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Two Models of Non-Euclidean Spaces Generated by Associative Algebras

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We present a nontrivial example how to generate non-Euclidean geometries from associative unital algebras. We consider bundles of the sphere of the degenerate non-Euclidean space and its two models. The first (conformal) model is obtained by the mapping S onto a plane passing through the origin. It is analogous to the stereographic mapping. The second model (projective) is constructed by the Norden normalization method, where we project the sphere onto a plane of normalization defining the metric and Christoffel symbols which allow us to find geodesic curves.

1. Introduction

A lot of models of non-Euclidean spaces were studied in the past, especially spaces of a constant curvature, projective spaces and the conformal planes (e.g. [10], [11], [12], [19]). There exists a lot of studies on how these models can be generated by algebras. It is well known that algebras define some structures in bundle manifolds of different types (e.g. [5], [9], [13]). In the literature, we can find many applications of this approach on the cases of non-Eucledian spaces (e.g. [4], [6], [16], [17], [20]).

We would like to present non-standard models within this framework. In the preliminaries we describe how an associative algebra generates a vector space and we also discuss some of its properties. In the next section we define a sphere and the map S in this vector space and we use it to construct a conformal model. In the last section we remind the reader of some facts about the Norden normalization method [7] and we use it for the construction of a projective model.

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Foundations of the theory of finite-dimensional associative algebras were made by E. Cartan (1898), Wedderburn (1908) and F. E. Molin (1983), who described the structure of any algebra over an arbitrary base field [2]. E. Study and E. Cartan in [15] classified all 3 and 4-dimensional unital associative irreducible[1] algebras up to an isomorphism. This classification can be also found in [18]. In this paper we consider only one type of 3-dimensional algebra $\mathfrak{A}$.

2. Preliminaries

Let $\mathfrak{A}$ be an unital associative 3-dimensional algebra and $\{\mathbf{1}, e_1, e_2\}$ be its basis with the identity element $\mathbf{1}$. The multiplication rules are:

$$(e_1)^2 = \mathbf{1}, \quad (e_2)^2 = 0, \quad e_1 e_2 = -e_2 e_1 = e_2. \tag{1}$$

The algebra $\mathfrak{A}$ is the set of upper triangular matrices

$$\begin{pmatrix} x_0 & x_2 \\ 0 & x_1 \end{pmatrix} = x_0 \cdot \mathbf{1} + x_1 \cdot e_1 + x_2 \cdot e_2, \text{ where}$$

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2}$$

are basic elements [2].

The algebra $\mathfrak{A}$ admits the following conjugation

$$x = x_0 + x_1 e_1 + x_2 e_2 \quad \to \quad \overline{x} = x_0 - x_1 e_1 - x_2 e_2$$

with the property $\overline{xy} = \overline{y}\,\overline{x}$.

We consider the bilinear form (x, y) which takes real values and determines a degenerate scalar product:

$$(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x}) = x_0 y_0 - x_1 y_1. \tag{3}$$

It defines the structure of a degenerate pseudo-Euclidean vector space of rank 2 on $\mathfrak{A}$. (It is also possible to call this space "semi-pseudo-Euclidean", but later we will call it just "pseudo-Euclidean".) The set of invertible elements $G = \{x \in \mathfrak{A} \mid \mid (x_0)^2 - (x_1)^2 \neq 0\}$ is a non-Abelian Lie group with the same multiplication rule ([1]). Its underlying manifold is obtained from $\mathbb{R}^3$ by removing two transversal planes, hence it consists of 4 connected components.

The distance is defined as usual, $d(x, y)^2 = (x - y, x - y)$. The geodesic curves $x(t)$ are then

$$x_0 = a_0 t + b_0 \qquad x_1 = a_1 t + b_1 \qquad x_2 = f(t)$$

where $f(t)$ is an arbitrary function of t and a_0, a_1, b_0, b_1 are the numerical coefficients.

In the basis (2) we can find two subalgebras: $R(e_1)$ with basis $\{\mathbf{1}, e_1\}$, it is an algebra of double numbers, and a subalgebra $R(e_2)$ with basis $\{\mathbf{1}, e_2\}$, it is an algebra of dual

[1] *Irreducible* means indecomposable into a direct sum of algebras.

numbers. The set of their invertible elements $H_1 = \{x_0 + x_1 e_1 \in R(e_1) \mid x_0^2 - x_1^2 \neq 0\}$ and $H_2 = \{x_0 + x_2 e_2 \in R(e_2) \mid x_0 \neq 0\}$ are Lie subgroups of the Lie group G.

The space of right cosets $H_1 x$ defines a trivial bundle $(G, \pi, M = G/H_1)$ over the real line $\mathbb{R}$ with the structure group H_1, where π is a canonical projection ([3]). The fiber is a plane without two transversal lines and the structure group is H_1. The manifold of the group G is diffeomorphic to direct sum $\mathbb{R} \times H_1$. The coordinate view of the canonical projection π is:

$$\pi(x) = \frac{x_2}{x_0 - x_1}. \tag{4}$$

The equation of fibers is:

$$u(x_0 - x_1) - x_2 = 0, \ u \in \mathbb{R}. \tag{5}$$

Let us investigate the isometry group of the pseudo-Euclidean space G. We can easily find that it has no dilations and inversions while there is a vertical translation $x \to x + a$, $a \in G$. Furthermore, the isometry group includes rotations, resp. anti-rotations,

$$x' = ax \qquad \text{or} \qquad x' = xa$$

with $|a|^2 = 1$, resp. $|a|^2 = -1$. These elements can be represented as:

$$a = \cosh\varphi \pm \sinh\varphi\, e_1 + u \sinh\varphi\, e_2,$$

$$\text{resp.} \quad a = \sinh\varphi \pm \cosh\varphi e_1 + u \cosh\varphi\, e_2,$$

where $u \in \mathbb{R}$. The anti-rotations map the elements with the positive norms into the elements with the negative norms and visa versa.

The bilinear form (3) in the algebra $\mathfrak{A}$ takes the real values, therefore it is possible to present it as: $(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x}) = \frac{1}{2}(\overline{x}y + \overline{y}x)$. Consequently, in the case of rotations the hyperbolic cosine of an angle between x and x' is equal to

$$\cosh(x, x') = \frac{(x, ax)}{|x||ax|} = \frac{1/2(x\overline{ax} + ax\overline{x})}{|x|^2} =$$

$$\frac{1/2(x\overline{x}\,\overline{a} + ax\overline{x})}{|x|^2} = \frac{1}{2}(\overline{a} + a) = \cosh\varphi, \tag{6}$$

and the same for the right multiplication. Similarly we get $\sinh\varphi$ for anti-rotations. Note that the angle φ does not depend on x.

Isometries

$$x' = axb, \tag{7}$$

where $|a|^2 = \pm 1$, $|b|^2 = \pm 1$, are compositions of rotations and/or anti-rotations $x' = ax$ and $x' = xb$. We see that (7) defines *proper* rotations and anti-rotations.

Similarly,

$$x' = a\overline{x}b \tag{8}$$

are compositions of the reflection $x' = \overline{x}$ and transformations (7). These are *improper* rotations and anti-rotations.

Proposition. *Any proper or improper rotation/anti-rotation of the pseudo-Euclidean space G can be represented by (7) or (8).*

Proof. Rotations and anti-rotations (7), (8) are compositions of odd and even numbers of reflections of planes passing through the origin. There corresponds an orthonormal vector n to each plane. If vectors x_1 and n are collinear, then $\overline{x}_1 n = \overline{n} x_1$ and $x_1' = -n\overline{x}_1 n = -n\overline{n}x_1 = -x_1$. If vectors x_2 and n are orthogonal, then $\overline{x}_2 n + \overline{n} x_2 = 0$ and $x_2' = -n\overline{x}_2 n = n\overline{n}x_2 = x_2$. On the other hand, any vector x can be represented by a sum of vectors x_1 and x_2. It means, that a reflection of the plane is: $x' = -n\overline{x}n$. Therefore, the composition of even, resp. odd number of reflections of planes are isometries (7), resp. (8). □

Corollary. *Only translations, rotations and anti-rotations are isometries of G.*

They all can be written in a known form (for further discussion see e.g. [19])

$$\begin{cases} x_0' = x_0 \cosh\varphi + x_1 \sinh\varphi + a_0 \\ x_1' = x_1 \cosh\varphi + x_0 \sinh\varphi + a_1 \\ x_2' = u_0 x_0 + u_1 x_1 + u_2 x_2 + a_2 \end{cases} \tag{9}$$

where $a = a_i e_i \in G$ and $u_i \in \mathbb{R}$.

Let us introduce adapted coordinates (u, λ, φ) of the bundle in semi-Euclidean space, here u is a basic coordinate, λ, φ are fiber coordinates. If $|x|^2 > 0$, we denote $\lambda = \pm\sqrt{x_0^2 - x_1^2} \neq 0$, the sign of λ is equal to the sign of x_0. The adapted coordinates of the bundle in this case are:

$$x_0 = \lambda \cosh\varphi, \quad x_1 = \lambda \sinh\varphi, \quad x_2 = u\lambda \exp\varphi, \tag{10}$$

where $\lambda \in \mathbb{R}_0, \quad u, \varphi \in \mathbb{R}$.

If $|x|^2 < 0$, then we write $\lambda = \pm\sqrt{x_1^2 - x_0^2}$, the sign of λ is equal to the sign of x_1:

$$x_0 = \lambda \sinh\varphi, \quad x_1 = \lambda \cosh\varphi, \quad x_2 = u\lambda \exp\varphi. \tag{11}$$

The structure group acts as follows:

$$u' = u, \quad \lambda' = \lambda\rho, \quad \varphi' = \varphi + \psi, \tag{12}$$

where the element $a(0, \rho, \psi)$ of the structure group acts on the element $x(u, \lambda, \varphi) \in G$. This group consists of 4 connected components.

3. Conformal model of a sphere

Definition. We call *semi-Euclidean sphere with an unit radius* the set of all elements of algebra $\mathfrak{A}$ whose square is equal to one,

$$S^2(1) = \{x \in \mathfrak{A} \mid x_0^2 - x_1^2 = 1\}.$$

Analogously, the set of elements with an imaginary unit module $|x|^2 = -1$ we call *semi-Euclidean sphere with an imaginary unit radius* $S^2(-1)$.

One of these spheres can be obtained from another by rotation. The isometries (9) are now constrained by additional relation $x_0^2 - x_1^2 = 1$, therefore, only rotations and vertical translations remain, $a_0 = a_1 = 0$.

We consider the subbundle of the bundle $(G, \pi, M = G/H_1)$ of semi-Euclidean sphere $S^2(1)$, i.e. the bundle $\pi : S^2(1) \to M$. The fibers of the new bundle are intersections of $S^2(1)$ and planes (5). The restriction of the group of double numbers H_1 to $S^2(1)$ is a Lie subgroup S_1 of double numbers with an unit module

$$S_1 = \{a_0 + a_1 e_1 \in H_1 \mid a_0^2 - a_1^2 = 1\}.$$

This group consists of two connected components. The bundle $(S^2(1), \pi, M)$ is a trivial bundle of the group $S^2(1)$ by the Lie subgroup S_1 to right cosets.

We define coordinates adapted to the bundle on semi-Euclidean sphere $S^2(1)$. If $x \in S^2(1)$ then from (10) we get $\lambda = \varepsilon$, $\varepsilon = \pm 1$. The parametric equation of semi-Euclidean sphere in the adapted coordinates (u, φ) is:

$$\mathbf{r}(u, \varphi) = \varepsilon(\cosh\varphi, \sinh\varphi, u \exp\varphi), \tag{13}$$

where u is a basis coordinate, φ is a fiber coordinate. Different values of ε correspond to different connected components of semi-Euclidean sphere $S^2(1)$.

Let us define the action of the structure group S_1 on semi-Euclidean sphere. From (12) and using the adapted coordinates of elements $a(0, \varepsilon_1, \psi)$, $x(u, \varepsilon, \varphi) \in S^2(1)$ we get:

$$u' = u, \quad \varepsilon' = \varepsilon\varepsilon_1, \qquad \varphi' = \varphi + \psi.$$

This group also consists of two connected components.

The metric tensor for semi-Euclidean sphere has the matrix representation:

$$(g_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

The linear element of the metric is:

$$ds_1^2 = -d\varphi^2. \tag{14}$$

Now, we want to define the conformal model of the bundle $(S^2(1), \pi, \mathbb{R})$. For that we need to introduce the conformal map of the sphere to a disconnected plane $f : S^2(1) \to Q \in \mathbb{R}^2$. Q is located at $x_0 = 0$. We know that the sphere consists of two disconnected components, one with $x_0 > 0$, and other with $x_0 < 0$. We choose a pole at the first one, $N(1, 0, 0)$. All points of $S^2(1)$ except the line passing through the pole N are stereographically projected to Q such that the first component of the sphere with $x_0 > 0$ is mapped on $\{(0, x_1, x_2) | \, x_1 \in (-\infty, -1) \cup (1, \infty), x_2 \in \mathbb{R}\}$ while the second component with $x_0 < 0$ is mapped on the strip $\{(0, x_1, x_2) | \, x_1 \in (-1, 1), x_2 \in \mathbb{R}\}$. We denote x, y coordinates on Q such that the x axis lies along x_1 while the y axis along x_2. Then

$$x = \frac{x_1}{1 - x_0}, \quad y = \frac{x_2}{1 - x_0}. \tag{15}$$

An inverse map $f^{-1} : Q \to S^2(1)$, where $x \neq \pm 1$, is:

$$x_0 = -\frac{1+x^2}{1-x^2}, \quad x_1 = \frac{2x}{1-x^2}, \quad x_2 = \frac{2y}{1-x^2}\,. \tag{16}$$

If we substitute formulas (15) into (13) then we obtain the relations between coordinates x, y and adapted coordinates u, φ which are on semi-Euclidean sphere:

$$f: \quad x = \frac{\sinh\varphi}{\varepsilon - \cosh\varphi}, \quad y = \frac{u\exp\varphi}{\varepsilon - \cosh\varphi}\,.$$

Then the inverse map is:

$$\varphi = \ln\Big(\varepsilon\frac{x-1}{x+1}\Big), \quad u = -\frac{2y}{(1-x)^2}\,. \tag{17}$$

Note that the lines $x = \pm 1$ are not included in the mapping and Q consists of three disconnected components. Also, the line $x_0 = 1, x_1 = 0$ has no image in this mapping. We add it by hand, identifying the image of this line with the points $\{(x,y)|\, x = = \pm\infty, y \in \mathbb{R}\}$ on Q. Then two disconnected parts $\{(x,y)|\, x \in (-\infty,-1), y \in \mathbb{R}\}$ and $\{(x,y)|\, x \in (1,\infty), y \in \mathbb{R}\}$ are connected and we call this plane C^2.

In particular, after enlarging Q into C^2 by the infinitely distant point and ideal line crossing this point, then the stereographic map f becomes diffeomorphism S. Note that the infinitely distant point is the image of point N. The ideal line is the image of the straight line belonging to $S^2(1)$ and crossing the pole: $x_0 = 1,\ x_1 = 0$.

Let us now consider the commutative diagram:

$$\begin{array}{ccccc} S^2(1) & & \xrightarrow{S} & & C^2 \\ & \pi \searrow & & \swarrow p & \\ & & \mathbb{R} & & \end{array}$$

The map $p = \pi \circ S^{-1} : C^2 \to R$ is defined by this diagram. We find the coordinate form of this map:

$$u = -\frac{2y}{(1-x)^2}\,.$$

The map $p : C^2 \to \mathbb{R}$ defines the trivial bundle with the base $\mathbb{R}$ and the structure group S_1.

Theorem. *Let S is the map : $S^2(1) \to C^2$ as described before. Then S is a conformal map.*

Proof. The metric on G induces the metric on C^2. In the coordinates x, y it has the form:

$$d\tilde{s}^2 = -dx^2. \tag{18}$$

Let us find the metric of semi-Euclidean sphere from the metric on C^2. From (17) we get $d\varphi = \frac{2}{x^2-1}dx$ and using (14) and (18) we find:

$$ds_1^2 = \frac{4}{(x^2-1)^2}d\tilde{s}^2.$$

Hence, the linear element of semi-Euclidean sphere differs from the linear element of C^2 by a conformal factor and therefore, the map S is conformal. □

We find the equation of fibers on C^2. The 1-parametric fibers family of the bundle $(S^2(1), \pi, \mathbb{R})$ in the adaptive coordinates (13) is: $u = c,\ c \in \mathbb{R}$. From (17) we get the image of this family under the map S:

$$y = -c/2 \cdot (x-1)^2. \tag{19}$$

The C^2 plane is also fibred by this 1-parametric family of parabolas.

4. The projective conformal model

Now we construct the projective semi-conformal model of the sphere $S^2(1)$ and the principal bundle on it. We use a normalization method of A.P.Norden [7], [8]. A. P. Shirokov in his work [14] constructed conformal models of Non-Euclidean spaces with this method.

Definition. A hypersurface X_{n-1} as an absolute in a projective space P_n is called *normalized* if with every point $Q \in X_{n-1}$ there is associated:
1) a line P_I which has the point Q as the only intersection with the tangent space T_{n-1}, and
2) a linear space P_{n-2} that belongs to T_{n-1}, but it does not contain the point Q.
We call them *normals of the first and second types,* P_I and P_{II}.

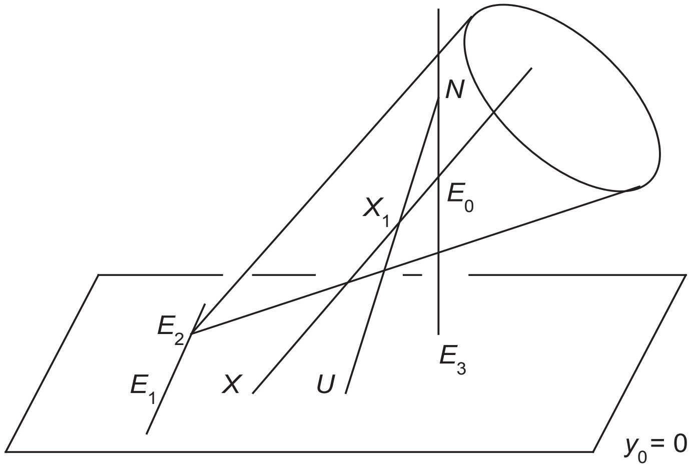
In order to have a polar normalization, P_I and P_{II} must be polar with respect to the absolute X_{n-1}.

We enlarge the semi-Euclidean space ${}_2E^3_1$ to a projective space P^3. Here ${}_kE^n_l$ denotes a n-dimensional semi-Euclidean space with the metric tensor of rank k, and l is the number of negative inertia index in a quadric form. We consider homogeneous coordinates $(y_0 : y_1 : y_2 : y_3)$ in P^3, where $x_i = \frac{y_i}{y_3}$, $i = 0, 1, 2$. Thus $S^2(1) :\ x_0^2 - x_1^2 = 1$ describes the hyperquadric in P^3:

$$y_0^2 - y_1^2 - y_3^2 = 0\,. \tag{20}$$

Here the projective basis (E_0, E_1, E_2, E_3) is chosen in the following way. The vertex E_0 of basis is inside the hyperquadric. The other vertices E_1, E_2, E_3 are on its polar plane, $y_0 = 0$. The line E_0E_3 crosses the hyperquadric at poles $N(1:0:0:1)$, $N'(1:0:0:-1)$. Vertices E_1, E_2 lie on the polar of the line E_0E_3. The vertex of the hyperquadric coincides with the vertex E_2.

The stereographic map of the projective plane $P^2 : y_0 = 0$ to the hyperquadric (20) from the pole $N(1:0:0:1)$ is shown on the picture. Let $U(0 : y_1 : y_2 : y_3) \in P^2$. If $y_3 = 0$, then the line UN belongs to the tangent plane $T_N : y_0 - y_3 = 0$ of the hyperquadric (20) at the point N and in this case the intersection point of the line UN with the hyperquadric is not uniquely determined. If $y_3 \neq 0$, then the intersection point of the line UN with the hyperquadric is unique. So, we choose the line E_1E_2 :



: $y_3 = 0$ as the line at infinity. In the area $y_3 \neq 0$ we consider the Cartesian coordinates $x_1 = \frac{y_1}{y_3}, x_2 = \frac{y_2}{y_3}$. Then the plane $\alpha : y_0 = 0, y_3 \neq 0$ becomes a plane with an affine structure A^2. It is possible to introduce the structure of semi-Euclidean plane ${}_1E^2$ with the linear element

$$ds_0^2 = dx_1^2 . \tag{21}$$

The hyperquadric and the plane α do not intersect each other or intersect in two imaginary parallel lines

$$x_1^2 = -1 . \tag{22}$$

The restriction of the stereographic projection to the plane α maps the point $U(0 : x_1 : x_2 : 1)$ into the point X_1

$$X_1(-1 - x_1^2 : 2x_1 : 2x_2 : 1 - x_1^2) . \tag{23}$$

So, the Cartesian coordinates x_i can be used as the local coordinates at the hyperquadric except the point of its intersection with the tangent plane T_N.

We construct an autopolar normalization of the hyperquadric. As a normal of the first type we take lines with the fixed center E_0 and as a normal of the second type we take their polar lines which belong to the plane α and cross the vertex E_2 of the hyperquadric. The line E_0X_1 intersects the plane α at the point

$$X(0 : 2x_1 : 2x_2 : 1 - x_1^2) .$$

In this normalization the polar of the point X intersects the plane α on the normal P_{II}. Thus for any point X in the plane α there corresponds a line which does not cross this point. It means that the plane α is also normalized. The normalization of α is defined by an absolute quadric (22).

We consider the derivative equations of this normalization. If we take normals of the first type with fixed center E_0, then the derivative equations ([7], p. 204) have the form:

$$\begin{aligned} \partial_i X &= Y_i + l_i X\,, \\ \nabla_j Y_i &= l_j Y_i + p_{ji} X\,. \end{aligned} \tag{24}$$

The points X, Y_i, E_0 define a family of projective frames. Here Y_i are generating points of the normal P_{II}.

We can calculate the values (X, X), (X, Y_i) on the plane α using the quadric form, which is in the left part of equation (20). So, $(X, X) = -(1 + x_1^2)^2$.

Let us find coordinates of the metric tensor on the plane α. Hence, we take the Weierstrass standardization

$$(\widetilde{X}, \widetilde{X}) = -1, \quad \widetilde{X} = \frac{X}{1 + x_1^2}\,.$$

Then the coordinates of the metric tensor are the scalar products of partial derivatives $g_{ij} = -(\partial_i \widetilde{X}, \partial_j \widetilde{X})$:

$$(g_{ij}) = \begin{pmatrix} \frac{4}{(1+x_1^2)^2} & 0 \\ 0 & 0 \end{pmatrix}.$$

We got the conformal model of the polar normalized plane $\alpha : y_0 = 0, y_3 \neq 0$ with a linear element

$$ds^2 = \frac{dx_1^2}{(1 + x_1^2)^2}\,. \tag{25}$$

It means that the following theorem is true:

Theorem. *The non-Euclidean plane α is conformally equivalent to semi-Euclidean plane ${}_1E^2$.*

The points X and Y_i are conjugated with respect to the polar (20) and $(X, Y_i) = 0$. From this equation and the derivative equations (24) we can get the non-zero connection coefficients:

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^2_{21} = -\frac{2x_1}{1 + x_1^2}\,, \quad \Gamma^2_{11} = \frac{2x_2}{1 + x_1^2}\,.$$

The sums $\Gamma^s_{ks} = \partial_k \ln \frac{c}{(1+x_1^2)^2}$ ($c = const$) are gradients, so the connection is equiaffine. Curvature tensor has the following non-zero elements:

$$R_{121\cdot}{}^{2} = -R_{211\cdot}{}^{2} = -\frac{4}{(1 + x_1^2)^2}\,.$$

Ricci curvature tensor $R_{sk} = R_{isk\cdot}{}^{i}$ is symmetric: $R_{11} = \frac{4}{(1+x_1^2)^2}$. Metric g_{ij} and curvature $R_{rsk\cdot}{}^{i}$ tensors are covariantly constant in this connection: $\nabla_k g_{ij} = 0$, $\nabla_l R_{rsk\cdot}{}^{i} = 0$.

This can be summarized into a proposition:

Proposition. *The autopolar normalization of the hyperquadric (20) constructed above defines an equiaffine connection on it with symmetric Ricci curvature tensor and covariantly constant metric and curvature tensors.*

The infinitesimal linear operators for the quadric are

$$\begin{cases} L_1 = y_0 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_0}, \\ L_2 = y_0 \frac{\partial}{\partial y_3} + y_3 \frac{\partial}{\partial y_0}, \\ L_3 = y_1 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_1}. \end{cases} \tag{26}$$

Solving geodesic equations we find parametric solutions

$$\begin{cases} x_1 = \tan(\omega t + \phi), \\ x_2 = (c_1 e^{2i\omega t} + c_2 e^{-2i\omega t}) \sec^2(\omega t + \phi). \end{cases} \tag{27}$$

where c_1, c_2, ω, ϕ are integration constants. Eliminating the parameter t we can rewrite these equations in a simple form

$$x_2 = A(x_1^2 - 1) + Bx_1,$$

where A and B are arbitrary constants. We see that the solution represents parabolas and lines in x_1x_2-plane.

Let us consider the bundle of this plane by the double numbers subalgebra. We write the equations of fibers of semi-Euclidean sphere $S^2(1)$ in homogeneous coordinates:

$$\begin{cases} (y_0 - y_1)v - y_2 = 0, \\ y_0^2 - y_1^2 - y_3^2 = 0. \end{cases} \tag{28}$$

This 1-parametric family of curves fibers the hyperquadric and it defines a bundle on it. The image of these fibers under the stereographic projection from the pole N to the plane α is:

$$x_2 = -v/2 \cdot (x_1 + 1)^2.$$

It is 1-parametric family of parabolas (compare with (19)).

Conclusion

General program is to study non-Euclidean spaces generated by unital associative algebras. In this paper we give an example of pseudo-Euclidean space and then we present two ways how to construct models of its fibration, which are conformal and projective models of pseudo-Euclidean sphere. For both models we get metric and images of a fibration. For the second we use a normalization method to construct an equiaffine connection, then we find infinitesimal linear operators and geodesics.

We would obtain the similar results for the space of right cosets by the Lie subgroup H_2 (it is the subgroup of invertible dual numbers) and the bundle of the group G by H_2. However, H_2 is a normal divisor of the group G. Therefore, the spaces of right and left cosets coincide.

A c k n o w l e d g e m e n t

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