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Probability Limit Identification Functions on Separable Metric Spaces

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In the present article we show the existence of a probability limit identification function on any separable metrizable topological space and an application of such function in stochastic analysis. The convergence in probability on topological spaces is studied as well.

1. Introduction

Probability limit identification function (*PLIF*) is a function $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ such that for every sequence $X_1, X_2, \ldots$ of real valued random variables defined on a probability space $(\Omega, \mathscr{A}, P)$, whose coordinates converge in probability (denote the probability limit X), the set

$$\{\omega \in \Omega : f(X_1(\omega), X_2(\omega), \ldots) \neq X(\omega)\} \tag{1.1}$$

is contained in a P-null set of $\mathscr{A}$. Function f does not depend on the underlying probability space and does not have to be mesurable. The concept of the PLIF was introduced by G. Simons in [3]. He showed that a PLIF exists iff a SPLIF does, where the SPLIF is a PLIF for 0-1 valued random variables converging in probability to a constant (note that each PLIF is also a SPLIF). The existence of a PLIF under the continuum hypothesis was proved by Štěpán in [5] by a transfinite construction strongly

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supported by the continuum hypothesis. To the authors' best knowledge the result has not yet been proved without using the hypothesis. Later on Blackwell showed in [1] that there is no Borel SPLIF. The proof was based on Oxtoby's category 0-1 law.

Consider now a topological space $(T, \mathscr{G})$ and a function $f : T^{\mathbb{N}} \to T$. The function f is called a *PLIF on T* if for any probability space $(\Omega, \mathscr{A}, P)$ and for any sequence $X_1, X_2, \dots$ of (Borel) random variables $X_n : (\Omega, \mathscr{A}) \to (T, \mathscr{B}(T))$, $n \in \mathbb{N}$ converging in probability to a random variable X the set (1.1) is contained in a P-null set of $\mathscr{A}$. In the present article we will show that such function exists under the continuum hypothesis if the topological space is separable and metrizable. For this purpose we will make use of the fact that such a space and the Hilbert cube are homeomorphic.

PLIFs may be used for construction of functional representations in stochastic analysis. First we will show the functional representation of the quadratic variation of any continuous local martingale. Let f be a PLIF on $\mathbb{R}$ and M any continuous local martingale with quadratic variation $\langle M \rangle$. Applying f on the sequence that approximates quadratic variation in probability (such approximation sequence may be found in [2], Proposition 17.17), we get a.s. representation of $\langle M \rangle_t$. Repeating this procedure in all positive rational points we almost surely get $\langle M \rangle(\omega)$ from $M(\omega)$ on the set of positive rational numbers and $\langle M \rangle$ can be then uniquely continuously completed on the whole nonnegative real line. This shows the existence of the mapping V such that for every continuous local martingale M we have

$$V(M(\omega)) = \langle M \rangle(\omega) \text{ for almost all } \omega.$$

Next we can construct the functional representation of a stochastic integral. Consider $\mathscr{C} = \mathscr{C}(\mathbb{R}_+)$ the space of all continuous functions defined on the nonnegative real line with metric d defined as follows

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \Big(\max_{t \leq n} |f(t) - g(t)| \wedge 1 \Big).$$

The space $(\mathscr{C}, d)$ is separable and let f_C be a PLIF on $\mathscr{C}$. Consider again M any continuous local martingale and any progressive process X such that $\int_0^t X_s^2 d\langle M \rangle_s < \infty$ a.s. for all $t \geq 0$. By using functional representation of quadratic variation we can create the sequence X^n of predictable step processes such that $\int X^n dM \to^P \int X dM$ in (C, d) (see [2], Lemma 17.23). The construction of these approximation processes is based on trajectories $X(\omega)$ and $M(\omega)$ only and the integrals of step processes can by determined from trajectories as well. Applying the PLIF f_C on the sequence $\int X^n dM$ results in functional representation of stochastic integral, that is the mapping I with the following property

$$I\Big(X(\omega), M(\omega)\Big) = \Big(\int X dM \Big)(\omega) \text{ for almost all } \omega,$$

for any continuous local martingale M and process X such that $\int XdM$ exists. If the filtration of the underlying stochastic base is complete, it easily follows that the process $I\Big(X(\omega), M(\omega)\Big)$ is an adapted continuous local martingale and therefore we may consider it to be the integral $\int XdM$.

Functional representation of stochastic integral can be further used to construct functional representation of weak solutions of stochastic differential equations. Consider an equation

$$dX_t = \sigma(t, X)dB_t + b(t, X)dt, \tag{1.2}$$

where B is a Brownian motion and σ, b are progressive coefficients. It is shown in [2] (Lemma 21.8) that there exists a measurable mapping $\hat{F}$ such that, if process X with distribution $\mathscr{L}(X)$ is a solution of the local martingale problem for $(\sigma\sigma', b)$ and u is a uniformly distributed random variable on $(0, 1)$ independent of X, then

$$B(\omega) = \hat{F}\Big(\mathscr{L}(X), X(\omega), u(\omega)\Big)$$

is a Brownian motion and (X, B) with induced filtration solves equation (1.2). In the same monograph (Theorem 21.7 and Theorem 18.12) we can find that B can be constructed as a stochastic integral $B = \int g_1(\sigma(s, X))dM + \int g_2(\sigma(s, X))dW$, where M is a continuous local martingale defined as a function of trajectories $X(\omega)$ and $b(t, X(\omega))$ and W is (some) Brownian motion independent of M. Using functional representation of stochastic integrals in the construction of B and Lemma 3.22 in [2] we get a mapping F (for the coefficients σ, b) such that for every process X which solves the local martingale problem for $(\sigma\sigma', b)$ and u a uniformly distributed random variable on $(0, 1)$ independent of X there exists a Brownian motion B such that

$$B(\omega) = F\Big(X(\omega), u(\omega)\Big) \text{ for almost all } \omega$$

and the pair (X, B) with induced filtration solves the equation (1.2). Considering complete filtration the process $F\Big(X(\omega), u(\omega)\Big)$ itself is the searched Brownian motion. In comparison with $\hat{F}$, the mapping F does not need the distribution of X as a parameter, on the other hand F need not be measurable.

2. The convergence in probability on topological spaces

Let $(T, \mathscr{G})$ be a topological space and $(\Omega, \mathscr{A}, P)$ a probability space. Consider Borel random variables $X, X_1, X_2, \ldots$ with values in T defined on $(\Omega, \mathscr{A})$. We say that the sequence $X_1, X_2, \ldots$ converges in probability to X (write $X_n \to^P X$), if for any open set $G \in \mathscr{G}$

$$P\Big(X_n \notin G, X \in G\Big) \to 0 \quad \text{for } n \to \infty. \tag{2.1}$$

Let us state a pair of simple (perhaps well known) observations in connection with the above definition.

Lemma 2.1 *Assume a separable metrizable space T and T-valued random variables $X, X_1, X_2, \dots$ Fix an equivalent metric d on T. Then (2.1) holds if and only if $d(X_n, X) \to 0$ in probability.*

Hence, (2.1) extends consistently the standard definition of the convergence in probability. For the case of completeness we offer a possible proof.

Proof. Assume (2.1) and consider $\epsilon > 0$ and $\delta > 0$. It follows by separability of T that

$$P[X \in \bigcup_{j=1}^{k} G_j] > 1 - \delta,$$

where $G_j = \{x \in T : d(x, x_j) < \epsilon\}$ for some $x_j \in T$ and $k \in \mathbb{N}$. Then

$$P[d(X, X_n) > 2\epsilon] \leq \delta + P[d(X, X_n) > 2\epsilon, X \in \bigcup_{j=1}^{k} G_j] \leq \delta + \sum_{j=1}^{k} P[X \in G_j, X_n \notin G_j]$$

and that proves that $d(X_n, X) \to 0$ in probability. If this is assumed, if F is a closed set in T and $\epsilon > 0$ then, denoting by F^ϵ the closed ϵ-neighborhood of F, we get

$$\begin{aligned} P[X \in T \setminus F, X_n \in F] &\leq P[X \in T \setminus F^\epsilon, X_n \in F] + P[X \in F^\epsilon \setminus F] \leq \\ &\leq P[d(X, X_n) > \epsilon] + P[X \in F^\epsilon \setminus F]. \end{aligned}$$

Letting $n \to \infty$ and the $\epsilon \to 0$ we prove (2.1). □

Lemma 2.2 *Assume that (2.1) holds for $G \in \mathcal{Q}$ where $\mathcal{Q}$ is a subbase for $\mathcal{G}$ and that the probability distribution of X is a Radon probability measure. Then $X_n \to X$ in probability.*

Hence, assuming that T is a Polish space it suffices to verify (2.1) for $G \in \mathcal{Q}$ where $\mathcal{Q}$ is an arbitrary topological subbase.

Proof. First observe that the property (2.1) is closed on finite unions and intersections. Indeed, if $G_j \in \mathcal{G}$ for $j = 1, \dots, k$ are sets such that (2.1) holds then

$$P\Big[X_n \notin \bigcap_{j=1}^{k} G_j, X \in \bigcap_{j=1}^{k} G_j\Big] \leq \sum_{j=1}^{k} P[X_n \notin G_j, X \in G_j]$$

and

$$\begin{aligned} P\Big[X_n \notin \bigcup_{j=1}^{k} G_j, X \in \bigcup_{j=1}^{k} G_j\Big] &\leq \sum_{j=1}^{k} P\Big[X_n \notin \bigcup_{j=1}^{k} G_j, X \in G_j\Big] \leq \\ &\leq \sum_{j=1}^{k} P[X_n \notin G_j, X \in G_j]. \end{aligned}$$

Thus we may assume w.l.g. in our proof that $\mathcal{Q}$ is a subbase closed on finite unions and intersections. Consider $G \in \mathcal{G}$ and $\epsilon > 0$. It follows that there is a compact set $K \subset G$ such that $P[X \in G \setminus K] < \epsilon$ and that $K \subset G_\epsilon \subset G$ for a set $G_\epsilon \in \mathcal{Q}$. This concludes the proof as

$$P[X_n \notin G, X \in G] \leq \epsilon + P[X_n \notin G_\epsilon, X \in K] \leq \epsilon + P[X_n \notin G_\epsilon, X \in G_\epsilon].$$

That proves (2.1) for all $G \in \mathcal{G}$. □

Recall that T is an uniformizable space if it is a Hausdorff space the topology of which is generated by the subbase $\mathcal{G}^0$ that consists of the sets

$$\{x \in T : |f(x) - f(x_0)| < \epsilon\}, \quad x_0 \in T, \quad f \in C(T), \quad \epsilon > 0. \tag{2.2}$$

Hence, assuming that T is an uniformizable Radon space it is enough to verify (2.1) for $G \in \mathcal{G}^0$ where $\mathcal{G}^0$ is the topological subbase defined by (2.2).

Lemma 2.3 *Assume that T is an uniformizable topological space and that P_X is a Radon probability measure on T. Then*

$$X_n \xrightarrow{P} X \iff f(X_n) \xrightarrow{P} f(X) \quad \forall f \in C(T). \tag{2.3}$$

Hence assuming that T is an uniformizable Radon space then (2.3) is equivalent to (2.1).

Proof. The implication $\Rightarrow$ is obvious. Assuming (2.3) we prove that (2.1) holds for all $G \in \mathcal{G}^0$ where $\mathcal{G}^0$ is the subbase (2.2). Use Lemma 2.2 to conclude the proof. □

Lemma 2.4 *Consider a cube $T = [0, 1]^J$ where $J \neq \emptyset$, denote as $p_j : T \to$ $\to [0, 1]$ the projections and consider T-valued random variables X_n, X such that P_X is a Radon probability measure on T. Then*

$$X_n \xrightarrow{P} X \iff p_j(X_n) \xrightarrow{P} p_j(X) \quad \forall j \in J. \tag{2.4}$$

Recall that P_X is a Radon measure automatically if J is at most countable set. To prove Lemma 2.4 apply Lemma 2.2 observing that the sets

$$p_j^{-1}(G), \quad G \subset [0, 1] \quad \text{an open set}, \quad j \in J,$$

form a topological subbase for T.

The probability limit X in sense of (2.1) is not determined uniquely almost surely generally. We shall scrutinize the problem in the case of a metrizable space T.

Lemma 2.5 *Assume that $Y, X, X_1, X_2, \ldots$ are T-valued random variables where T is a metrizable space and P_Y and P_X are Radon measures on T. Then*

$$X_n \xrightarrow{P} Y, \quad X_n \xrightarrow{P} X \quad \implies X = Y \quad \textit{almost surely.}$$

Proof. Denote by μ the joint probability distribution of (X, Y) in T^2 and observe that μ is a probability measure on $\mathscr{B}(T) \times \mathscr{B}(T) \subset \mathscr{B}(T \times T)$. It follows that having $\epsilon > 0$ there is a compact rectangle $K = K_1 \times K_2$ such that $\mu(K) \geq 1 - \epsilon$ holds. Further, the restriction of μ to K denoted as μ_K is obviously a Radon measure, especially τ-additive measure, on the compact metric space K since

$$[\mathscr{B}(K_1) \times \mathscr{B}(K_2)] \cap K = \mathscr{B}(K).$$

It follows by Lemma 2.3 that $f(X) = f(Y)$ almost surely for all $f \in C(T)$ and therefore

$$\mu_K(K_f) = \mu(K_f) = \mu(K), \quad \text{where} \quad K_f := \{(x, y) \in K : f(x) = f(y)\} \quad \forall f \in C(T).$$

Since each K_f is a closed set we apply the τ-additivity of μ_K to get

$$\mu\Big(\bigcap_{f \in C(T)} K_f\Big) = \inf_{f \in C(T)} \mu(K_f) = \mu(K) \geq 1 - \epsilon$$

and therefore $P[f(X) = f(Y), \forall f \in C(T)] \geq 1 - \epsilon$, hence

$$P[f(X) = f(Y), \forall f \in C(T)] = 1 \quad \text{and} \quad X = Y \quad \text{almost surely.}$$

□

3. The existence of a PLIF on separable metrizable spaces

To prove the existence of a PLIF on any separable metrizable space we will need three useful lemmas. First of them shows how to construct PLIFs on subspaces, the second shows how to construct PLIFs on product spaces and the last constructs PLIFs on homeomorphic spaces.

Lemma 3.1 *Let $(T, \mathscr{G})$ be a topological space and $(U, \mathscr{Q})$ its nonempty subspace with the induced topology, i.e. $\emptyset \neq U \subset T$ and $\mathscr{Q} = \{G \cap U : G \in \mathscr{G}\}$. If there exists a PLIF on T then there exists a PLIF on U as well.*

Proof. Consider f_T a PLIF on T. For a sequence $(u_1, u_2, \dots) \in U^{\mathbb{N}}$ define mapping $f_U : U^{\mathbb{N}} \to U$ as follows

$$f_U(u_1, u_2, \dots) = \begin{cases} f_T(u_1, u_2, \dots), & \text{if } f_T(u_1, u_2, \dots) \in U, \\ c & \text{otherwise,} \end{cases}$$

where $c \in U$ is any fixed constant. Then f_U is the PLIF on U. The argument is as follows: Consider U-valued Borel random variables $X, X_1, X_2, \dots$ such that $X_n \to X$ in probability, i.e. such that $P[X_n \notin G \cap U, X \in G \cap U] \to 0$ holds for arbitrary $G \in \mathscr{G}$. Observing that $X, X_1, X_2, \dots$ are at same time T-valued Borel random variables we conclude the proof. □

Lemma 3.2 *Let $(T_i, \mathscr{G}_i)$, $i \in \mathbb{N}$ be Polish topological spaces. If there exist PLIFs on each T_i then there exists a PLIF on $\prod_{i \in \mathbb{N}} T_i$ with the product topology.*

Proof. Label $T = \prod_{i\in\mathbb{N}} T_i$ the product space and $\mathscr{G}$ the product topology on T. Consider any sequence $\{x_n\}_{n=1}^{\infty} = \{(x_n^1, x_n^2, \dots)\}_{n=1}^{\infty}$ in T (i.e. $x_n^i \in T_i$). Define the mapping $f : T^{\mathbb{N}} \to T$ in the following way

$$f(x_1, x_2, \dots) = \Big(f_1(x_1^1, x_2^1, \dots), f_2(x_1^2, x_2^2 \dots), \dots\Big),$$

where f_i is a PLIF on T_i.

We can easily show that f is a PLIF on T. Consider any sequence of T-valued Borel random variables $\{X_n\}_{n=1}^{\infty}$ converging in probability to Y. It follows that $X_n^i \to\ \to^P Y^i$ with $n \to \infty$ for all $i \in \mathbb{N}$. Applying PLIFs f_i we get for each $i \in \mathbb{N}$ the following

$$f_i(X_1^i(\omega), X_2^i(\omega), \dots) = Y^i(\omega) \quad \text{for almost all } \omega.$$

As a result we have for the mapping f

$$f(X_1(\omega), X_2(\omega), \dots) = Y(\omega) \quad \text{for almost all } \omega.$$

□

Lemma 3.3 *If H and T are homeomorphic topological spaces and H has a PLIF then T possesses a PLIF as well.*

Proof. Consider an homeomorphism $v : T \leftrightarrow H$ and a PLIF f_H on H. Then $f_T : T^{\mathbb{N}} \to T$ defined by

$$f_T(x_1, x_2, \dots) = v^{-1}(f_H(v(x_1), v(x_2), \dots)), \quad x_j \in T,$$

is a PLIF on T. The argument reads as follows: Consider a sequence $X_1, X_2, \dots$ of T-valued Borel random variables that converges in probability to a variable X. Further set

$$Z = v(X) \quad Z_n = v(X_n) \quad \forall n \in \mathbb{N}.$$

Note that $Z, Z_1, Z_2 \dots$ are Borel random variables with values in H and $Z_n \to^P Z$. Therefore we have

$$f_H(Z_1(\omega), Z_2(\omega), \dots) = Z(\omega) \quad \text{for almost all } \omega.$$

Using that and the definition of f_T we get

$$\begin{aligned} f_T(X_1(\omega), X_2(\omega), \dots) &= v^{-1}\Big(f_H\Big(v(X_1(\omega)), v(X_2(\omega)), \dots\Big)\Big) = \\ &= v^{-1}\Big(f_H\Big(Z_1(\omega), Z_2(\omega), \dots\Big)\Big) = v^{-1}\Big(Z(\omega)\Big) = \\ &= v^{-1}\Big(v(X(\omega))\Big) = X(\omega) \quad \text{for almost all } \omega. \end{aligned}$$

□

Now we can prove the main theorem of this article. We will be using the fact that any separable metrizable space is homeomorphic to a subspace of the Hilbert cube (see [4], Theorem 2.1.32) and preceding lemmas.

Theorem 3.4 *Under the continuum hypothesis there is a PLIF on any separable metrizable topological space.*

Proof. Consider any separable metrizable topological space T. Let $v : T \to \mathbb{H}$ be a homeomorphism T into the Hilbert cube $\mathbb{H} = [0,1]^{\mathbb{N}}$ (with the product topology). Denote $H_T = v(T)$ subspace of $\mathbb{H}$ with the induced topology. As under the continuum hypothesis there is a PLIF on $\mathbb{R}$ (see [5]), we get by applying lemmas 3.1 and 3.2 the existence of a PLIF on H_T. Lemma 3.3 then shows the existence of a PLIF on T. $\square$

4. Conclusion

In the Theorem 3.4 we showed the existence of a PLIF on a separable metrizable space. As Lemma 2.5 suggests there is a chance that a PLIF exists on any Radon metrizable space, i.e. any metrizable topological space T such that arbitrary finite Borel measure on T is inner regular with respect to compact subsets of T. As far as we know this problem is left open.

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