b^{-1})^{-1}) \cdot (a^{-1} \vee b^{-1}) = a(a^{-1} \vee b^{-1}) \wedge 1 = (1\vee ab^{-1})\wedge 1 = 1$, $a\wedge(a^{-1}\vee b^{-1})^{-1} = (a^{-1}\vee b^{-1})^{-1}$, $b\wedge(a^{-1}\vee b^{-1})^{-1} = (a^{-1}\vee b^{-1})^{-1}$, and hence $(a\wedge b)\wedge(a^{-1}\vee b^{-1})^{-1} = (a^{-1}\vee b^{-1})^{-1}$. On the other hand, $a\wedge a\wedge b = a\wedge b$, $(a\wedge b)^{-1}a\wedge 1 = 1$, $(a\wedge b)^{-1}\wedge b^{-1} = b^{-1}$. Now, $(a\wedge b)^{-1}\vee a^{-1} = (a\wedge b)^{-1}\vee((a\wedge b)^{-1}\wedge a^{-1}) = (a\wedge b)^{-1}$ and $(a\wedge b)^{-1}\vee b^{-1} = (a\wedge b)^{-1}$. Thus $a^{-1}\vee b^{-1}\vee(a\wedge b)^{-1} = (a\wedge b)^{-1}$, $(a^{-1} \vee b^{-1})(a \wedge b) \vee 1 = 1$, $(a \wedge b) \vee (a^{-1} \vee b^{-1})^{-1} = (a^{-1} \vee b^{-1})^{-1}$ and, finally, $a \wedge b = (a \wedge b) \wedge ((a \wedge b) \vee (a^{-1} \vee b^{-1})^{-1}) = (a \wedge b) \wedge (a^{-1} \vee b^{-1})^{-1}$. Consequently, $a \wedge b = (a^{-1} \vee b^{-1})^{-1}$ and, dually, $a \vee b = (a^{-1} \wedge b^{-1})^{-1}$. Now, it is clear that the algebraic structures $G(\vee, \cdot)$ and $G(\wedge, \cdot)$ are conjugate semirings.

Finally, let $G = G(\cdot, \leq)$ be an ordered group, i.e., $\leq$ is a reflexive, antisymmetric and transitive relation defined on $G$ and $a \leq b$ implies $ca \leq cb$ and $ac \leq bc$. Assume that the ordered set $G(\leq)$ is a lattice and put $a \vee b = \sup(a, b)$, $a \wedge b = \inf(a, b)$, so that $G(\vee, \wedge)$ is an algebraic lattice. Now, $ca \leq c(a \vee b)$, $cb \leq c(a \vee b)$, and hence $ca\vee cb \leq c(a\vee b)$ and $c^{-1}(ca\vee cb) \leq a\vee b = c^{-1}ca\vee c^{-1}cb$. From this, $d(e\vee f) \leq de\vee df$ for all $d, e, f \in G$ and we have proved that $w(u \vee v) = wu \vee wv$ for all $u, v, w \in G$. Quite similarly, $w(u \wedge v) = wu \wedge wv$. This means that $G(\cdot, \vee, \wedge)$ is a lattice ordered group in the algebraic sense.

**1.5 Proposition.** *$Z(S(\cdot))$ is a subsemiring of $S$.*

*Proof.* It is easy. □

**1.6 Lemma.** *$S$ is additively idempotent if and only if $1_S + 1_S = 1_S$.*

*Proof.* It is easy. □

**1.7 Proposition.** *Let $1_S + 1_S \neq 1_S$ (see 1.2, 1.6) and let $Q$ be the subgroup of $S(\cdot)$ generated by all the elements $n1_S$, $n \geq 1$. Then $Q$ is a subsemiring of $S$ and $Q \cong \mathbb{Q}^+$ (the parasemifield of positive rational numbers). Moreover, $Q \subseteq Z(S(\cdot))$ (see 1.5).*

*Proof.* We have $n1_S \cdot a = (1_S + \cdots + 1_S)a = a + \cdots + a = na = a \cdot n1_S$ for all $a \in S$ and all positive integers $n$. Thus $n1_S \in Z(S(\cdot))$ and $Q \subseteq Z(S(\cdot))$ follows from 1.5.

Let $n_1, n_2, m_1, m_2$ be positive integers. Then $(n_1 1_S)(m_1 1_S) + (n_2 1_S)(m_2 1_S)^{-1} = (n_1 1_S)(m_2 1_S)(m_1 1_S)^{-1}(m_2 1_S)^{-1} + (n_2 1_S)(m_1 1_S)(m_1 1_S)^{-1}(m_2 1_S)^{-1} = ((n_1 m_2 1_S) + (m_2 m_1 1_S))(m_1 m_2 1_S)^{-1}$ and, further, $(n_1 1_S)(m_1 1_S)^{-1} \cdot (n_2 1_S)(m_2 1_S)^{-1} = (n_1 n_2 1_S)(m_1 m_2 1_S)^{-1}$. If $n_1 m_1^{-1} = n_2 m_2^{-1}$ then $n_1 m_2 = n_2 m_1$, $n_1 m_2 1_S = n_2 m_1 1_S$ and $(n_1 1_S)(m_1 1_S)^{-1} = (n_2 1_S)(m_2 1_S)^{-1}$. Using these observations, we get a semiring homomorphism $\varphi : \mathbb{Q}^+ \to Q$ such that $\varphi(nm^{-1}) = (n1_S)(m1_S)^{-1}$. Clearly, $\varphi(1) = 1_S$ and $\varphi(\mathbb{Q}^+) = Q$. Since the parasemifield $\mathbb{Q}^+$ is congruence-simple, the homomorphism $\varphi$ is an isomorphism. □

## 2. Elementary observation (b)

In this section, let $S$ be a left-ideal-simple semiring.

**2.1 Proposition.** *Just one of the following four cases takes place:*

(1) *$Sa = S$ for every $a \in S$;*

(2) *$S$ contains a multiplicatively absorbing element $w$ and $Sa = S$ for every $a \in S \setminus \{w\}$;*

(3) *$S$ is a zero multiplication ring of finite prime order;*

(4) *$S$ is isomorphic to one of $\mathrm{Z}_1, \mathrm{Z}_3, \mathrm{Z}_4, \mathrm{Z}_9$ (see I.2.1).*

*Proof.* Assume that (1) is not true. Then $A = \{a \in S \mid Sa \neq S\} \neq \emptyset$. If $a \in A$ then $Sa$ is a proper left ideal, and therefore $Sa = \{w_a\}$, where $w_a$ is right multiplicatively absorbing. The set $B$ of right multiplicatively absorbing elements is an ideal. If $B = S$ then $uv = v$ for all $uv, \in S$ and every subsemigroup of $S(+)$ is a left ideal. Then $S(+)$ has no non-trivial proper subsemigroups and we see that either $|S| = 2$ or $S(+)$ is a $p$-element group for some prime number $p$. Since $uv = v$, $v = uv = (u + u)v = uv + uv = v + v$, $S(+)$ is idempotent and $S \cong Z_9$ (see I.2.1). On the other hand, if $B \neq S$ then $B = \{w\}$, $w$ being multiplicatively absorbing. We have $Sa = \{w\}$ for every $a \in A$ and it is easy to see that $A$ is an ideal of $S$. If $|A| = 1$ then $A = \{w\}$ and (2) is true. Finally, if $|A| \geq 2$ then $A = S$, $|SS| = 1$ and I.5.3 applies. □

**2.2 Proposition.** *Just one of the following seven cases takes place:*

(1) $Sa = S$ *for every* $a \in S$*;*

(2) $0_S \in S$ *is voth additively neutral and multiplicatively absorbing,* $T = S \setminus \{0\}$ *is a subsemiring of* $S$ *and* $Ta = T$ *for every* $a \in T$*;*

(3) $o_S \in S$ *is bi-absorbing,* $T = S \setminus \{o\}$ *is a subsemiring of* $S$ *and* $Ta = T$ *for every* $a \in T$*;*

(4) $o_S \in S$ *is bi-absorbing,* $Ta = T$ *and* $o \in T + a$ *for every* $a \in T = S \setminus \{o\}$*,*

(5) $S$ *is a skew-field;*

(6) $S$ *is a zero multiplicaiton ring of finite prime order;*

(7) $S$ *is isomorphic to one of* $\mathrm{Z}_1, \mathrm{Z}_3, \mathrm{Z}_4, \mathrm{Z}_9$.

*Proof.* Assume that neither (1) nor (6) nor (7) is true. According to 2.1, $S$ contains a multiplicatively absorbing element $w$ and $Sa = S$ for every $a \in T = S \setminus \{w\}$.

First, let $w = 0_S$ be additively neutral. For every $a \in T$, $Sa = S$ and the set $B_a = \{\, b \in S \mid ba = 0 \,\}$ is a left ideal. Since $B_a \neq S$, we have $B_a = \{0\}$ and it follows that $TT \subseteq T$ and $Ta = T$ for every $a \in T$. If $S$ is not a ring then (2) follows from I.3.6. On the other hand, if $S$ is a ring then, for every $a \in T$, there is $l_a \in T$ with $l_a = a$ and, for every $b \in T$, we have $bl_a a = ba$, $(bl_a - b)a = 0$ and $bl_a = b$. It follows that $l_a = 1_S$ is the unity of the ring $S$ ($b((l_{a_1} - l_{a_2}) = 0$ and $l_{a_1} = 1_S = l_{a_2}$). For every $c \in T$ there is $d \in T$ with $dc = 1$. Then $(cd - 1)c = cdc - c = c - c = 0$ and $cd = 1$. Thus $S$ is a skew-field.

Next, let $w \neq 0_S$. By I.3.6, $w = o_S$ is bi-absorbing. For every $a \in T$, we have $Sa = S$ and the set $C_a = \}\, c \in S \mid ca = o \,\}$ is a left ideal. Since $C_a \neq S$, we have $C_a = \{o\}$ and it follows that $TT \subseteq T$ and $Ta = T$. If $T + T \nsubseteq T$ then the set $D = \{\, d \in T \mid o \in d + T \,\}$ is non-empty. But $D \cup \{o\}$ is an ideal of $S$. □

**2.3 Lemma.** *Assume that* $Sa = S$ *for every* $a \in S$ *such that* $a$ *is not multiplicatively absorbing (see 2.1). Define a relation* $\varrho$ *on* $S$ *by* $(a, b) \in \varrho$ *iff* $xa = xb$ *for every* $x \in S$*. Then:*

(i) $\varrho$ *is a congruence of the semiring* $S$*.*

(ii) *If* $a, b, c \in S$ *are such that* $ab = ac$ *then either* $(b, c) \in \varrho$ *or* $a$ *is multiplicatively absorbing.*

(iii) *If* $a, b, c \in S$ *are such that* $(ab, ab) \in \varrho$ *then either* $(b, c) \in \varrho$ *or* $a$ *is multiplicatively absorbing.*

*Proof.* It is easy. □

**2.4 Proposition.** *Let* $S$ *be finite. Then just one of the following eight cases takes place:*

(1) $S$ *is additively idempotent,* $o_S \in S$ *is bi-absorbing,* $Ta = T$*,* $Sa = S$ *and* $o \in T + a$ *for every* $a \in T = S \setminus \{o\}$*;*

(2) $S$ *is additively idempotent,* $o_S \in S$ *is bi-absorbing,* $T = S \setminus \{o\}$ *is a subsemiring of* $S$ *and* $ab = a$ *for all* $a, b \in T$*;*

(3) $S$ *is additively idempotent and* $ab = a$ *for all* $a, b \in S$;
(4) $S$ *is additively idempotent* $0_S \in S$ *is additively neutral and multiplicatively absorbing,* $T = S \setminus \{0\}$ *is a subsemiring of* $S$ *and* $ab = a$ *for all* $a, b \in T$;
(5) $S$ *is additively constant,* $o_S \in S$ *is bi-absorbing and* $Ta = T$, $Sa = S$ *for every* $a \in T = S \setminus \{o\}$;
(6) $S$ *is a (finite) field;*
(7) $S$ *is a zero multiplication ring of (finite) prime order;*
(8) $S$ *is isomorphic to one of* $Z_1, Z_3, Z_4, Z_9$ *(see I.2.1).*

*Proof.* We can assume that either 2.1(1) or 2.1(2) is true. The rest of the proof is divided into four parts (use 2.2).
(i) Let 2.1(1) be true and let $\varrho \neq S \times S$, where $\varrho$ is defined in 2.3. Then $R = S/\varrho$ is a non-trivial semiring and the multiplicative semigroup $R(\cdot)$ is left cancellative. Since $Sa = S$ for every $a \in S$, both the semigroups $S(\cdot)$ and $R(\cdot)$ are right divisible. Since $S$ is finite, the semigroups are right quasigroups. Consequently, $R(\cdot)$ is a quasigroup, and hence a group. This contradicts 1.1.
(ii) Let 2.1(1) be true and let $\varrho = S \times S$. Then $ab = ac$ for all $a, b, c \in S$ and (3) follows from I.5.2.
(iii) Let 2.1(2) be true and let $w = 0_S$ be additively neutral and multiplicatively absorbing. If $S$ is a ring then $S$ is a (finite) field by 2.2. If $S$ is not a ring then $T = S \setminus \{0\}$ is a subsemiring of $S$ and $Ta = T$ for every $a \in T$. Now, by (i) and (ii), we have $ab = a$ for all $a, b \in T$. Thus (4) is true.
(iv) Let 2.1(2) be true and let $w = o_S$ be bi-absorbing. If $T + T \subseteq T$, where $T = S \setminus \{o\}$, then $T$ is a subsemiring of $S$ and $Ta = T$ for every $a \in T$ (use 2.2). Again, $ab = a$ for all $a, b \in T$ and (2) is true. Finally, assume that $T + T \nsubseteq T$. By 2.2, $Ta = T$ for every $a \in T$ and $o \in T + a$. If $S$ is additively idempotent then (1) is true. If $S$ is not additively idempotent then $S$ is additively constant by I.3.10. Thus (5) is true. □

**2.5 Proposition.** *Let* $S$ *be left-ideal-free. Then* $Sa = S$ *for every* $a \in S$*. Moreover, if* $S$ *is finite then* $ab = a$ *for all* $a, b \in S$.

*Proof.* Use 2.2 and 2.4. □

### 3. Elementary observations (c)

In this section, let $S$ be a non-trivial finite semiring containing a bi-absorbing element $o_S$ such that $Ta = T$ and $o \in T + a$ for every $a \in T = S \setminus \{o\}$ (see 2.4).

**3.1 Lemma.** (i) $S$ *is left-ideal-simple.*
(ii) *The multiplicative semigroup* $T(\cdot)$ *is a right quasigroup.*
(iii) $A = \{\, a \in T \mid a^a = a \,\} \neq \emptyset$.
(iv) *Every element from* $A$ *is right multiplicatively neutral in* $S$.

(v) $ab = a$ *for all* $a, b \in A$.
(vi) *For every* $a \in T$ *there is a uniquely determined element* $\underline{l}(a) \in A$ *with* $\underline{l}(a)a = a$.

*Proof.* The semigroup $T(\cdot)$ is right divisible and it is a right quasigroup, since it is finite. Consequently, for every $a \in T$ there is a uniquely determined element $\underline{l}(a) \in T$ with $\underline{l}(a)a = a$. We have $b\underline{l}(a)a = ba$ for every $b \in T$, so that $b\underline{l}(a) = b$ and $\underline{l}(a)$ is right multiplicatively neutral in $S$. Of course, $A = \{\, a \in T \mid \underline{l}(a) = a \,\}$ and the rest is clear. $\square$

**3.2 Lemma.** *Let* $f \in A$. *Then:*
(i) $(fT)(\cdot)$ *is a group.*
(ii) $(fT) \cup \{o\}$ *is a subsemiring of* $S$.
(iii) $fa + fb = o$ *for all* $a, b \in T$ *such that* $fa \neq fb$.

*Proof.* By 3.1(iv), $fafb = fab$, $ffa = fa$ and $faf = fa$ for all $a, b \in S$. Consequently, $fT$ is a subsemigroup of $T(\cdot)$ and $f$ is the neutral element of $fT$. We have $fTfa = fTa = fT$ for every $a \in T$, and so $(fT)(\cdot)$ is a right quasigroup. Now, it is clear that, in fact, it is a group. Furthermore, $fa + fb = f(a + b) \in (fT) \cup \{o\}$ for all $a, b \in T$ and $R = (fT) \cup \{o\}$ is a subsemiring of $S$. The assertions (i) and (ii) are proved. If $S$ is additively constant then (iii) is clear.

Assume that $S$ is not additively constant. By 2.4, $S$ is additively idempotent. Put $Q = \{\, a \in R \mid a \notin a + (R \setminus \{a\}) \,\}$. Then $o \notin Q$ and $bQ \subseteq Q$ for every $b \in R \setminus \{o\}$.

Let $Q \neq \emptyset$. Then $Q = R \setminus \{o\} = fT$ and if $a, b \in R$ are such that $a + b \neq o$ then $a + b \in Q$. But $a + b = (a + b) + a$ and $a + b = (a + b) + b$ It means that $a = a + b = b$ and (iii) is true.

Finally, let $Q = \emptyset$. Choose $a_1 \in R \setminus \{o\}$. Since $a_1 \notin Q$, there is $a_2 \in R \setminus \{a_1\}$ with $a_1 = a_1 + a_2$. Clearly, $a_2 \neq o$ and there is $a_3 \in R \setminus \{a_2\}$ with $a_2 = a_2 + a_3$. Since $a_1 \neq a_2$, we have $a_3 \neq a_1$. Proceeding in this way, we find an infinite sequence of pair-wise different elements $a_1, a_2, a_3, \dots$, a contradiction. $\square$

**3.3 Lemma.** *Let* $f \in A$. *Then:*
(i) $(\underline{l}(a), fa) \neq (\underline{l}(b), fb)$ *for all* $a, b \in T$, $a \neq b$.
(ii) $\underline{l}(a)\underline{l}(b) = \underline{l}(a) = \underline{l}(ab)$ *and* $fafb = fab$ *for all* $a, b \in T$.

*Proof.* (i) If $\underline{l}(a) = \underline{l}(b)$ and $fa = fb$ then $a = \underline{l}(a)a = \underline{l}(a)fa = \underline{l}(a)fb = \underline{l}(a)b = {}$
$= \underline{l}(b)b = b$ (use 3.1).
(ii) $fafb = fab$ by 3.1(iv). $\square$

**3.4 Lemma.** *Assume that* $S$ *is additively idempotent. The following conditions are equivalent for* $a, b \in T$:

(i) $a + b \neq o$ *(i.e.,* $a + b \in T$*).*
(ii) $a + b \neq o$ *and* $\underline{l}(a + b) = \underline{l}(a) + \underline{l}(b)$.
(iii) $fa = fb$ *for some* $f \in A$.

(iv) *$fa = fb$ for every $f \in A$.*
(v) *$ca = cb$ for some $c \in T$.*
(vi) *$ua = ub$ for every $u \in S$.*

*Proof.* If $ca = cb$ for some $c \in T$ then $S = Sc$ implies (vi). Consequently, the conditions (iii),…,(vi) are equivalent. If (iv) is true then $f(a+b) = fa + fb = fa \neq o$ ($S$ is additively idempotent), and hence $a + b \neq o$ and (i) is true. Finally, if $a + b \neq o$ then $fa + fb = f(a + b) \neq o$ and $fa = fb$ by 3.2(iii). Hence (iv) is true and $(\underline{l}(a)+\underline{l}(b))\cdot(a+b) = \underline{l}(a)a+\underline{l}(a)b+\underline{l}(b)a+\underline{l}(b)b = a+b$. Thus $\underline{l}(a+b) = \underline{l}(a)+\underline{l}(b)$. □

**3.5 Lemma.** *Let $e, f \in A$ and $a \in T$. Then $\underline{l}(ea) = e$ and $fea = fa$ (i.e., $(\underline{l}(ea), fea) =$ $= (e, fa)$).*

*Proof.* It is obvious. □

## 4. Examples

**4.1** Every two-element semiring is both left- and right-ideal-simple (see I.2.1).

**4.2** Let $S(+)$ be semilattice (i.e., an idempotent commutative semigroup). Define a multiplication on $S$ by $ab = a$ for all $a, b \in S$. Them $S = S(+,\cdot)$ becomes a bi-idempotent semiring. If $|S| \geq 2$ then this semiring is left-ideal-free and contains no right multiplicatively absorbing element.

**4.3** Let $S_1$ be a semiring of type 4.2. Let $0 \notin S_1$ and put $S = S_1 \cup \{0\}$, where 0 is additively neutral and multiplicatively absorbing. Then $S$ becomes a bi-idempotent semiring that is left-ideal-simple. If $|S_1| \geq 2$ then the semiring $S$ is not congruence-simple.

**4.4** Let $S_1$ be a semiring of type 4.2. Let $o \notin S_1$ and put $S = S_1 \cup \{o\}$, where $o$ is bi-absorbing. Then $S$ becomes a bi-idempotent semiring that is left-ideal-simple. If $|S_1| \geq 2$ then $S$ is not congruence-simple.

**4.5** Let $G = G(\cdot)$ be a group, $o \notin G$, $S = G \cup \{o\}$, $a + b = a + o = o + a = o$, $a + a = a$ and $ao = o = oa$ for all $a, b \in G$, $a \neq b$. Then $S$ is an additively idempotent semiring, $o = o_S$ is bi-absorbing and the unity of $G$ is multiplicatively neutral. The semiring $S$ is both left- and right-ideal-simple. If $|G| \geq 2$ then $S$ has no additively neutral element. If $|G| = 1$ then $S \cong Z_6$ (see I.2.1).

**4.6** Let $A = A(+)$ be a semilattice, $G = G(\cdot)$ a group and $S = (A \times G) \cup \{o\}$, where $o \notin A \times G$. Define an addition and a multiplication on $S$ by the following rules:
(a) $x + o = o + x = xo = ox = o$ for every $x \in S$;
(b) $(a, u) + (b, v) = o$ for all $a, b \in A$, $u, v \in G$, $u \neq v$;
(c) $(a, u) + (b, u) = (a + b, u)$ for all $a, b \in A$, $u \in G$;
(d) $(a, u) \cdot (b, v) = (a, uv)$ for all $a, b \in A$, $u, v \in G$.

It is moderately tedious but easy to check that $S$ becomes an additively idempotent semiring and $o = o_S$ is bi-absorbing. Furthermore, $Sx = S$ for every $x \in T = S \setminus \{o\}$, and hence $S$ is left-ideal-simple. We have $o \in T + y$ for every $y \in S$. For every $a \in A$, the set $(\{a\} \times G) \cup \{o\}$ is a right ideal. Consequently, $S$ is right-ideal-simple if and only if $|A| = 1$ (see 4.4). For every $a \in A$, the element $(a, 1)$ is right multiplicatively neutral. If $|A| \geq 2$ then the semiring $S$ has no multiplicatively neutral element. The semiring $S$ has an additively neutral element if and only if $|G| = 1$ and $A(+)$ has such an element (then $0_S = (0_A, 1_G)$).

**4.7** Let $G = G(\cdot)$ be a group, $o \notin G$, $S = G \cup \{o\}$ and $x + y = xo = ox = o$ for all $x, y \in S$. Then $S$ is an additively constant semiring, $o = o_S$ is bi-absorbing and the unity of $G$ is multiplicatively neutral. The semiring $S$ is both left- and right-ideal-simple and has no additively neutral element. If $|G| = 1$ then $S \cong Z_2$ (see I.2.1).

**4.8** Let $A$ be a non-empty set, $o \notin A$ and $S = A \cup \{o\}$. Put $x + y = xo = ox = o$ for all $x, y \in S$ and $ab = a$ for all $a, b \in A$. Then $S$ is an additively constant semiring, $o = o_S$ is bi-absorbing and $Sa = S$ for every $a \in A$. Consequently, $S$ is left-ideal-simple.

**4.9** Let $A$ be a non-empty set, $G$ $(= G(\cdot))$ a group and $S$ $=)A \times G) \cup \{o\}$, where $o \notin A \times G$. Define an addition and a multiplication on $S$ by the following rules:
(a) $x + y = o = ox = xo$ for all $x, y \in S$;
(b) $(a, u)(b, v) = (a, uv)$ for all $a, b \in A$, $u, v \in G$.

It is easy to check that $S$ becomes an additively constant semiring and that $o = o_S$ is bi-absorbing. Furthermore, $Sx = S$ for every $x \in T = S \setminus \{o\}$, and hence $S$ is left-ideal-simple. We have $o \in T + y$ for every $y \in S$. For every $a \in A$, the set $(\{a\} \times G) \cup \{o\}$ is a right ideal. Consequently, $S$ is right-ideal-simple if and only if $|A| = 1$ (see 4.7). For every $a \in A$, the element $(a, 1)$ is right multiplicatively neutral. If $|A| \geq 2$ then the semiring $S$ has no multiplicatively neutral element. The semiring $S$ has no additively neutral element.

**4.10** Let $R$ be a semiring such that the multiplicative semigroup $R(\cdot)$ is a group. Let the group $R(\cdot)$ be subgroup of a group $T(\cdot)$. Let $o \notin T$ and $S = T \cup \{o\}$. Define an addition on $S$ by $x + o = o = o + x$ for every $x \in S$, $a + b = o$ for $a, b \in T$, $a^{-1}b \notin R$ (equivalently, $b^{-1} \notin R$) and $a + b = a(1 + a^{-1}b)$ for $a, b \in T$, $a^{-1}b \in R$ (equivalently, $b^{-1}a \in R$). Setting $xo = o = ox$ for every $x \in S$, we get an algebraic structure $S = S(+, \cdot)$ with two binary operations, where $o = o_S$ is apparently bi-absorbing. $T(\cdot)$ is a subgroup of $S(\cdot)$ and $S(\cdot)$ is a monoid, $1_S = 1_T$. If $a, b \in T$ are such that $a^{-1}b \notin R$ then $b^{-1}a \notin R$ and $a + b = o = b + a$. If $a^{-1}b \in R$ then $b^{-1}a \in R$, $a^{-1}b(1 + b^{-1}a) = a^{-1}b + 1$ and $a + b = a(1 + a^{-1}b) = b(1 + b^{-1}a) = b + a$. It means that $S(+)$ is a commutative groupoid.
(i) If $x, y, z \in S$ are such that $o \in \{x, y, z\}$ then $x + (y + z) = o = (x + y) + z$.
(ii) Let $a, b, c \in T$ be such that $a^{-1}b \in R$ and $b^{-1}c \in R$. Then $a^{-1}c \in R$, $a + b = a(1 + a^{-1}b)$, $b + c = b(1 + b^{-1}c)$, $c^{-1}a(1 + a^{-1}b) = c^{-1}a + c^{-1}b \in R$, $a^{-1}b(1 + b^{-1}c) = a^{-1}b + a^{-1}c \in R$ and $(a + b) + c = c + (a + b) = c(1 + c^{-1}a(1 + a^{-1}b)) = c(1 + c^{-1}a + c^{-1}b) \neq o$, $a + (b + c) = a(1 + a^{-1}b(1 + b^{-1}c)) = a(1 + a^{-1}b + a^{-1}c) \neq o$.

But $a^{-1}c(1 + c^{-1}a + c^{-1}b) = a^{-1}c + 1 + a^{-1}b$ and $(a + b) + c = c(1 + c^{-1}a + c^{-1}b) = = a(1 + a^{-1}b + a^{-1}c) = a + (b + c)$.
(iii) Let $a, b, c \in T$ be such that $a^{-1}b \in R$ and $b^{-1}c \notin R$. Then $a + b = a(1 + a^{-1}b)$, $c^{-1}a = c^{-1}bb^{-1}a \notin R$, $1 + a^{-1}b \in R$, $c^{-1}a(1 + a^{-1}b) \notin R$ and $(a + b) + c = c + (a + b) = = o = a + o = a + (b + c)$.
(iv) Let $a, b, c \in T$ be such that $a^{-1}b \notin R$ and $b^{-1}c \in R$. Similarly as in (iii), we have $(a + b) + c = o = a + (b + c)$.
(v) Let $a, b, c \in T$ be such that $a^{-1}b \notin R$ and $b^{-1}c \notin R$. Then $(a + b) + c = o + c = o = = a + o = a + (b + c)$.
(vi) Combining (i),. . . ,(v), we have verified that $S(+)$ is a commutative semigroup.
(vii) If $x, y, z \in S$ are such that $0 \in \{x, y, z\}$ then $x(y + z) = o = xy + xz$ and $(y + z)x = = o = yx + zx$.
(viii) Let $a, b, c \in T$ be such that $b^{-1}c \in R$. Then $(ab)^{-1}(ac) = b^{-1}c \in R$ and $a(b + c) = ab(1 + b^{-1}c) = ab(1 + (an)^{-1}(ac)) = ab + ac$.
(ix) Let $a, b, c \in T$ be such that $b^{-1}c \notin R$. Then $(ab)^{-1}(ac) = b^{-1}c \notin R$ and $a(b + c) = = o = ab + ac$.
(x) Combining (vii), (viii) and (ix), we have verified that the multiplication is left distributive over the addition. It means that $S = S(+, \cdot)$ is a left near-semiring.
(xi) Let $a, b, c \in T$ be such that $b^{-1}c \in R$. Then $b + c = b(1 + b^{-1}c)$ and $(b + + c)a = a^{-1}b^{-1}ca$. If $a^{-1}b^{-1}ca \notin R$ then $ba + ca = o \neq (b + c)a$. If $a^{-1}b^{-1}ca \in R$ then $ba + ca = ba(1 + a^{-1}b^{-1}ca) \neq o$. In the latter case, $ba + ca = (b + c)a$ iff $a(1 + a^{-1}b^{-1}ca)a^{-1} = 1 + b^{-1}c$ or $1 + a^{-1}b^{-1}ca = a^{-1}(1 + b^{-1}c)a$.
(xii) Let $a, b, c \in T$ be such that $b^{-1}c \notin R$. Then $b + c = o$ and $(b + c)a = o$. If $a^{-1}b^{-1}ca \in R$ then $ba + ca \neq o = (b + c)a$.
(xiii) Let $a, b \in T$ and $u \in R$. then $b^{-1}bu = u$, $b + bu = b(1 + u)$, $(b + bu)a = b(1 + u)a$. If $a^{-1}ua \in R$ then $ba + bua = ba(1 + a^{-1}ua)$. Thus $(b + bu)a = ba + bua$ iff $a^{-1}(1 + u)a = = 1 + a^{-1}ua$. If $a^{-1}ua \notin R$ then $ba + bua = o$.
(xiv) Combining (xi), (xii) and (xiii), we have verified that the near-semiring $S$ is a semiring if and only if $R(\cdot)$ is a normal subgroup of $T(\cdot)$ and $a^{-1}(1 + ua) = 1 + a^{-1}ua$ for all $a \in T$ and $u \in R$. (Notice that these conditions are satisfied if $R \subseteq Z(T(\cdot))$.)
(xv) Assume that $S$ is a semiring (see (xiv)). The unity $1_T = 1_S$ is multiplicatively neutral and $o = o_S$ is bi-absorbing. If $|T| = 1$ then $S \cong Z_6$. If $|T| \geq 2$ then $S$ has no additively neutral element. If $|R| = 1$ then $S$ is as in 4.5.

We have $a + a = (1 + 1)a = 2a$ for every $a \in T$. Consequently, either $S$ is additively idempotent or $o$ is the only additively idempotent element of $S$. We have also $Sa = S - aS$ for every $a \in T$. It follows that $S$ is both left- and right-ideal-simple. Of course, $R = \{ a \in T \mid 1 + a \neq o \}$. If $R \neq T$ then $o \in T + x$ for every $x \in S$.

If $S$ is not additively idempotent then $2_S = 1_S + 1 + S \neq 1_S$ and the subgroup $Q$ of the semiring $R$ generated by all $n_S$, $n \geq 1$, is a subsemiring of $R$ and $Q \cong \mathbb{Q}^+$ (the parasemifield of positive rational numbers). Clearly, $Q \subseteq Z(T(\cdot))$.

In particular, if $R(\cdot)$ is an infinite cyclic group then $S$ is additively idempotent and we claim that $R \subseteq Z(T(\cdot))$.

Indeed, if $R \nsubseteq Z(T(\cdot))$ then there is $a \in T$ such that $a^{-1}ua = u^{-1}$ for every $u \in R$. Now, $(1+u)^{-1} = a^{-1}(1+u)a = 1 + a^{-1}ua = 1 + u^{-1}$, $1 = (1+u)(1+u^{-1}) = 1 + u^{-1} + u + 1 = 1 + u + u^{-1}$, $1 + u = 1$, $1 + u^{-1} = 1$ for every $u \in R$. On the other hand, $1 + u^{-1} = 1$ implies $u = u + 1$, so that $u = 1$ and $|R| = 1$, a contradiction.

## 5. Finite left-ideal-simple semirings

In this part, let $S$ be a finite left-ideal-simple semiring.

**5.1 Theorem.** *Just one of the following eleven cases takes place:*

(1) *$S$ is additively idempotent, $o_S \in S$ is bi-absorbing, $1_S \in S$ is multiplicatively neutral, $T(\cdot)$ is a subgroup of $S(\cdot)$, where $T = S \setminus \{o_S\}$ and $a + b = o$ for all $a, b \in T$, $a \neq b$* (see 4.5)*;*

(2) *$S$ is additively idempotent, $o_S \in S$ is bi-absorbing, $S$ has no multiplicatively neutral element, $T(\cdot)$ is a subgroup of $S(\cdot)$, where $T = S \setminus \{o_S\}$, $o \in a + T$ for every $a \in T$ and $S$ is constructed in the way described in 4.6 (where $|A| \geq 2$ and $|G| \geq 2$; then $|S| \geq 5$);*

(3) *$S$ is additively idempotent, $o_S \in S$ is bi-absorbing, $T = S \setminus \{o_S\}$ is a subsemiring of $S$ and $ab = a$ for all $a \in S$, $b \in T$* (see 4.4)*;*

(4) *$S$ is additively idempotent and $ab = a$ for all $a, b \in S$* (see 4.2)*;*

(5) *$S$ is additively idempotent, $0_S \in S$ is additively neutral and multiplicatively absorbing, $T = S \setminus \{0_S\}$ is a subsemiring of $S$ and $ab = a$ for all $a \in S$, $b \in T$* (see 4.3)*;*

(6) *$S$ is additively constant, $o_S \in S$ is bi-absorbing, $1_S \in S$ is multiplicatively neutral and $T(\cdot)$ is a subgroup of $S(\cdot)$, where $T = S \setminus \{o_S\}$* (see 4.7)*;*

(7) *$S$ is additively constant, $o_S \in S$ is bi-absorbing and $ab = a$for all $a \in S$ and $b \in S \setminus \{o_S\}$* (see 4.8)*;*

(8) *$S$ is additively constant, $o_S$ is bi-absorbing, $S$ has no multiplicatively neutral element, $ab \neq a$ for some $a, b \in S \setminus \{o_S\}$ and $S$ is constructed in the way described in 4.9 (where $|A| \geq 2$ and $|G| \geq 2$; then $|S| \geq 5$);*

(9) *$S$ is a (finite) field;*

(10) *$S$ is a zero multiplication ring of prime order;*

(11) $S \cong Z_1, Z_3, Z_4, Z_9$ (see I.2.1).

*Proof.* In view of 2.4, we will assume that either 2.4(1) or 2.4(5) is satisfied. In both cases, $S$ contains a bi-absorbing element $o$ and we put $T = S \setminus \{o\}$. We have $Ta = T$, $o \in T + a$ and $Sa = S$ for every $a \in T$. Now, as in 3.1, put $A = \{a \in T \mid a^2 = a\}$ and choose $f \in A$. Define a mapping $\varphi : T \to A \times G$, where $G = fT$, by $\varphi(a) = (\underline{l}(a), fa)$. By 3.3(i), the mapping $\varphi$ is injective and, by 3.5, it is projective as well. Thus $\varphi$ is a biunique mapping of $T$ onto $A \times G$ and, by 3.3(ii), $\varphi(ab) = (\underline{l}(a), fab)$ for all $a, b \in T$. The rest of the proof is divided into six parts.

(i) Let $S$ be additively idempotent and let $|A| = 1$. We have $A = \{f\}$, where $f = 1_S$

is multiplicatively neutral, and hence $T = 1T = fT = G$ and $T(\cdot)$ is a group. Now, using 3.4, we see that (1) is true.
(ii) Let $S$ be additively idempotent and let $|G| = 1$. We have $G = fT = \{f\}$, so that $fa = f$ for every $a \in T$. Then $e = ef = efa = ea$ for all $e \in A$, $a \in T$, and hence $\underline{l}(a) = \underline{l}(a)a = a$. Thus $A = T$, $ab = a$ for all $a, b \in T$. Moreover, $a(a + b) = a^2 + ab = a + a = a$ and it follows that $a + b \in T$. Now, it is clear that (3) is true.
(iii) Let $S$ be additively idempotent and let $|A| \geq 2$ and $|G| \geq 2$. By 3.2(i), $G(\cdot)$ is a group. If $e_1, e_2 \in A$ then $e_1e_2 = e_1$, $e_1(e_1 + e_2) = e_1^2 + e_1e_2 = e_1 + e_1 = e_1$, so that $e_1 + e_2 \neq o$. Of course, $(e_1 + e_2)^2 = e_1^2 + e_1e_2 + e_2e_1 + e_2^2 = e_1 + e_1 + e_2 + e_2 = e_1 + e_2$ and $e_1 + e_2 \in A$. Thus $A$ is a subsemiring of $S$. By 3.4, if $a, b \in T$ then $a + b \neq o$ iff $fa = fb$; then $\underline{l}(a + b) = \underline{l}(a) + \underline{l}(b)$. By 3.1, $\underline{l}(ab) = \underline{l}(a) = \underline{l}(a)\underline{l}(b)$ and $fab = fafb$ for all $a, b \in T$. Now, it is clear that (2) is true.
(iv) Let $S$ be additively constant and let $|A| = 1$. By 3.2(i), $T(\cdot) = G(\cdot)$ is a group and (6) is true.
(v) Let $S$ be additively constant and let $|G| = 1$. We have $G = fT = \{f\}$ and $A = T$, $ab = a$ for all $a, b \in T$. Thus (7) is true.
(vi) Let $S$ be additively constant, $|A| \geq 2$ and $|G| \geq 2$. Then (8) is true. □

**5.2 Corollary.** *Assume that $0_S \in S$. Then either 5.1(3) is true (and $0_T \in T$) or 5.1(4) is true (and $0 \in S(+)$) or 5.1(5) is true or 5.1(9) is true or 5.1(10) is true or $S \cong Z_3, Z_4, Z_9$.* □

**5.3 Corollary.** *Assume that $0_S \in S$ is multiplicatively absorbing. Then either 5.1(5) is true or 5.1(9) is true or 5.1(10) is true or $S \cong Z_4$.* □

**5.4 Corollary.** *Assume that $1_S \in S$. Then either 5.1(1) is true or 5.1(6) is true or 5.1(9) is true or $S \cong Z_5, Z_6$.* □

**5.5 Corollary.** *If $0_S \in S$ and $1_S \in S$ then either $S$ is a field or $S \cong Z_5, Z_6$.* □

**5.6 Corollary.** *If $S$ has no multiplicatively absorbing element then either 5.1(4) is true or $S \cong Z_9$.* □

**5.7 Corollary.** *If $S$ is left-ideal-free then 5.1(4) is true.* □

**5.8 Corollary.** *If $S$ is both left- and right-ideal-simple then either 5.1(1) is true or 5.1(6) is true or 5.1(9) is true or 5.1(10) is true or 5.1(11) is true or $S \cong Z_5, Z_6, Z_{10}$.*
□

## 6. Left- and right-ideal-simple semirings

Let $S$ be a left- and right-ideal-simple semiring.

**6.1 Theorem.** *Just one of the following nine cases takes place:*

(1) $S(\cdot)$ *is a group (then $S$ is infinite);*

(2) $0_S \in S$ *is multiplicatively absorbing,* $T = S \setminus \{0_S\}$ *is a subsemiring of* $S$ *and* $T(\cdot)$ *is a group (then either* $S \cong Z_5$ *or* $S$ *is infinite;*

(3) $o_S \in S$ *is bi-absorbing,* $T = S \setminus \{o_S\}$ *is a subsemiring of* $S$ *and* $T(\cdot)$ *is a group (then either* $S \cong Z_6$ *or* $S$ *is infinite);*

(4) $S$ *is additively idempotent,* $o_S \in S$ *is bi-absorbing,* $T(\cdot)$ *is a group and* $o_S \in \in T + a$ *for every* $a \in T = S \setminus \{o_S\}$*;*

(5) $o_S \in S$ *is bi-absorbing,* $o_S$ *is the only additively idempotent element of* $S$*,* $T(\cdot)$ *is a group and* $2a \neq o_S \in T + a$ *for every* $a \in T = S \setminus \{o_S\}$*;*

(6) $S$ *is additively constant,* $o_S \in S$ *is bi-absorbing and* $T(\cdot)$ *is a group, where* $T = S \setminus \{o_S\}$ *(see 4.7; then either* $S \cong Z_2$ *or* $S$ *is infinite);*

(7) $S$ *is a skew-field;*

(8) $S$ *is a zero-multiplication ring of prime order;*

(9) $S \cong Z_1, Z_3, Z_4, Z_9, Z_{10}$.

*Proof.* Assume that none of (1), (2), (3), (7), (8), and (9) is true. Now, considering 2.2 and the right hand form of 2.2, we conclude that $o = o_S \in S$ is bi-absorbing, $T(\cdot)$ is a group and $o \in T + a$ for every $a \in T = S \setminus \{o\}$. If $S$ is additively idempotent then (4) is true. If $S$ is additively constant then (6) is true. Henceforth, assume that $a+b \neq o$ and $2c \neq c$ for some $a, b, c \in S$. Clearly, $a, b, c \in T$, $1+a^{-1}b \neq o \neq 1+b^{-1}a$, $o \neq (1+a^{-1}b)(1+b^{-1}a) = 1+1+a^{-1}b+b^{-1}a$, $1+1 \neq o$, $d+d \neq o$ and $2d = 2cc^{-1}d \neq \neq cc^{-1}d = d$ for every $d \in T$. Thus (5) is true. □

**6.2 Theorem.** *Let $S$ satisfy 6.1(4) or (5). Then:*

(i) *The set* $R = \{\, a \in T \mid 1 + a \neq o \,\}$ *is a subsemiring of* $S$*,the multiplicative semigroup* $R(\cdot)$ *is a subgroup of the group* $T(\cdot)$*,* $|T| \geq 2$ *and* $T \neq R$*.*

(ii) $R(\cdot)$ *is a normal subgroup of* $T(\cdot)$*.*

(iii) $a^{-1}(1 + u)a = 1 + a^{-1}ua$ *for all* $a \in T$ *and* $u \in R$*.*

(iv) *If* $a, b \in T$ *then* $a + b \neq o$ *if and only if* $a^{-1}b \in R$ *(*$a + b = a(1 + a^{-1}b)$*).*

(v) $1_T = 1_S$ *is multiplicatively neutral.*

(vi) $S$ *has no additively neutral element.*

(vii) *If* $1 + 1 = 1$ *then* $S$ *is additively idempotent.*

(viii) *If* $1 + 1 \neq 1$ *then* $o$ *is the only additively idempotent element of* $S$*.*

(ix) *If* $R = \{1\}$ *then* $S$ *is additively idempotent and* $a + b = o$ *for all* $a, b \in T$*,* $a \neq b$ *(see 4.5; of course,* $|T| \geq 2$*).*

(x) *If* $|R| \geq 2$ *then* $S$ *is infinite. (Notice that* $S$ *is constructed in the way described in 4.10.)*

*Proof.* (i) We have $1 + 1 \neq o$, and therefore $1 \in R$. If $a, b \in R$ then $a^{-1} + 1 = a^{-1}(1 + a) \neq o$, $1 + a + b + ab = (1 + a)(1 + b) \neq o$, and so $1 + ab \neq o$ and $1 + a + b \neq o$. It follows that $R$ is a subsemiring of $S$ and $R(\cdot)$ is a subgroup of the group $T(\cdot)$. Since $1 + 1 \neq o \in T + 1$, we have $|T| \geq 2$ and $T \neq R$.
(ii) If $a \in T$ and $u \in R$ then $o \neq a^{-1}(1 + u)a = 1 + a^{-1}ua$. Thus $a^{-1}ua \in R$.
The remaining assertions are now easy. $\square$

**6.3 Corollary.** *Assume that $0_S \in S$. Then either 6.1(2) is true or 6.1(7) is true or 6.1(8) is true or $S \cong Z_3, Z_4, Z_6, Z_9, Z_{10}$. (Notice that $0_S$ is multiplicatively absorbing in the first three cases and also if $S \cong Z_4$.)* $\square$

**6.4 Corollary.** *If $1_S \notin S$ then either 6.1(8) is true or $S \cong Z_1, Z_3, Z_4, Z_9, Z_{10}$.* $\square$

**6.5 Corollary.** *Assume that $0_S \in S$ and $1_S \in S$. Then either 6.1(2) is true or 6.1(7) is true or $S \cong Z_6$. (If either $0_S$ is not multiplicatively absorbing or $0_S = 1_S$ then $S \cong Z_6$.)* $\square$

**6.6 Corollary.** *Assume that $0_S \in S$, $1_S \in S$, $a + b = 0_S$ for some $a \in S$ and $b \in S \setminus \{0_S\}$ and either $0_S \neq 1_S$ or $|S| \geq 3$. Then $S$ is a skew-field.* $\square$

**6.7 Corollary.** *If $S$ is left- and right-ideal-free then $S(\cdot)$ is a group.* $\square$

## References

[1] T. Kepka and P. Němec: *Ideal-simple semirings I.*