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Note on the Stability of the Dirichlet Problem and the Poisson Equation

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Arendt and Daners (2008) characterized stability of regular open sets in terms of solutions of Poisson equation in the sense of distributions. We shall prove a different version of their theorem with relaxed conditions.

1. Introduction

If A is a subset of $\mathbb{R}^N$ then we denote by $\overline{A}$ the closure of A and by A° the interior of A. Let $C(K)$ denote the space of all continuous functions on a compact set K in $\mathbb{R}^N$. Let U be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, and let $\mathbf{H}(U)$ denote the family of all functions in $C(\overline{U})$ which are harmonic on U. We say that U is *regular* if for every function $f \in C(\partial U)$ there exists a *classical solution* for the Dirichlet problem, that is, a function $h_f^U \in \mathbf{H}(U)$ such that $h_f^U = f$ on ∂U. More generally, we denote h_f^U the so-called *Perron-Wiener-Brelot* (PWB) solution for the Dirichlet problem which can be assigned to each $f \in C(\partial U)$ even if the set U is not regular. If the classical solution exists then the PWB solution h_f^U coincides with the classical one. If it is clear from the context which set U we mean then we will write simply h_f instead of h_f^U.

We shall need some facts about distributions. We denote by $D(U)$ the space of all functions which are infinitely differentiable on U and have a compact support in U. Each member of its dual space $D(U)'$ is called a *distribution*. Let us recall that

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every locally integrable function f and Radon measure μ can be identified with a corresponding distribution Λ_f, resp. Λ_μ in a well-known manner and each distribution is infinitely differentiable (in a certain sense). We shall usually write just f or μ instead of Λ_f or Λ_μ if no confusion can arise.

We introduce two notions of stability of the set U.

Definition 1.1 By $\mathbf{H}(\overline{U})$ we denote the family of all functions on $\overline{U}$ which can be harmonically extended to some neighbourhood of $\overline{U}$. We say that the set U is *H-stable* if $\mathbf{H}(U) = \overline{\mathbf{H}(\overline{U})}$ where the closure is taken under the usual supremum norm.

Definition 1.2 By $H^1(U) = W^{1,2}(U)$ we denote the Sobolev space of all locally integrable functions on U which together with their first distributional derivatives belong to $L^2(U)$. The space $H_0^1(U)$ is defined as the closure of $D(U)$ in $H^1(U)$ (under the usual norm) and we define yet another space

$$H_0^1(\overline{U}) = \{f|_U : f \in H^1(\mathbb{R}^N),\ u(x) = 0 \text{ almost everywhere on } \mathbb{R}^N \setminus U\}$$

We call the set U *S-stable* if $H_0^1(U) = H_0^1(\overline{U})$.

We shall need some more definitions before proceeding further. We denote by $C_0(U)$ the space of all continuous functions on U such that for each $\varepsilon > 0$ there exists a compact subset K of U such that $|f| < \varepsilon$ on $U \setminus K$. If Ω is an open subset of $\mathbb{R}^N$, we denote by $C^\infty(\Omega)$ the space of all functions which are infinitely differentiable on Ω. Finally, for $A \subset \mathbb{R}^N$ arbitrary, we denote by $C^\infty(A)$ the space of all functions which can be extended to some open neighbourhood W of A and this extension lies in $C^\infty(W)$.

Definition 1.3 Let U be a bounded open subset of $\mathbb{R}^N$. We say that $\{U_n\}$ is an *upper sequence with limit* $\overline{U}$ if, for each $n \in \mathbb{N}$,

$$U_n \text{ is a bounded open subset of } \mathbb{R}^N,\ U_{n+1} \subset U_n \text{ and } \bigcap_{n\in\mathbb{N}} U_n = \overline{U}.$$

We say that $\{U_n\}$ is a *regular upper sequence with limit* $\overline{U}$ if $\{U_n\}$ is an upper sequence with limit $\overline{U}$ and each U_n is a regular set. We note that U itself need not to be regular (in both cases).

Arendt and Daners [1] proved the following theorem:

Theorem 1.4 *Let U be a bounded open subset of $\mathbb{R}^N$ and $\{U_n\}$ be an upper sequence with limit $\overline{U}$. Consider the following statements:*

(a) *U is S-stable.*

(b) *If $\Phi \in C(\overline{U}_1)$, $h_n \in \mathbf{H}(U_n)$ and $h_n|_{\partial U_n} = \Phi|_{\partial U_n}$ then h_n converges to the PWB solution h_φ for the Dirichlet problem on U with a boundary condition $\varphi = \Phi|_{\partial U}$ locally uniformly on U.*

(c) *If $\Phi \in C(\overline{U}_1)$, $h_n \in \mathbf{H}(U_n)$, $h_n|_{\partial U_n} = \Phi|_{\partial U_n}$ and $h \in \mathbf{H}(U)$, $h|_{\partial U} = \Phi|_{\partial U}$ then $h_n \to h$ uniformly on $\overline{U}$.*

(d) *U is H-stable.*

(e) *If $f \in L^p(U_1)$ for some $p > N/2$, $u_n \in C_0(U_n)$ such that $-\triangle u_n = f$ in $D(U_n)'$ and $u \in C_0(U)$ such that $-\triangle u = f$ in $D(U)'$ then $u_n \to u$ uniformly on $\overline{U}$.*

Then $(a) \Leftrightarrow (b) \Rightarrow (c) \Leftrightarrow (d) \Rightarrow (e)$. *If, moreover, U is regular then all of the statements are equivalent.*

As noted in the article of Arendt and Daners [1], Keldyš [7] called the set U *stable* if the statement (b) is true (for each choice of $\{U_n\}$ and Φ satisfying the assumptions). Arendt and Daners also mentioned the paper of Hedberg [5] where he showed the equivalence of (b) and (d) for a special case of so called *topologically regular* set U, that is, when U satisfies a condition $(\overline{U})^\circ = U$. In fact, this is a classical result and can be find in various books and articles concerning potential theory and harmonic approximation. Apart from the Keldyš paper [7], let us mention at least the article of Deny [3] and books of Landkof [8], Gardiner [4] and Armitage and Gardiner [2] where the proof of this statement can be found in various alternatives.[1] However, the case of a general bounded open set seems to be usually omitted in the literature. The only actually written proof known to the author is in the article of Vincent-Smith (1969) but the proof there deals with axiomatic potential theory.

Our plan for the paper is this. In Section 2, we summarize necessary definitions and facts about potential theoretic notions needed later. In Section 3, we show that $(b) \Leftrightarrow (d)$ for *all* bounded open subsets of $\mathbb{R}^N$. As already mentioned, this is not a new result, but our proof deals only with classical potential theory and is simpler than that of Vincent-Smith (1969). We shall use it then in Section 4 to relax conditions of the statement (e) of the previous theorem in the case of regular sets. Namely, we will show that it is enough to consider only smooth functions and classical solutions instead of larger set of functions in Lebesgue spaces and distributional solutions. Section 5 concludes this article with some remarks and comments on the case of open sets which are not regular.

2. Preliminaries

Throughout the rest of the article, we assume that $N \geq 2$. The so-called fine topology on $\mathbb{R}^N$ is the coarsest topology which makes every superharmonic function on $\mathbb{R}^N$ continuous. Topological notions such as open sets etc. with respect to the fine topology will be shortly called finely open sets etc.

Let Ω be an open subset of $\mathbb{R}^N$ which is assumed to be bounded if $N = 2$. If μ is a nonnegative measure on Ω then we define

$$G_\Omega\mu(x) = \int_\Omega G_\Omega(x, y)\, d\mu(y).$$

[1] The last one mentioned, Theorem 7.9.5 of [2], shall be used in this article.

Here, $G_\Omega : \Omega \times \Omega \to [0, +\infty]$ is called the *Green function* of Ω and is defined by

$$G_\Omega(x, y) = U_y(x) - h_y(x)$$

where $U_y(x)$ is the fundamental solution of the Laplace equation, namely

$$U_y(x) = \begin{cases} -\log\|x-y\| & x \neq y,\ N = 2 \\ \|x-y\|^{2-N} & x \neq y,\ N > 2 \\ +\infty & x = y \end{cases}$$

and h_y is the greatest harmonic minorant of U_y on Ω. For our choice of Ω, such a minorant always exists. See, for example, Definition 4.1.1 and Theorem 4.1.2 in [2].

We denote $\mathscr{U}_+(\Omega)$ the collection of all nonnegative superharmonic functions on Ω. For $u \in \mathscr{U}_+(\Omega)$ and $E \subset \Omega$, we define the *reduced function* of u relative to E in Ω by

$$R_u^E(x) = \inf\{v(x) : v \in \mathscr{U}_+(\Omega) \text{ and } v \geq u \text{ on } E\}, \qquad x \in \Omega$$

and the *balayage* of u relative to E in Ω by

$$\widehat{R}_u^E(x) = \liminf_{y \to x} R_u^E(y), \qquad x \in \Omega.$$

Next we introduce the notion of thinness. We say that the set $A \subset \mathbb{R}^N$ is *thin at the point* $x \in \mathbb{R}^N$ if there is a finely open set containing x which does not intersect $A \setminus \{x\}$. Obviously, if $B \supset A$ and A is not thin at x then B is not thin at x too. Vice versa, if $A \subset B$ and B is thin at x then A is thin at x as well. We remark that an arbitrary set is obviously thin at any point in the interior of its complement.[2]

A set E in $\mathbb{R}^N$ is called *polar* if there exists a superharmonic function u on some open set $\omega \supset E$ such that

$$E \subset \{x \in \omega : u(x) = +\infty\}.$$

Proposition 2.1 *A set in $\mathbb{R}^N$ is polar if and only if it is thin at each of its points.*

Proof. See Theorem 7.3.7 of [2]. □

Proposition 2.2 *Let E be a relatively closed polar subset of Ω. If h is harmonic on $\Omega \setminus E$ and for each $x \in E$ there exists a neighbourhood V of x such that h is bounded on $V \setminus E$ then h has a unique harmonic extension to Ω.*

Proof. See Corollary 5.2.3 of [2]. □

Proposition 2.3 *Let K be a compact subset of $\mathbb{R}^N$. Then the following are equivalent:*

(i) *For each u continuous on K and harmonic on K° and each $\varepsilon > 0$, there exists a function v harmonic on some open neighbourhood of K such that $|v - u| < \varepsilon$ on K.*

(ii) *The sets $\mathbb{R}^N \setminus K$ and $\mathbb{R}^N \setminus K^\circ$ are thin at the same points.*

Proof. See Theorem 7.9.5 of [2]. □

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[2] In fact, it is thin in the fine interior of its complement. Since the fine topology is (strictly) finer than the Euclidean topology, the fine interior of the set always contains its interior (with respect to Euclidean topology).

3. Equivalences of Different Versions of Stability

The aim of this section is to prove the equivalence of (a) and (d) in Theorem 1.4 for a general bounded open subset U of $\mathbb{R}^N$. We shall do it in two steps.

Proposition 3.1 *Let U be a bounded open subset of $\mathbb{R}^N$. Then the following statements are equivalent:*
(i) U is H-stable.
(ii) The sets $\mathbb{R}^N \setminus U$ and $\mathbb{R}^N \setminus \overline{U}$ are thin at the same points.

Let us note that (for now) we shall need only the implication $(i) \implies (ii)$.

Proof. If $(\overline{U})^\circ = U$ then the assertion is a direct consequence of Proposition 2.3, simply put $K = \overline{U}$.

Now, let us assume that U is arbitrary and (ii) is true. Obviously, the set $\mathbb{R}^N \setminus \overline{U}$ is thin at any point of $\overline{U}^\circ$. Hence, we see from the thinness condition that $\mathbb{R}^N \setminus U$ is thin at any point of $\overline{U}^\circ$. Therefore, the set $(\overline{U})^\circ \setminus U$ (as a subset of $\mathbb{R}^N \setminus U$) is thin at each of its points and. Thus it is a polar set in view of Proposition 2.1.

It is obvious that $(\overline{U})^\circ \setminus U$ is a relatively closed subset of $(\overline{U})^\circ$. So if we have a function $f \in \mathbf{H}(U)$ then the Proposition 2.2 ensures that it is harmonic on $(\overline{U})^\circ$ and therefore is in $\mathbf{H}((\overline{U})^\circ)$. So it is enough to prove the equality of $\mathbf{H}(\overline{U})$ and $\overline{\mathbf{H}((\overline{U})^\circ)}$. Once again, we can use Proposition 2.3 since the assumption (ii) and the inclusions

$$\left(\mathbb{R}^N \setminus \overline{U}\right) \subset \left(\mathbb{R}^N \setminus \overline{U}^\circ\right) \subset \left(\mathbb{R}^N \setminus U\right)$$

immediately yield that the sets $\mathbb{R}^N \setminus (\overline{U})^\circ$ and $\mathbb{R}^N \setminus \overline{U}$ are thin at the same points.

If U is not topologically regular and (i) is true then we know from Proposition 2.3 that $\mathbb{R}^N \setminus \overline{U}$ and $\mathbb{R}^N \setminus \overline{U}^\circ$ are thin at the same points since

$$\overline{\mathbf{H}(\overline{U})} \subset \mathbf{H}(\overline{U}^\circ) \subset \mathbf{H}(U) = \overline{\mathbf{H}(\overline{U})}.$$

The inclusions are obvious and the last equality is assumed. Hence, $\mathbf{H}(\overline{U}^\circ) = \overline{\mathbf{H}(\overline{U})}$. If (ii) is not true then $\overline{U}^\circ \setminus U$ must not be thin at some point.

Hence $\overline{U}^\circ \setminus U$ is not polar. Let us choose an open ball $\Omega \subset \mathbb{R}^N$ such that $U \subset \Omega$. By the proof of Theorem 5.3.7 of [2], there exists a bounded continuous potential $G_\Omega\mu$ on Ω where the measure μ is nonnegative, $\mu \not\equiv 0$ and its support $\operatorname{spt}\mu$ is a compact subset K of $\overline{U}^\circ \setminus U$. Hence $G_\Omega\mu$ is harmonic on $\Omega \setminus K$ and by Theorem 5.7.4 of [2] it is not a constant function and attains its strict maximum at some point of K. Hence $G_\Omega\mu \notin \overline{\mathbf{H}(\overline{U})}$ because (due to the maximum principle) any function in $\overline{\mathbf{H}(\overline{U})}$ cannot attain its strict maximum in $\overline{U}^\circ$. □

Let us note that the following proposition remains true and the proof almost unchanged if h_n are considered to be PWB solutions instead of classical solutions.

Proposition 3.2 *Let U be a bounded open subset of $\mathbb{R}^N$ and $\{U_n\}$ be an upper sequence with limit $\overline{U}$. Let $\mathbb{R}^N \setminus U$ and $\mathbb{R}^N \setminus \overline{U}$ be thin at the same points. Further,* *let $f \in C(\mathbb{R}^N)$, h_n be the classical solution of the Dirichlet problem on U_n with the boundary condition $f|_{\partial U_n}$ and h be the PWB solution of the Dirichlet problem on U with the boundary condition $f|_{\partial U}$. Then*

$$h_n(x) \to h(x) \qquad \textit{for all } x \in U.$$

Our proof follows closely the proof of similar Lemma 7.9.4 of [2].

Proof. Let us choose a bounded open set $\Omega \in \mathbb{R}^N$, such that $\Omega \supset \overline{U}_1$. Let $\varepsilon > 0$. Due to the Stone-Weierstrass theorem, there exist $u_1 \in \mathscr{U}_+(\Omega)$ and $u_2 \in \mathscr{U}_+(\Omega)$ such that (see also Lemma 7.9.1 of [2])

$$|f - (u_1 - u_2)| < \varepsilon/3 \quad \text{ on } \overline{U}_1.$$

In view of the maximum principle for harmonic functions we have

$$|h_f^{U_n} - (h_{u_1}^{U_n} - h_{u_2}^{U_n})| < \varepsilon \quad \text{ on } U_n.$$

Since PWB solutions of the Dirichlet problem on U are harmonic on U, then, if we apply the maximum principle again for each $x \in U$ on a suitable closed ball with the center x in U, we get

$$|h_f^{U} - (h_{u_1}^{U} - h_{u_2}^{U})| < \varepsilon \quad \text{ on } U.$$

Using a simple triangle inequality, we get

$$|h_f^{U_n} - h_f^U| \le |h_f^{U_n} - h_{u_1}^{U_n} + h_{u_2}^{U_n}| + |h_{u_1}^{U_n} - h_{u_1}^U| + |h_{u_2}^{U_n} - h_{u_2}^U| + |h_f^U - h_{u_1}^U + h_{u_2}^U| \le$$

$$\le 2\varepsilon + |h_{u_1}^{U_n} - h_{u_1}^U| + |h_{u_2}^{U_n} - h_{u_2}^U| \quad \text{ on } U$$

so it is sufficient to show that

$$h_{u_i}^{U_n}(x) \to h_{u_i}^U(x) \qquad \text{where } n \to \infty,\ x \in U,\ i = 1, 2.$$

We refer to Theorem 6.9.1. and Theorem 5.7.3. of [2] to see that

$$h_{u_i}^{U_n}(x) = \widehat{R}_{u_i}^{\Omega\setminus U_n}(x) \quad \text{ for each } x \in U_n, \qquad h_{u_i}^{U}(x) = \widehat{R}_{u_i}^{\Omega\setminus U}(x) \quad \text{for each } x \in U$$

and, by virtue of Theorem 5.7.3(iv) of [2], we have

$$h_{u_i}^{U_n}(x) = \widehat{R}_{u_i}^{\Omega\setminus U_n}(x) \to \widehat{R}_{u_i}^{\Omega\setminus \overline{U}}(x).$$

Hence, it is sufficient to show that

$$\widehat{R}_{u_i}^{\Omega\setminus \overline{U}} = \widehat{R}_{u_i}^{\Omega\setminus U} \qquad \text{on } U, \text{ for } i = 1, 2. \tag{$*$}$$

Let now $i = 1, 2$ be fixed. If $v \in \mathscr{U}_+(\Omega)$ and $v \ge u_i$ on $\Omega \setminus \overline{U}$ then, by fine continuity of superharmonic functions, $v \ge u_i$ on those points of $\partial\overline{U}$ where $\Omega \setminus \overline{U}$ is not thin (every fine neighbourhood of such a point intersects $\Omega \setminus \overline{U}$). Due to the assumption on thinness, we have that $v \ge u_i$ on those points of $\partial\overline{U}$ where $\Omega \setminus \overline{U}^\circ$ is not thin. Since the set of points in $\partial\overline{U}$ where $\Omega \setminus \overline{U}^\circ$ is thin forms a set which is thin at any of its points and thus is polar and the same is true for the set $\overline{U}^\circ \setminus U$, the equality $(*)$ follows from Theorem 5.7.3(ii) of [2]. □

Remark 3.3 We observe that there exists a sequence of regular open sets U_n satisfying the assumption in the previous lemma. It is, for example, a simple consequence of the following fact. If $K \subset V \subset \mathbb{R}^N$, K is compact and V is open then there exists a regular open set W such that $K \subset W \subset V$. See M. Hervé [6], Proposition 7.1.

Remark 3.4 If the sequence $\{h_n\}$ is bounded and $h_n \to h$ pointwise on U then by Harnack inequalities the convergence is locally uniform on U. See, for example, Theorem 1.5.8 of [2].

Now, we can state our main result of this section.

Theorem 3.5 *Let U be a bounded open subset of $\mathbb{R}^N$. Then the following statements are equivalent:*
(i) *U is S-stable.*
(ii) *U is H-stable.*

Proof. $(i) \implies (ii)$: This is the part $(a) \implies (d)$ in Theorem 1.4.
$(ii) \implies (i)$: If U is H-stable then, by Proposition 3.1, $\mathbb{R}^N \setminus U$ and $\mathbb{R}^N \setminus \overline{U}$ are thin at the same points. Obviously, the truth of the statement (b) of Theorem 1.4 now follows directly from Proposition 3.2 and Remark 3.4. □

4. Stability via Poisson Equation and Infinitely Smooth Functions

Let us recall that we denote by $C^\infty(A)$ the family of all functions $f : A \to \mathbb{R}$ which can be extended to a C^∞ function on some open set containing A.

Theorem 4.1 *Let U be a bounded open regular subset of $\mathbb{R}^N$ and $\{U_n\}$ be a regular upper sequence with limit $\overline{U}$. Then the following statements are equivalent:*
(i) U is S-stable.
(ii) If $f \in C^\infty(\overline{U}_1)$, $u_n \in C_0(U_n)$ such that $-\triangle u_n = f$ and $u \in C_0(U)$ such that $-\triangle u = f$ then $u_n \to u$ uniformly on $\overline{U}$.

Proof. We shall prove that $(i) \implies (ii)$. It can be done by discussing the regularity of distributional solutions in Theorem 1.4 or by a simple argument that follows. Without any loss of generality we may assume that $f \in C^\infty(\mathbb{R}^N)$. We put

$$\Phi(x) = \int_{\mathbb{R}^N} U_x(y)\, f(y)\, \mathrm{d}y, \qquad x \in \mathbb{R}^N.$$

We get by Corollary 4.5.4 of [2] that $\Phi \in C^\infty(\mathbb{R}^N)$ and, by Corollary 4.5.5 of [2], we have

$\triangle\Phi = -c_N f$ where c_N is a constant dependent only on the dimension N.

We put

$$h_n = \frac{1}{c_N}\Phi - u_n \qquad \text{and} \qquad h = \frac{1}{c_N}\Phi - u.$$

It follows that $h_n \in \mathbf{H}(U_n)$ and $h \in \mathbf{H}(U)$ and it is enough to show that $h_n \to h$ uniformly on $\overline{U}$. But this is true in view of Theorem 1.4, part $(a) \implies (c)$.

We shall now prove that $(ii) \implies (i)$ if U is regular. In view of Corollary 3.5, it is enough to show that U is H-stable, that is,

$$\mathbf{H}(U) = \overline{\mathbf{H}(\overline{U})}.$$

Let $h \in \mathbf{H}(U)$ and choose $\varepsilon > 0$. Since U is regular, there exists $g \in \mathbf{H}(U)$ such that $g|_{\partial U}$ has an extension

$$\Phi_g \in C^\infty(\overline{U}_1) \qquad \text{with} \qquad |g - h| < \varepsilon/2 \ \text{ on } \overline{U}.$$

We denote h_n the solutions of the Dirichlet problem on U_n with the boundary condition $\Phi_g|_{\partial U_n}$. Then

$$u_n = h_n - \Phi_g \in C_0(U_n) \qquad \text{and} \qquad -\triangle u_n = \triangle \Phi_g$$

and

$$u = g - \Phi_g \in C_0(U) \qquad \text{and} - \triangle u = \triangle \Phi_g.$$

If we put

$$f = \triangle \Phi_g \in C^\infty(\overline{U}_1)$$

then (b) implies that $u_n \to u$ uniformly on $\overline{U}$ and hence

$$h_n \to g \qquad \text{uniformly on } \overline{U}.$$

For some $n \in \mathbb{N}$, we have

$$h_n|_{\overline{U}} \in \mathbf{H}(\overline{U}) \quad \text{such that} \quad |h_n - g| < \varepsilon/2 \ \text{ on } \overline{U},$$

therefore

$$|h_n - h| < \varepsilon \qquad \text{on } \overline{U}.$$

Hence, $h \in \overline{\mathbf{H}(\overline{U})}$ and the proof is complete. □

Perhaps it is appropriate to note that instead of functions with $\mathscr{C}^\infty$ extension, we could use any dense subset of $C(\partial U)$ which contains functions extendable to $\mathbb{R}^N$ with the extension at least twice continuously differentiable. This motivates the first remark in the following section.

5. Remarks and Questions

5.1 Approximation Property

Let U be a bounded open subset of $\mathbb{R}^N$. We say that U has a C^2-*approximation property* if for every $h \in \mathbf{H}(U)$ and $\varepsilon > 0$ there exists $g \in \mathbf{H}(U)$ such that $|g - h| < \varepsilon$ on $\overline{U}$ and if there exists a neighborhood V of $\overline{U}$ and $\Phi \in C^2(V)$ such that $\Phi = g$ on $\overline{U}$.

Remark 5.1 Let U be a bounded open subset of $\mathbb{R}^N$, not necessarily regular, and let $\{U_n\}$ be a regular upper sequence with limit $\overline{U}$. Then the following statements are equivalent:

(i) U is H-stable.

(ii) The set U has a C^2-approximation property and if $f \in C^2(\overline{U}_1)$, $u_n \in C_0(U_n)$ such that $-\triangle u_n = f$ and $u \in C_0(U)$ such that $-\triangle u = f$ then $u_n \to u$ uniformly on $\overline{U}$.

Proof. The proof $(ii) \implies (i)$ is nearly the same as the one above in Theorem 4.1. But if (i) is true then it is obvious that the set U has the approximation property since for an arbitrary $h \in \mathbf{H}(U)$ and $\varepsilon > 0$ there exists a function $g \in \mathbf{H}(\overline{U})$ such that $|g - h| < \varepsilon$ on $\overline{U}$. □

Remark 5.2 Whether every bounded open subset of $\mathbb{R}^N$ has the C^2-approximation property seems to be an open question. The regular one has it and this was an essential tool in proving Theorem 4.1. But neither regularity nor stability is a necessary condition for U having the C^2-approximation property. For example, if U is the Lebesgue spine then U is a stable set (for harmonic functions) and thus it has C^2-approximation property yet it is not regular. On the other hand, let U be an open unit ball and choose some compact set $K \subset U$, such that K is not thin at any of its point and has an empty interior. Then $V = U \setminus K$ is regular but not stable (in view of Proposition 3.1).

Question 5.3 Let U be a bounded open subset of $\mathbb{R}^N$ which has the C^2-approximation property. Does U has to be either regular or stable?

5.2 Potential Condition of Stability

We recall that Ω is an open subset of $\mathbb{R}^N$, assumed to be bounded if $N = 2$. If μ is a nonnegative measure on Ω then by Definition 4.3.1, Corollary 4.3.3 and Theorem 4.3.8(i) in [2] we have

$$-\Delta G_\Omega \mu = c_N \mu \qquad \text{(in the sense of distributions).} \tag{5.1}$$

The constant c_N has the same meaning as it had in the proof of Theorem 4.1.

Let $U \subset \Omega$ be a bounded open subset of $\mathbb{R}^N$ and let us once again return to the condition of H-stability, that is,

$$\mathbf{H}(U) = \overline{\mathbf{H}(\overline{U})}.$$

We define

$$P = \{G_\Omega \mu : \mu \textit{ is a nonnegative measure on } \Omega \textit{ and } G_\Omega \mu \in \mathbf{H}(U)\}.$$

In other words, the set P is the set of all continuous (Green) potentials on Ω which are harmonic on U. We proved in Proposition 3.1 a thinness condition for H-stability. This proposition can be slightly refined in the following way:

Proposition 5.4 *Let U be a bounded open subset of Ω. Then the following statements are equivalent:*

(i) U is H-stable.

(ii) $P \subset \overline{\mathbf{H}(\overline{U})}$.

(iii) The sets $\mathbb{R}^N \setminus U$ and $\mathbb{R}^N \setminus \overline{U}$ are thin at the same points.

Proof. If U is topologically regular then we know that $(i) \implies (ii)$ and $(iii) \implies (i)$. For the remaining implication $(ii) \implies (iii)$, we can use word by word the proof of part $(a) \implies (b)$ in Theorem 7.9.5 in [2].

If U is not topologically regular, it also is enough to proof $(ii) \implies (iii)$. From the topologically regular case, it follows that $\mathbb{R}^N \setminus \overline{U}$ and $\mathbb{R}^N \setminus \overline{U}^{\circ}$ has to be thin at the same points. Hence, if (iii) is not true then $\overline{U}^{\circ} \setminus U$ has not to be thin at some point and the rest follows word by word again as in the last section of the proof of Proposition 3.1.[3] □

Now, we can provide another variation of the condition (e) in Theorem 1.4.

Theorem 5.5 *Let U be a bounded open subset of $\mathbb{R}^N$ (not necessarily regular) and let $\{U_n\}$ be a regular upper sequence with limit $\overline{U}$. Let Ω be a bounded open subset of $\mathbb{R}^N$ which contains $\overline{U_1}$. Then the following statements are equivalent:*

(i) U is S-stable.

(ii) If $G_\Omega \mu \in P$, $u_n \in C_0(U_n)$ such that $-\triangle u_n = c_N \mu$ in the sense of distributions and $u \in C_0(U)$ such that $-\triangle u = c_N \mu$ in the sense of distributions, then $u_n \to u$ uniformly on $\overline{U}$.

We note that, for U regular, Theorem 1.4 of Arendt and Daners states that it is sufficient in (ii) to consider (signed) measures which are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^N$ and with density in L^p, $p > N/2$. Our previous generalization of the regular case dealt with (signed) measures with infinitely smooth densities.

We emphasize the fact that we do not need to assume that U is regular in the theorem above.

Proof. We shall prove that $(i) \implies (ii)$. By (5.1), we have

$$\triangle G_\Omega \mu = -c_N \mu, \qquad \text{in the sense of distributions.}$$

We put

$$h_n = \frac{1}{c_N} G_\Omega \mu - u_n \qquad \text{and} \qquad h = \frac{1}{c_N} G_\Omega \mu - u.$$

It follows that $h_n \in \mathbf{H}(U_n)$ and $h \in \mathbf{H}(U)$ and it is enough to show that $h_n \to h$ uniformly on $\overline{U}$. But this is true in view of Theorem 1.4, part $(a) \implies (c)$.

We shall now prove that $(ii) \implies (i)$. In view of Corollary 3.5, it is enough to show that U is H-stable, and, by Proposition 5.4, it is enough to show that each function

[3] This is another reason why we gave an independent proof of Proposition 3.1.

in P lies in $\overline{\mathbf{H}(\overline{U})}$. Let h_n be the classical solution of the Dirichlet problem on U_n with the boundary condition $G_\Omega\mu|_{\partial U_n}$ and h be the classical solution of the Dirichlet problem on U with the boundary condition $G_\Omega\mu|_{\partial U}$ (which is, in fact, the function $G_\Omega\mu$ exactly, due to the assumption that $G_\Omega\mu \in P$). Then we put

$$u_n = h_n - G_\Omega\mu, \qquad u = h - G_\Omega\mu = 0$$

and, by (ii), we see that $u_n \to u$ uniformly on $\overline{U}$. Hence, $h_n \to h = G_\Omega\mu$ uniformly on $\overline{U}$, but $h_n \in \mathbf{H}(\overline{U})$. The proof is complete. □

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