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AVERAGING PROBLEM IN GENERAL RELATIVITY AND COSMOLOGY

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Praha

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It is tradition in cosmology to use the homogeneous and isotropic FRW (Friedmann-Robertson-Walker) spacetime. However, the real universe is inhomogeneous and anisotropic on small scales so if we want to retain the FRW approach, we should at least perform some averaging procedure. Because of the nonlinearity of the Einstein field equations, we will in general obtain a nonzero correlation term, which does not necessarily obey the energy condition and so it can mimic the dark energy term. In this article I will try to review different approaches to the averaging problem with the emphasis on cosmology.

1. Introduction

In General relativity (GR) the evolution of the metric tensor is driven by the Einstein field equations. As emphasized in 1980s by *Ellis* [1984], averaging and evolution do not commute, i.e. $\left\langle E_{\mu\nu}(g_{\mu\nu})\right\rangle \neq E_{\mu\nu}(\left\langle g_{\mu\nu}\right\rangle)$. $E_{\mu\nu}$ is the Einstein tensor, $g_{\mu\nu}$ is the metric tensor and $\langle\rangle$ is some unspecified averaging procedure. On the other hand, in cosmology one usually uses the homogeneous and isotropic Friedmann-Robertson-Walker (FRW) metric and the smooth stress energy tensor of the perfect fluid. If we want to use a simple model and represent the dynamics of the universe by one single scale function $a(t)$ (not to use more general inhomogeneous cosmological model), we should put a new correlation term $C_{\mu\nu}$ into the equations.

$$E_{\mu\nu}(\left\langle g_{\mu\nu}\right\rangle) = 8\pi\left\langle T_{\mu\nu}\right\rangle + C_{\mu\nu}, \tag{1}$$

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which is defined by the construction

$$C_{\mu\nu} = E_{\mu\nu}(\left\langle g_{\mu\nu} \right\rangle) - \left\langle E_{\mu\nu}(g_{\mu\nu}) \right\rangle . \tag{2}$$

It does not necessarily obey the usual energy condition and it can act as dark energy [*Buchert*, 2008]. Averaging can be considered over some spacelike hypersurface, which depends on the selected slicing or over some spacetime interval, which can be covariantely defined. There are two main goals concerning averaging—the first is to construct averaged metric and the second is to obtain correlation term modifying Einstein equations.

There is a technical problem in a definition of an averaged tensor: Integrating a tensor field in curved spacetime does not result in a new tensor field (this is because of the addition of the tensors living in different spaces). In the next sections we will show some attempts how to solve this problem.

2. Isaacson's approach

Following the work of *Brill and Hartle* [1964], Isaacson used an averaging method for computing the effective gravitational stress energy tensor [*Isaacson*, 1968]. In order to compute an average value of the general tensor over the domain $\mathscr{D}$ at the base point x, he parallel transports tensors from points in $\mathscr{D}$ to x and then integrates.

$$\left\langle A_{\mu\nu}(x) \right\rangle_{BH} = \frac{1}{V_{\mathscr{D}}} \int\limits_{\mathscr{D}} g_{\mu}^{\alpha'}(x, x') g_{\nu}^{\beta'}(x, x') A_{\alpha'\beta'}(x') \sqrt{-g(x')} d^4 x' . \tag{3}$$

$g(x')$ denotes the determinant of the metric. $g_{\mu}^{\alpha'}(x, x')$ is the bivector of geodesic parallel displacement that serves to parallel transport of $A_{\alpha'\beta'}(x')$ and $V_{\mathscr{D}}$ is the volume of $\mathscr{D}$. Integration over x' is justified because of the contraction over the prime indices. It can be shown that the following properties hold:

- One can ignore the terms $\left\langle A_{\mu\nu}{}^{\rho}{}_{;\rho} \right\rangle_{BH}$.
- One can integrate by parts.
- Covariant derivatives commute.

3. Macroscopic Gravity

Another promising approach to the averaging problem is the method (valid for n-dimensional manifolds) developed by Zalaletdinov who also gives several conditions for the correlation term to be fulfilled [*Zalaletdinov*, 1992, 1993, 2004]. One of the big problem of BH averaging scheme is that it leaves the metric tensor unchanged. To overcome this trouble, Zalaletdinov introduced a bilocal averaging operator $\mathscr{W}_{\beta}^{\alpha'}(x', x)$ which transforms as a vector at the point x' and as a covector at the point x. Its construction follows from the demanded properties:

$$\lim_{x'\to x}\mathscr{W}^{\alpha'}_{\beta}(x',x)=\delta^{\alpha}_{\beta}, \tag{4}$$

$$\mathscr{W}^{\alpha'}_{\gamma''}(x',x'')\mathscr{W}^{\gamma''}_{\beta}(x'',x)=\mathscr{W}^{\alpha'}_{\beta}(x',x). \tag{5}$$

It can be shown that these two properties are equivalent to the following form of the bilocal operator:

$$\mathscr{W}^{\alpha'}_{\beta}(x',x)=F^{\alpha'}_{\gamma}(x')F^{-1\gamma}_{\beta}(x). \tag{6}$$

Now it is possible for a given compact region $\mathscr{D}$ of a differentiable space-time manifold $(\mathscr{M},g_{\alpha\beta})$ with a volume n-form to define the average value of the tensor field $t^{\alpha\dots}_{\beta\dots}(x)$, $x\in\mathscr{M}$ as

$$\bar{t}^{\alpha\dots}_{\beta\dots}(x)=\frac{1}{V_{\mathscr{D}}}\int_{\mathscr{D}}\tilde{t}^{\alpha\dots}_{\beta\dots}(x,x')\sqrt{-g'}d^nx', \tag{7}$$

g'=det$(g_{\alpha\beta}(x'))$, $V_{\mathscr{D}}$ is the volume of $\mathscr{D}$ and the object $\tilde{t}^{\alpha\dots}_{\beta\dots}(x,x')$ is the bilocal extension of the tensor $t^{\alpha\dots}_{\beta\dots}(x)$ using the bivector $\mathscr{W}^{\alpha'}_{\beta}(x',x)$

$$\tilde{t}^{\alpha\dots}_{\beta\dots}(x,x')=\mathscr{W}^{\alpha}_{\alpha'}(x',x)\dots\mathscr{W}^{\beta'}_{\beta}(x',x)\dots t^{\alpha'\dots}_{\beta'\dots}(x'). \tag{8}$$

Now it is possible to bilocally extend Einstein equations and then perform averaging. The theory of Macroscopic Gravity not only averages Einstein equations but also geometry itself. From the consistent procedure how to average Cartan structure equations and their integrability equations it is possible to find a system of algebraic and differential equations that must be fulfilled by the correlation term term.

The first exact solution of Macroscopic Gravity was published by *Coley et al.* [2005]. Resulting correlation term can be interpreted as an additional space curvature.

4. Buchert equations

In the last two sections we have seen that it isn't very obvious how to average tensors. However, averaging scalars has a clear rule. In most of the cosmological models there is a preferred timelike vector (cosmic time) so it is useful to perform 3+1 splitting of the variables. Here we will restrict ourselves only to the dust source [*Buchert*, 2000] [it can be generalized for the perfect fluid *Buchert*, 2001].

For the metric $ds^2=-dt^2+g_{ij}dX^idX^j$ spatial averaging of the scalar field Ψ over the domain $\mathscr{D}$ is defined by

$$\left\langle\Psi(t,X^i)\right\rangle_{\mathscr{D}}:=\frac{1}{V_{\mathscr{D}}}\int_{\mathscr{D}}Jd^3X\Psi(t,X^i), \tag{9}$$

$$V_{\mathscr{D}}=\int_{\mathscr{D}}Jd^3X, \tag{10}$$

where $J := \sqrt{detg_{ij}}$, g_{ij} is the metric of the spacelike hypersurface and X^i are the comoving coordinates. Taking time derivative of this definition we can obtain the following important commutation rule:

$$\partial_t \left\langle \Psi(t, X^i) \right\rangle_{\mathscr{D}} - \left\langle \partial_t \Psi(t, X^i) \right\rangle_{\mathscr{D}} = \left\langle \Psi(t, X^i) \right\rangle_{\mathscr{D}} \langle \Theta \rangle_{\mathscr{D}} - \left\langle \Psi(t, X^i) \Theta \right\rangle_{\mathscr{D}}, \tag{11}$$

where the expansion rate Θ is related to the velocity of the fluid u^μ according to the definition by $\Theta = u^\mu{}_{;\mu}$. Next we introduce in analogy with FRW spacetime a dimensionless scale factor $a_{\mathscr{D}}$ and the effective Hubble parameter $H_{\mathscr{D}}$

$$a_{\mathscr{D}} = \left(\frac{V_{\mathscr{D}}}{V_{\mathscr{D}i}} \right)^{\frac{1}{3}}, \tag{12}$$

$$\langle \Theta \rangle_{\mathscr{D}} = \frac{\dot{V_{\mathscr{D}}}}{V_{\mathscr{D}}} = 3 \frac{\dot{a_{\mathscr{D}}}}{a_{\mathscr{D}}} =: 3H_{\mathscr{D}}. \tag{13}$$

A dot denotes partial derivative with respect to time, $V_{\mathscr{D}i}$ is the volume of the initial domain which geodetically evolved to $V_{\mathscr{D}}$. Now we have a formalism how to average scalars. To obtain scalar equation from the Einstein equation, we have to contract it with available tensors—i.e. $g^{\mu\nu}, u^\mu$ and ∇^μ. After contraction we obtain the Raychaudhuri equation, the Hamiltonian constraint and the continuity equation. Now we perform averaging and use the commutation rule (11).

$$3 \frac{\ddot{a_{\mathscr{D}}}}{a_{\mathscr{D}}} + 4\pi G \langle \rho \rangle_{\mathscr{D}} - \Lambda = \mathscr{Q}_{\mathscr{D}}, \tag{14}$$

$$\left(\frac{\dot{a}_{\mathscr{D}}}{a_{\mathscr{D}}} \right)^2 - \frac{8\pi G}{3} \langle \rho \rangle_{\mathscr{D}} + \frac{\langle \mathscr{R} \rangle_{\mathscr{D}}}{6} - \frac{\Lambda}{3} = -\frac{\mathscr{Q}_{\mathscr{D}}}{6}, \tag{15}$$

$$\partial_t \langle \rho \rangle_{\mathscr{D}} + 3 \frac{\dot{a_{\mathscr{D}}}}{a_{\mathscr{D}}} \langle \rho \rangle_{\mathscr{D}} = 0. \tag{16}$$

$\langle \mathscr{R} \rangle_{\mathscr{D}}$ denotes average value of the spatial Ricci scalar, $\langle \rho \rangle_{\mathscr{D}}$ means average density of the averaged fluid and $\mathscr{Q}_{\mathscr{D}}$ that shows possible backreaction (by present inhomogeneity and anisotropy) is defined by

$$\mathscr{Q}_{\mathscr{D}} := \frac{2}{3} \left\langle (\Theta - \langle \Theta \rangle_{\mathscr{D}})^2 \right\rangle_{\mathscr{D}} - 2 \left\langle \sigma^2 \right\rangle_{\mathscr{D}}. \tag{17}$$

The scalar $\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij}$ is constructed from the shear tensor. The time derivative of the averaged Hamiltonian constrain agrees with the Raychaudhuri equation when the integrability equation is fulfilled

$$\partial_t \mathscr{Q}_{\mathscr{D}} + 6 \frac{\dot{a}_{\mathscr{D}}}{a_{\mathscr{D}}} \mathscr{Q}_{\mathscr{D}} + \partial_t \langle \mathscr{R} \rangle_{\mathscr{D}} + 2 \frac{\dot{a}_{\mathscr{D}}}{a_{\mathscr{D}}} \langle \mathscr{R} \rangle_{\mathscr{D}} = 0. \tag{18}$$

In a similar way as in the FRW approach we can define dimensionless variables (omega factors)

$$\Omega_m^{\mathscr{D}} := \frac{8\pi G}{3H_{\mathscr{D}}^2} \langle \rho \rangle_{\mathscr{D}}; \quad \Omega_\Lambda^{\mathscr{D}} := \frac{\Lambda}{3H_{\mathscr{D}}^2}; \quad \Omega_{\mathscr{R}}^{\mathscr{D}} := -\frac{\langle \mathscr{R} \rangle_{\mathscr{D}}}{6H_{\mathscr{D}}^2}; \quad \Omega_{\mathscr{Q}}^{\mathscr{D}} := -\frac{\mathscr{Q}_{\mathscr{D}}}{6H_{\mathscr{D}}^2} \tag{19}$$

and Hamiltonian constraint will be written in the standard form

$$\Omega_m^{\mathscr{D}} + \Omega_\Lambda^{\mathscr{D}} + \Omega_{\mathscr{R}}^{\mathscr{D}} + \Omega_{\mathscr{Q}}^{\mathscr{D}} = 1. \tag{20}$$

The formalism can be extended [*Larena*, 2009] to arbitrary coordinate system. In addition to the fluid 4-velocity u^μ, there is another velocity n^μ of the observer. In the Buchert equations there are together with the kinematic term $\mathscr{Q}_{\mathscr{D}}$ (and the dynamic term if the fluid has nonzero pressure) other corrections which complicate the resulting equations.

It is still not clear how big the correction to the Friedmann equations are. They are some claims that they are negligible [*Ishibashi*, 2006], however there some models which are able to explain observed acceleration of the universe [*Wiltshire*, 2007]. More references can be found e.g. in *Ellis* [2011]. For the scale issue see for example *Li and Schwarz* [2008].

5. Ricci flow

In the last section it was shown how to average scalars on an inhomogeneous manifold. However, cosmological data are most often interpreted in the FRW spacetime. In addition to the averaging (9), there should also be some procedure how to smooth geometry itself. The theory of Macroscopic Gravity uses averaging of the Cartan structure equations. There exists mathematically interesting alternative how to reach 3-spaces of constant curvature. Let g_{ab} be a given metric on the closed 3-manifold without boundary, which depends on the parameter β (typically cosmic time) and let it evolve in the direction of the Ricci tensor

$$\begin{aligned} \frac{\partial}{\partial\beta} g_{ab}(\beta) &= -2R_{ab}(\beta), \quad 0 \le \beta \le T_0 \\ g_{ab}(\beta = 0) &= g_{ab}. \end{aligned} \tag{21}$$

It can be shown that on the compact manifold for the sufficiently small β local solution exist and if the initial metric has a positive Ricci curvature, solution exists for all β converging exponentially to the space of the constant curvature [technical details and other references can be found in *Buchert and Carfora*, 2002].

By this procedure also the other parameters will change—the average value of the density will change after smoothing $\mathscr{D}_0$ to $\overline{\mathscr{D}}$ as $\langle\rho\rangle_{\overline{\mathscr{D}}} = M_{\overline{\mathscr{D}}}/V_{\overline{\mathscr{D}}}$. Similarly we would obtain a new set of the normalized omega factors, which can be very different from the original ones.

6. Averaging using scalar curvature invariants

If we can average scalars it is natural to ask how we can represent spacetime by scalar quantities. In *Coley et al.* [2009] it was proven that the class of four-dimensional Lorentzian manifolds that cannot be completely characterized by the scalar polynomial curvature invariants constructed from the Riemann tensor and its covariant derivatives must be of Kundt form (e.g. admitting geodetic null vector with a null expansion, rotation and shear).

For a given spacetime $(\mathcal{M}, g_{\alpha\beta})$ we define the set of scalar invariants [*Coley*, 2010]

$$\mathcal{I} \equiv \left\{R, R_{\mu\nu}R^{\mu\nu}, C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta}, R_{\mu\nu\alpha\beta;\gamma}R^{\mu\nu\alpha\beta;\gamma}, R_{\mu\nu\alpha\beta;\gamma\delta}R^{\mu\nu\alpha\beta;\gamma\delta}, ...\right\}. \tag{22}$$

Integrating over the domain $\mathcal{D}$ we obtain another set $\bar{\mathcal{I}}$ characterizing a smoother geometry. As we can see from relations like $\overline{R_{\mu\nu}R^{\mu\nu}} \neq \bar{R}_{\mu\nu}\bar{R}^{\mu\nu}$ it is possible that there does not exist any metric tensor $\bar{g}_{\mu\nu}$ which would be constructed from the set $\bar{\mathcal{I}}$. To overcome this difficulty we will first remove the scalars which are not algebraically independent. It means that we will restrict our discussion to the subset $\mathcal{I}_A \subseteq \mathcal{I}$. Then we will omit any scalars that can be computed from the equations ("syzygies") characterizing particular spacetimes (e.g. defining the algebraic type of the spacetime, like the Segre type or the Petrov type). We will obtain the new set $\mathcal{I}_{SA} \subseteq \mathcal{I}_A$ and by averaging we will get $\bar{\mathcal{I}}_{SA}$. By the inverse procedure we will acquire a complete set $\bar{\mathcal{I}}$ (here we suppose that averaging will not change the form of the equations which allowed the construction $\mathcal{I}_{SA} \subseteq \mathcal{I}_A$).

7. Averaging Cartan scalars

In the last section geometry was characterized by the curvature scalars. This procedure works well only in four dimensions and it is rather difficult to obtain the metric or the Ricci tensor from the averaged scalars. It can be shown [*Cartan*, 1946] that the geometry may be completely characterized by the Riemann tensor and the finite number of its covariant derivatives (Cartan scalars). Because the Einstein tensor consists of the sum of the Riemann tensor, it is possible to average geometry and the Einstein equations together.

We will start with the construction of the Cartan scalars (for the texts concerning equivalence problem [see, e.g., *Karlhede*, 1980, 2006]. Let $(\mathcal{M}, g)$ be n-dimensional differentiable manifold with a metric

$$\mathbf{g} = \eta_{ij}\boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^j, \tag{23}$$

η_{ij} is constant symmetric matrix and $\boldsymbol{\omega}^i$, i=1,2,...,n form the base of the cotangent space at the point x^μ. The tetrad (frame) is defined up to generalized rotations

$$\boldsymbol{\omega}^i = \omega^i_\nu(x^\mu, \xi^\Upsilon)\mathbf{d}x^\nu, \tag{24}$$

ξ^Υ, $\Upsilon=1,...,\frac{1}{2}n(n-1)$ denotes the coordinates of the orthogonal group. In Macroscopic Gravity, theory uses (bilocally extended) Cartan equation. Now all the geometrical objects will be defined on the enlarged $\frac{1}{2}n(n+1)$ dimensional space $F(\mathscr{M})$—the frame bundle of $\mathscr{M}$. The exterior derivative will be extended to $d = d_x + d_\xi$ and the Cartan equations have the form

$$\boldsymbol{d}\omega^i = \omega^j \wedge \omega^i{}_j, \tag{25}$$

$$\boldsymbol{d}\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j + \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l. \tag{26}$$

with the condition

$$\eta_{ik}\omega^k{}_j + \eta_{jk}\omega^k{}_i = 0. \tag{27}$$

Applying next the exterior derivative we will obtain covariant derivatives of the curvature tensor.

$$\begin{array}{rcl} \mathbf{d}R_{ijkl} & = & R_{mjkl}\omega^m_i + R_{imkl}\omega^m_j + R_{ijml}\omega^m_k + R_{ijkm}\omega^m_l + R_{ijkl;m}\omega^m, \\ \mathbf{d}R_{ijkl;n} & = & R_{mjkl;n}\omega^m_i + R_{imkl;n}\omega^m_j + \cdots + R_{ijkl;nm}\omega^m, \\ & . & \\ & . & \\ & . & \end{array} \tag{28}$$

Let R^p denote the set $\left\{R_{ijkm}, R_{ijkm;n_1}, ..., R_{ijkm;n_1...n_p}\right\}$, p is the lowest number such that R^{p+1} contains no element that is functionally independent (over $F(\mathscr{M})$) of the elements in R^p (two functions f, g are functionally independent if the 1-forms $\mathbf{d}f$ and $\mathbf{d}g$ are linearly independent).

There exist a quite elaborate algorithm [*Karlhede*, 2006] how to compute Cartan scalars. It uses the structure of isotropy group of R^q and in every step it restrict the frame requiring that R^q takes a standard form.

Now we can use the same algorithm as in the previous section. It can be shown that the Cartan scalars satisfy some algebraic and differential relations which are in general nonlinear. It means that we have to restrict to the smaller set of the scalars, perform averaging and then construct a new set $\overline{R}^{p+1}$, from which it is possible to construct a new metric $\overline{g}_{\mu\nu}$.

8. Conclusion

The averaging problem in GR and especially in cosmology is of the fundamental importance. Backreaction term in the averaged Einstein equations will change the dynamic of the metric and affect the cosmological evolution. The question is how important these corrections are and when it is possible to neglect them [*Buchert*, 2008, *Ellis*, 2011].

There is also a problem how to define average of tensors. In this review we have introduced several different candidates how to average Einstein field equations and also spacetime geometry. Macroscopic Gravity is the promising model how to average inhomogeneities, but only a few simplified solutions are known because of the complexity of the equations. The most popular approach to the averaging problem are the Buchert equations. However, only scalar part of the equations are averaged so we have less equation then variables and we have to put some relation by hand.

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