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Diffusion with an Reflecting and Absorbing Level Set Boundary – A Simulation Study

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Praha

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A map $f \in \mathscr{C}^2(\mathbb{R}^n)$ is considered. Diffusions given by an n-dimensional stochastic differential equations $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ are constructed to stay in region $K = [f \leq c]$ forever in a way that the boundary $S = [f = c] = \partial K$ is either absorbing or reflecting. The purpose of the paper is to provide easy to apply conditions on the coefficients $b(x)$ and $\sigma(x)$ with the aim to exhibit simulations of the diffusions with above properties.

1. Introduction

Having a function $f \in \mathscr{C}^2(\mathbb{R}^n)$ and a constant $c \in \mathbb{R}$ we denote

$$K := \{x : f(x) \leq c\}, \quad K^e = \{x : f(x) \geq c\}$$

and

$$S := \partial K = \partial K^e = \{x : f(x) = c\},$$

calling the S a boundary.

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Further consider a stochastic differential equation (SDE)

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \tag{1}$$

where $B_t = (B_t^1, ..., B_t^n)$ is an n-dimensional Brownian motion, $b(x) = (b_1(x), ..., b_n(x))^T$ and $\sigma(x) = (\sigma_{ij}(x)_{1\le i,j\le n})$ are Borel functions. Recall that a continuous n-dimensional $\mathscr{F}_t$-adapted process $X = (X^1, X^2, ..., X^n)$ solves (1) if

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s \quad \text{holds almost surely for all} \quad t \ge 0,$$

where $\mathscr{F}_t$ is the augmented canonical filtration of the Brownian motion B_t. Since the above n-dimensional equation reads exactly as

$$X_t^i = X_0^i + \int_0^t b^i(X_s)ds + \sum_{k=1}^n \int_0^t \sigma_{i,k}(X_s)dB_s^k, \quad 1 \le i \le n,$$

we implicitly assume that b, σ and X are such that all coefficients $b^i(X)$ and $\sigma_{i,j}^2(X)$ are locally integrable on $\mathbb{R}^+$.

As for the definitions of standard concepts connected with stochastic differential equation theory we refer our reader to [3].

The purpose of this paper is to find easy to apply conditions on the coefficients b and σ that would force arbitrary solution to the equation (1) that starts in K

- to stay in K forever, hence to define a diffusion in K,
  and moreover
- to get the boundary S either absorbing or reflecting.

Absorbing and reflecting barriers for a diffusion has been for some time a frequented topic in stochastic analysis, see chapter 12 in [2], for example.

Coming back to the equation (1), the Itô formula yields

$$df(X_t) = Lf(X_t)dt + dM_t, \tag{2}$$

where

$$\begin{aligned} Lf(x) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)b_i(x) + \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x)a_{ij}(x) \\ &= \text{grad} f(x)^T \cdot b(x) + \frac{1}{2}\text{tr}\left(f''(x)\cdot a(x)\right), \end{aligned} \tag{3}$$

$$dM_t = \text{grad} f(X_t)^T \cdot \sigma(X_t)dB_t,$$

$$a(x) = \sigma(x)\sigma(x)^T, \quad \text{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), ..., \frac{\partial f}{\partial x_n}(x)\right)$$

and

$$f''(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{1\le i,j\le n}$$

is an $n \times n$ matrix.

Further, we get

$$d[f(X)]_t = d[M]_t = \text{grad} f(X_t)^T \cdot a(X_t) \cdot \text{grad} f(X_t) dt \tag{4}$$
$$= \sum_{i,j=1}^{n} \frac{\partial f}{\partial x_i}(X_t) \cdot \frac{\partial f}{\partial x_j}(X_t) \cdot a_{ij}(X_t) dt$$
$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X_t) \cdot \sigma_{ij}(X_t) \right)^2 dt,$$

where $[X]$ denotes the quadratic variation of X. The differential operator $Lf(x)$ and the coefficient $\text{grad} f(x)^T \cdot \sigma(x)$ are important when trying to study a boundary behavior of a solution X to (1).

2. Diffusion in $[f \leq c]$ and boundary equations

The following lemma provides sufficient conditions on the coefficients in (1) to define a diffusion in K.

Lemma 1 *Assume that there is an open neighborhood G of boundary S such that for all $x \in G \cap K^e$*

$$\text{grad} f(x)^T \cdot a(x) \cdot \text{grad} f(x) = 0 \tag{5}$$

and

$$Lf(x) \leq 0. \tag{6}$$

Then $X \in K$ almost surely for an arbitrary solution X with an initial condition $X_0 = x_0$, where $x_0 \in K$.

Proof. If X is a solution to (1) with $X_0 = x_0 \in K$ then

$$f(X_v) - f(X_u) = \int_u^v Lf(X_s) ds + \int_u^v \text{grad}^T f(X_s) \cdot \sigma(X_s) dB_s \tag{7}$$
$$\forall -\infty < u < v < \infty$$

hold outside a P-null set N.

What we have to prove is that $P(N_r) = 0$ for all $r \in \mathbb{Q}^+$ where $N_r = [f(X_r) > c]$. Hence, consider an $\omega \in N_r$ and assume that $\omega \notin N$. Put

$$u = u(\omega) = \sup\{s \leq r : X_s(\omega) \in K\}$$

and observe that there is some $u < v = v(\omega) < r$ such that

$$(X_u, X_v) \subset G \cap K^e, \quad f(X_u) = c, \quad f(X_v) > c$$

hold, where $(X_u, X_v) = \{X_s, s \in (u, v)\}$. Since $\omega \notin N$, it follows by (7) that $f(X_v) - f(X_u) = f(X_v) - c \leq 0$, hence a contradiction. It follows that $N_r \subset N$, therefore $P(N_r) = 0$. □

Lemma 1 motivates us to define a **boundary equation** for S as the equation (1) if there is a neighborhood $G \supset S$ such that

$$Lf(x) = 0 \tag{8}$$

and

$$\text{grad} f(x)^T \cdot \sigma(x) = 0 \tag{9}$$

hold for all $x \in G$.

Lemma 1 applied simultaneously to the pairs (f, c) and $(-f, -c)$ proves the following remark.

Remark 2 *Any solution X to a boundary equation (1) with $X_0 = x_0 \in S$ will stay in S forever almost surely.*

Because we plan to involve boundary equations when simulating a diffusion in K we need some procedure how to construct them for a given boundary S. In other words, we need to establish coefficients $b(x)$ and $\sigma(x)$ in (1) to exhibit a boundary equation.

Assume that there exists an $\epsilon > 0$ such that

$$K^\epsilon := \{x : f(x) \leq c + \epsilon\} \quad \text{is a bounded set}$$

and

$$\text{grad} f(x) \neq 0, \quad x \in S,$$

hold. Hence, there exists a number $0 < \delta < \epsilon$ such that

$$G^\delta := \{x : |f(x) - c| < \delta\}, \quad \text{is a bounded set}, \quad |\text{grad} f(x)| \geq \delta > 0, \quad \forall x \in G^\delta.$$

Define

$$b(x) = -\frac{1}{2} \cdot \text{div}\, n(x) \cdot n(x), \quad \sigma(x) = I_n - n(x) \cdot n(x)^T, \quad x \in G^\delta, \tag{10}$$

where

$$n(x) = \frac{\text{grad} f(x)}{|\text{grad} f(x)|}, \quad \text{div}\, n(x) = \sum_{i=1}^{n} \frac{\partial n_i}{\partial x_i}(x).$$

Assuming that $f \in \mathscr{C}^3(\mathbb{R}^n)$, then all coordinates $b^i(x)$ and $\sigma_{ij}(x)$ are $\mathscr{C}^1(G^\delta)$. It follows by the extension theorem proved by H. Whitney (see [1], p.50) that they possess extensions in $\mathscr{C}^1(\mathbb{R}^n)$. It follows that $b^i(x)$ and $\sigma_{ij}(x)$ are Lipschitz on G^δ because $G \subset H$, where H is a convex bounded set. Hence, the coefficient b, σ have globally Lipschitz extensions b^*, σ^*. Now, we prove that the equation with these coefficients b^* and σ^* is a boundary equation.

Denote $g = \text{grad} f(x)$ and $\zeta = g \cdot g^T$, then g is an eigenvector of matrix ζ associated with the eigenvalue $\lambda = |g|^2$. Hence, g is an eigenvector of σ^* associated with the eigenvalue $\lambda = 0$ for all $x \in G^\delta$. Especially, σ^* is an idempotent matrix and

$$\text{grad} f(x)^T \cdot \sigma^*(x) = 0, \quad \forall x \in G^\delta,$$

hence the condition (9) is satisfied.

Now, we have to verify the condition (8). To simplify our notation write

$$\partial_i f := \frac{\partial f}{\partial x_i} \quad \text{and} \quad \partial_{ij} f := \frac{\partial^2 f}{\partial x_i \partial x_j}$$

and compute

$$\partial_i |g| = \partial_i \sqrt{\sum_{j=1}^{n} (\partial_j f)^2} = \frac{1}{2} |g|^{-1} \cdot \sum_{j=1}^{n} (2 \partial_j f \cdot \partial_{ji} f)$$

$$\begin{aligned}
\operatorname{div} n &= \sum_{i=1}^{n} \partial_i \left(\frac{\partial_i f}{|g|} \right) = \sum_{i=1}^{n} \frac{\partial_{ii} f \cdot |g| - \partial_i f \cdot \partial_i |g|}{|g|^2} \\
&= \frac{1}{|g|^2} \left(\sum_{i=1}^{n} \partial_{ii} f \cdot |g| - \sum_{i=1}^{n} \partial_i f \sum_{j=1}^{n} |g|^{-1} \partial_j f \cdot \partial_{ji} f \right) \\
&= \frac{1}{|g|} \left(\sum_{i=1}^{n} \partial_{ii} f - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial_i f}{|g|} \cdot \frac{\partial_j f}{|g|} \partial_{ji} f \right)
\end{aligned}$$

$$\begin{aligned}
\operatorname{grad} f^T \cdot b &= -\frac{1}{2} \cdot g^T \cdot \operatorname{div} n \cdot n = -\frac{1}{2} \cdot \operatorname{div} n \cdot |g| \\
&= -\frac{1}{2} \cdot \left(\sum_{i=1}^{n} \partial_{ii} f - \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial_i f}{|g|} \cdot \frac{\partial_j f}{|g|} \partial_{ji} f \right) \\
&= -\frac{1}{2} \operatorname{tr} (f'' \cdot \sigma^*) = -\frac{1}{2} \operatorname{tr} \left(f'' \cdot \sigma^* \cdot \sigma^{*T} \right).
\end{aligned}$$

It has been verified for all $x \in G^\delta$, hence (8) and (9) are true statements.

Example 1 *The boundary equation on the unit circle in $\mathbb{R}^2$. In this case we have $f(x_1, x_2) = x_1^2 + x_2^2$ and $c = 1$. The construction suggested by (10) needs to compute*

$$\operatorname{grad} f(x) = (2x_1, 2x_2)^T, \quad n(x) = \frac{1}{\sqrt{x_1^2 + x_2^2}} (x_1, x_2)^T = \frac{(x_1, x_2)^T}{|x|}$$

$$\operatorname{div} n(x) = \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \frac{x_i}{|x|} = \frac{|x| - \frac{x_1^2}{|x|}}{|x|^2} + \frac{|x| - \frac{x_2^2}{|x|}}{|x|^2} = \frac{1}{|x|}.$$

Thus we get

$$b(x) = -\frac{1}{2} \cdot \frac{1}{|x|} \cdot \frac{(x_1, x_2)^T}{|x|} \quad \text{and} \quad \sigma(x) = \frac{1}{|x|^2} \begin{pmatrix} x_2^2 & -x_1 \cdot x_2 \\ -x_1 \cdot x_2 & x_1^2 \end{pmatrix},$$

and the boundary equation

$$dX_t = -\frac{1}{2} |X_t|^{-2} \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} dt + |X_t|^{-2} \begin{pmatrix} (X_{2,t})^2 & -X_{1,t} \cdot X_{2,t} \\ -X_{1,t} \cdot X_{2,t} & (X_{1,t})^2 \end{pmatrix} dB_t. \tag{11}$$

Another possibility is presented in [4], *Example 5.1.4., p. 67, where we find the following SDE:*

$$dY_t = -\frac{1}{2}\begin{pmatrix} Y_{1,t} \\ Y_{2,t} \end{pmatrix} dt + \begin{pmatrix} -Y_{2,t} & 0 \\ Y_{1,t} & 0 \end{pmatrix} dB_t. \tag{12}$$

We can easily verify, that

$$\operatorname{grad} f(x)^T \cdot \sigma(x) = 0 \quad \textit{and} \quad Lf(x) = 0 \quad \forall x \in \mathbb{R}^2,$$

hence the equation (12) is a boundary equation again.

A natural question arises: How many boundary equations may be constructed in this case? Looking into it in a detail observe that conditions (8) and (9) may be rewritten as

$$2(x_1 \cdot b_1(x) + x_2 \cdot b_2(x)) = -(\sigma_{11}^2(x) + \sigma_{12}^2(x) + \sigma_{21}^2(x) + \sigma_{22}^2(x)) \tag{13}$$

$$x_1 \cdot \sigma_{1i}(x) = -x_2 \sigma_{2i}(x), \quad i = 1, 2. \tag{14}$$

It follows by (13), that $b(x) = (b_1(x), b_2(x))$ could not be chosen arbitrarily, because we expect $(x_1 \cdot b_1(x) + x_2 \cdot b_2(x))$ as a nonnegative term. Further, it is obvious by (14) that $\sigma(x)$ has to be chosen as

$$\sigma(x) = \begin{pmatrix} -g_1(x) \cdot x_2 & -g_2(x) \cdot x_2 \\ g_1(x) \cdot x_1 & g_2(x) \cdot x_1 \end{pmatrix},$$

where $g_1(x)$ and $g_2(x)$ are arbitrary functions. Hence

$$a(x) = (g_1(x)^2 + g_2(x)^2)\begin{pmatrix} x_2^2 & -x_1 \cdot x_2 \\ -x_1 \cdot x_2 & x_1^2 \end{pmatrix}$$

and we get the equation

$$2(x_1 \cdot b_1(x) + x_2 \cdot b_2(x)) = -(g_1(x)^2 + g_2(x)^2)(x_1^2 + x_2^2). \tag{15}$$

We observe that given an arbitrary $b(x)$ such that $(x_1 \cdot b_1(x) + x_2 \cdot b_2(x)) \leq 0$ we may construct functions $g_1(x)$ and $g_2(x)$ to satisfy (15). Observe also that the function $(g_1(x)^2 + g_2(x)^2))$ in uniquely determined in $\mathbb{R}^2 \setminus \{(0,0)\}$. Hence, the matrix $a(x)$ is uniquely defined by the coefficient $b(x)$ and consequently $b(x)$ determines the distribution of a unique weak solution X to all possible boundary equations with the coefficient $b(x)$ (see [4], *p.149).*

Having a fixed solution X to (1), the boundary S is said to be reflecting for X if outside a P-null set, there is no pair $0 \leq u < v < \infty$ such that $X_s \in S$ for all $s \in (u, v)$ and we shall say that the boundary S is absorbing for X if outside a P-null set the implication

$$X_t \in S \Rightarrow X_{t+s} \in S, \quad s \geq 0, t \geq 0$$

holds.

Recall that having an equation (1) with a unique weak solution X then the above definitions coincide with the standard ones formulated in terms of the corresponding Markov semigroup $(P^x, x \in \mathbb{R}^n)$: The boundary S is said to be reflecting and absorbing if

$$P^x(Z^0 = \emptyset) = 1 \quad \text{and} \quad P^x(Z = \mathbb{R}^+) = 1 \quad \forall x \in S, \qquad \text{respectively,}$$

where $Z = \{t \geq 0 : X_t \in S\}$ and X_t is the corresponding canonical process.

Lemma 3 *Consider a solution X to (1) and assume that there is an open neighborhood $G \supset S$ such that (5) and (6) hold for all $x \in G \cap K^e$. Moreover suppose that $Lf(x) < 0$ is true for all $x \in S$. Then S is a reflecting boundary for X.*

Proof. We will apply the same idea as in the proof of Lemma 1. Let N is a P-null set such that (7) holds outside the set N. Assume that $\omega \notin N$ and that there exist times $u < v$ such that $X_s(\omega) \in S$ for all $s \in (u, v)$. Then

$$f(X_v) - f(X_u) = 0 = \int_u^v Lf(X_s)ds,$$

hence a contradiction. □

Lemma 4 *Consider an equation (1) that has a unique strong solution such that*

$$\tau := \inf\{t \geq 0 : X_t \in S\} < \infty \quad \textit{almost surely if} \quad X_0 \in K.$$

Moreover, assume that there is a boundary equation

$$dX_t = b^*(X_t)dt + \sigma^*(X_t)dB_t \tag{16}$$

such that

$$b^*(x) = b(x), \quad \sigma^*(x) = \sigma(x) \quad \textit{holds for all} \quad x \in S$$

where b^* and σ^* are Lipschitz continuous in an open neighborhood $G \supset S$. Then S is an absorbing boundary for X.

Proof. We may assume without loss of generality that the coefficients b^* and σ^* are globally Lipschitz. Hence, there is a solution X^* to (16) with $X^*_0 = X_\tau$. Define

$$Y_t = X_t \quad \text{if} \quad t \leq \tau \quad \text{and} \quad Y_t = X^*_{t-\tau} \quad \text{if} \quad t \geq \tau$$

and observe that Y_t is a solution to (1) with $Y_0 = X_0$ that is absorbed by S. Since $X = Y$ almost surely, the unique solution X possess the property, too. □

3. Simulations

In this section, we suggest a method how to define a diffusion in K with either absorbing or reflecting boundary $S = [f = c]$. Fix a function $f \in \mathscr{C}^2(\mathbb{R}^n)$, a constant c and an equation (1). We suggest the following two steps to modify (1) in order to get a diffusion in $K = [f \leq c]$.

- We consider an equation

$$dX_t = b^*(X_t)dt + \sigma(X_t)^* dB_t \tag{17}$$

where the coefficients are defined on an open neighborhood $G \supset S$ of S such that solutions to (17) do not leave K.
- Chose $\epsilon > 0$ and denote $K^\epsilon := \{x \in K : |x - y| \geq \epsilon,\ \forall y \in S\}$. Further construct the equation

$$dX_t = \hat{b}(X_t)dt + \hat{\sigma}(X_t)dB_t, \tag{18}$$

where

$$\begin{aligned} \hat{b}(x) &= b(x) && x \in K^\epsilon \\ &= b^*(x) && x \in G \setminus K \\ &= d(x) \cdot b(x) + (1 - d(x)) \cdot b^*(x) && x \in K \setminus K^\epsilon, \end{aligned}$$

and where $d(x) := \frac{1}{\epsilon}\inf_{y \in S}|x - y|$ and $\hat{\sigma}$ is constructed from σ and σ^* by the same way as $\hat{b}$ from b and b^*.

If (1) has coefficients that are Lipschitz in K and (17) coefficients with the property in G, then (18) has Lipschitz coefficients in $G \cup K$. It is obvious that any solution to (18) is diffusion in K.

Example 2 *Diffusion in the unit circle in $\mathbb{R}^2$. In this case, we have $f(x_1, x_2) = x_1^2 + x_2^2$ and $c = 1$. Therefore $K = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ and $S = \{x \in \mathbb{R}^2 : : |x| = 1\}$.*

Consider (1) in the form:

$$dX_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} dB_t.$$

In other words we plan to modify a 2-dimensional Brownian motion to get a diffusion in K with S either a reflecting or an absorbing boundary.

We start with an example of absorbing boundary. In this case we employ (12) as an equation (17), therefore

$$b^*(x) = \frac{1}{2}(-x_1, -x_2)^T, \quad \sigma^*(x) = \begin{pmatrix} -x_2 & 0 \\ x_1 & 0 \end{pmatrix}$$

and choosing $\epsilon = 0.1$ we define the coefficients in (18) by:

$$\begin{aligned} \hat{b}(x) &= (0, 0)^T && \textit{if} \quad |x| \leq 1 - \epsilon \\ &= \frac{|x| - 0.9}{0.1} \cdot b^*(x) && \textit{if} \quad 1 - \epsilon < |x| < 1 \\ &= b^*(x) && \textit{if} \quad |x| \geq 1 \end{aligned}$$

and

$$\hat{\sigma}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \textit{if} \quad |x| \le 1-\epsilon$$

$$= \frac{|x|-0.9}{0.1}\cdot\sigma^*(x) + \frac{1-|x|}{0.1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \textit{if} \quad 1-\epsilon < |x| < 1$$

$$= \sigma^*(x) \qquad \textit{if} \quad |x| \ge 1.$$

Simulation of a solution to equation (18) is shown on the left hand side of Figure 1.

Finally, we construct a diffusion with S as a reflecting boundary. First we need an equation (17). Choosing the $\sigma^*(x)$ as above we get

$$Lf(x) = 2(x_1\cdot b_1(x) + x_2\cdot b_2(x)) - x_1^2 - x_2^2.$$

Hence,a possible candidate to satisfy the requirements of Lemma 3 is given as $b^*(x) = (-x_1, -x_2)^T$, that provides an equation (17) in the form

$$dX_t = \begin{pmatrix} -X_{1,t} \\ -X_{2,t} \end{pmatrix} dt + \begin{pmatrix} -X_{2,t} & 0 \\ X_{1,t} & 0 \end{pmatrix} dB_t.$$

Thus we have constructed equation (18) whose diffusion coefficient coincides with $\hat{\sigma}$ employed in the previous example with S as an absorbing boundary, while its shift coefficient $\hat{b}$ that increases twofold than $\hat{b}$ used to construct the absorbing boundary. Simulation of the corresponding diffusion is shown by the right hand side of Figure 1.

Example 3 *Assume that*

$$f(x_1, x_2) = x_1^2 + x_2^2, \qquad x_1 \ge 0$$

$$= 4x_1^2 + x_2^2, \qquad x_1 < 0,$$

and choose $c = 1$. Consider (1) in the form

$$dX_t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} dB_t,$$

as in Example 2. Even though our f does not belong to $\mathscr{C}^2$ in this case, we still can construct diffusion in $K = [f \le 1]$. For an absorbing boundary we shall procced as follows: The coefficient σ^* has to be defined as

$$\sigma^*(x) = \begin{pmatrix} -x_2h_1(x) & -x_2g_1(x) \\ x_1h_1(x) & x_1g_1(x) \end{pmatrix} \qquad x_1 \ge 0$$

$$= \begin{pmatrix} -x_2h_2(x) & -x_2g_2(x) \\ 4x_1h_2(x) & 4x_1g_2(x) \end{pmatrix} \qquad x_1 < 0, \tag{19}$$

where h_1, h_2, g_1 and g_2 are arbitrary functions. It follows by (8) that

$$2x_1b_1^*(x) + 2x_2b_2^*(x) + (h_1^2(x) + g_1^2(x))\cdot(x_1^2 + x_2^2) = 0 \qquad x_1 \ge 0$$

$$8x_1b_1^*(x) + 2x_2b_2^*(x) + (h_2^2(x) + g_2^2(x))\cdot(x_1^2 + 4x_2^2) = 0 \qquad x_1 < 0.$$

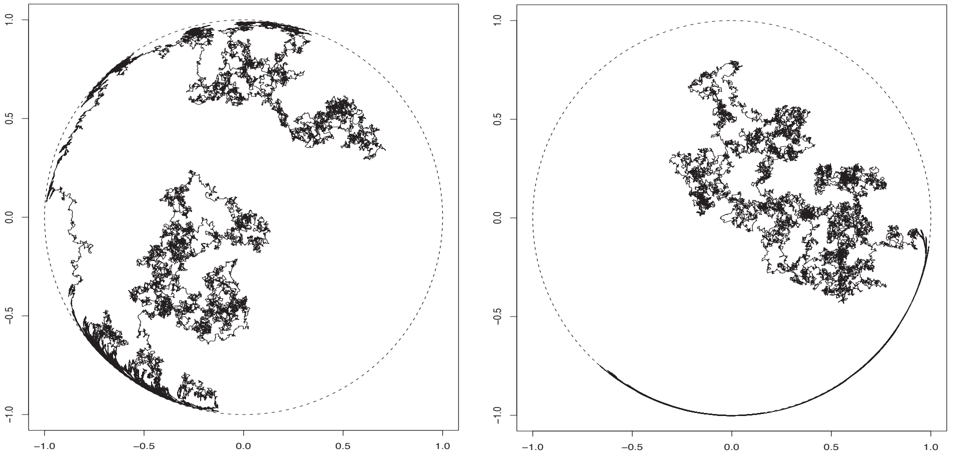

Figure 1: The left hand side shows a diffusion in the unit circle with absorbing boundary S while the right hand one provides a diffusion with S as a reflecting boundary

Wanting the coefficients b^* *and* σ^* *continuous we put*

$$h_1(0, x_2) = h_2(0, x_2) = g_1(0, x_2) = g_2(0, x_2) = b_2^*(0, x_2) = 0.$$

Hence, we choose $h_1(x) = h_2(x) = x_1$, $g_1(x) = g_2(x) = 0$. *Thus*

$$\begin{aligned} b^*(x) &= -\frac{1}{2}(x_1^3, x_1^2 x_2)^T, && x_1 \geq 0 \\ &= -2(x_1^3, x_1^2 x_2)^T, && x_1 < 0. \end{aligned}$$

and equation (17) is given as

$$\begin{aligned} dX_t &= -\frac{1}{2}\begin{pmatrix} X_{1,t}^3 \\ X_{1,t}^2 X_{2,1} \end{pmatrix} dt + \begin{pmatrix} -X_{1,t}X_{2,t} & 0 \\ X_{1,t}^2 & 0 \end{pmatrix} dB_t && X_{1,t} \geq 0 \\ &= -2\begin{pmatrix} X_{1,t}^3 \\ X_{1,t}^2 X_{2,1} \end{pmatrix} dt + \begin{pmatrix} -X_{1,t}X_{2,t} & 0 \\ 4X_{1,t}^2 & 0 \end{pmatrix} dB_t && X_{1,t} < 0. \end{aligned}$$

Choosing $\epsilon = 0.1$ *and denoting* $K^\epsilon := \{x \in K : f(x) \geq 1 - \epsilon\}$ *we define the coefficients in (18) as:*

$$\begin{aligned} \hat{b}(x) &= (0, 0)^T && \text{if} \quad f(x) \leq 1 - \epsilon \\ &= \frac{f(x) - 0.9}{0.1} \cdot b^*(x) && \text{if} \quad 1 - \epsilon < f(x) < 1 \\ &= b^*(x) && \text{if} \quad |x| \geq 1 \end{aligned}$$

and

$$\hat{\sigma}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \textit{if} \quad f(x) \le 1-\epsilon$$
$$= \frac{1-f(x)}{0.1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{f(x)-0.9}{0.1}\sigma^*(x) \qquad \textit{if} \quad 1-\epsilon < f(x) < 1$$
$$= \sigma^*(x) \qquad \textit{if} \quad f(x) \ge 1.$$

The equation (18) defines a diffusion in K with $S = [f = 1]$ *as an absorbing boundary . It is possible to show, that the points* $[0, 1]$ *and* $[0, -1]$ *absorb arbitrary solution X to (18).*

Finally, we shall construct a diffusion with S as an reflecting boundary. To construct the corresponding equation (17) choose σ^* *defined by (19) with* $h_1(x) = h_2(x) = x_1$ *and* $g_1(x) = g_2(x) = 0$. *Further define a shift coefficient* b^* *by*

$$b^*(x) = -2(x_1^3, x_1^2 x_2)^T, \qquad x_1 \ge 0$$
$$= -8(x_1^3, x_1^2 x_2)^T, \qquad x_1 < 0. \tag{20}$$

That quadruplicates the b^* *used to construct S as an absorbing boundary. Obviously the corresponding equation (18) defines a diffusion that has reflecting boundary S.*

Since $b^*(0, x_2) = \sigma^*(0, x_2) = 0$ *holds for the both reflecting and absorbing equation (17) we may combine the pair of them to get an equation that defines a diffusion absorbed by S whenever* $x_1 > 0$ *and reflected one if* $x_1 < 0$. *The corresponding simulations are presented by Figure 2.*

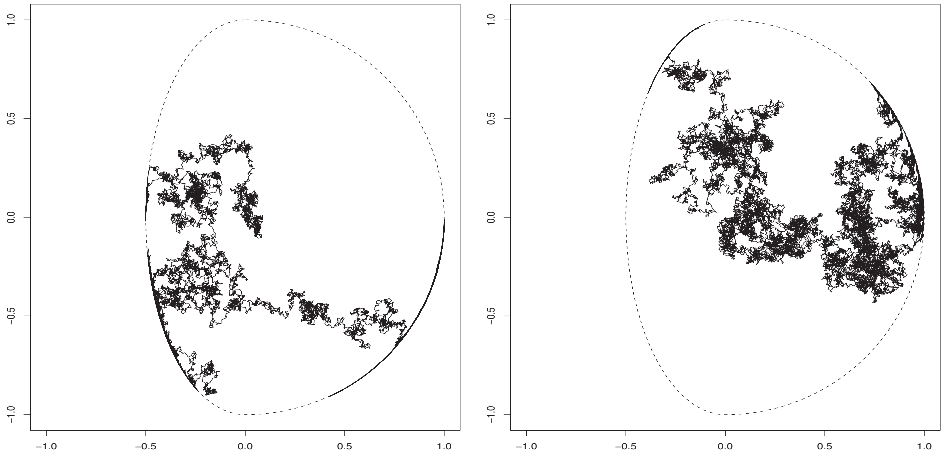

Figure 2: The left hand side visualizes a diffusion with $S \cap [x_1 < 0]$ and $S \cap [x_1 > 0]$ as an absorbing and reflecting boundary, respectively. A diffusion that is absorbed by S if $x_1 > 0$ and reflected by S if $x_1 < 0$ is shown on the left

Example 4 *Having* $f(x_1, x_2) = |x_1| + |x_2|$ *and* $c = 1$ *we get that* $K = [x : f(x) \leq 1]$ *is the square with vertices* $[0,1], [1,0], [-1,0]$ *and* $[0,-1]$. *Because* $f \notin \mathscr{C}^2$, *we shall proceed in a manner of Example 3 to construct a diffusion in the square K.*

First we exhibit a diffusion that makes the boundary $S = \partial K$ *absorbing. The boundary is a union of four* $\mathscr{C}^2$*-curves. In the first quadrant, the curve is given by* $x_1 + x_2 = 1$, *in the second one we have* $-x_1 + x_2 = 1$, *etc. The boundary equation for the boundary given by* $x_1 + x_2 = 1$ *is defined by*

$$dX_1 = h(x) \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} dt + \begin{pmatrix} g_1(x) & g_2(x) \\ -g_1(x) & -g_2(x) \end{pmatrix} dB_t, \tag{21}$$

where h, g_1 *and* g_2 *can be chosen arbitrarily. We chose* h, g_1 *and* g_2 *so that the coefficients of (21) are equal to zero for all x such that* $x_1 = 0$ *or* $x_2 = 0$, *therefore the vertices* $[1,0]$ *and* $[0,1]$ *will be the absorbing points of solution to (21). This equation will be employed as a boundary equation (17) in the first quadrant. Boundary equations for the remaining three quadrants are constructed in the same way. Therefore the equation (17) is defined by*

$$\begin{aligned} dX_1 &= b^*(X_t)dt + \sigma^*(X_t)dB_t \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} X_{1,t}X_{2,t} & 0 \\ -X_{1,t}X_{2,t} & 0 \end{pmatrix} dB_t \qquad \text{if} \quad X_{1,t}X_{2,t} \geq 0 \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} dt + \begin{pmatrix} X_{1,t}X_{2,t} & 0 \\ X_{1,t}X_{2,t} & 0 \end{pmatrix} dB_t \qquad \text{if} \quad X_{1,t}X_{2,t} < 0. \end{aligned}$$

Choose $\epsilon = 0.1$ *and define the coefficients in (18) by*

$$\hat{b}(x) = (0,0)^T$$

and

$$\begin{aligned} \hat{\sigma}(x) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if} \quad & f(x) \leq 1 - \epsilon \\ &= \frac{1 - f(x)}{0.1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{f(x) - 0.9}{0.1} \sigma^*(x) & \text{if} \quad & 1 - \epsilon < f(x) < 1 \\ &= \sigma^*(x) & \text{if} \quad & f(x) \geq 1. \end{aligned}$$

The above equation generates a diffusion in K with absorbing boundary S. It is obvious that the vertices of K are absorbing points for arbitrary solution to (18). A simulation of this diffusion is presented on the left side in Figure 3.

Now, we construct a diffusion with S as a reflecting boundary. We shall first specify the coefficients σ^* *and* b^* *in (17). One can easily verify that a possible choice is*

$$\sigma^*(x) = 0 \tag{22}$$

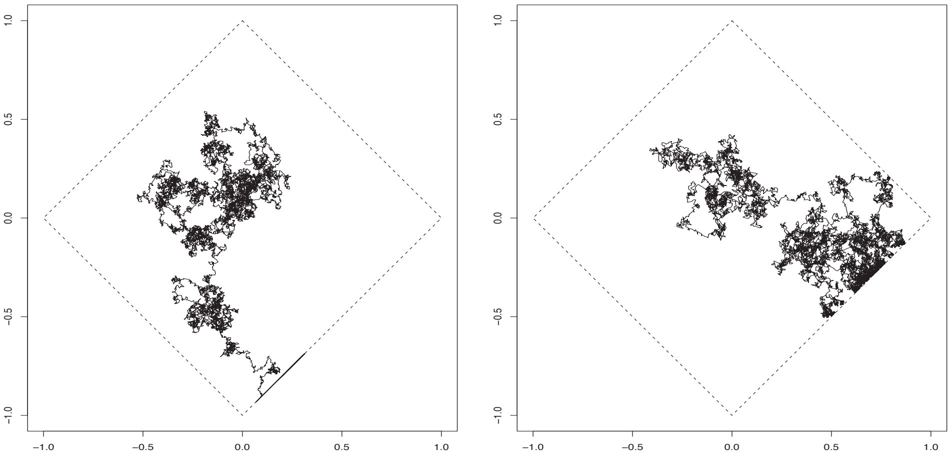

Figure 3: The left hand side shows a diffusion with S as an absorbing boundary, the right hand one a diffusion with reflecting boundary S

and

$$\begin{aligned} b^*(x) &= (-x_1x_2, -x_1x_2)^T && \text{if} \quad x_1 \geq 0, x_2 \geq 0 \\ &= (-x_1x_2, x_1x_2)^T && \text{if} \quad x_1 < 0, x_2 \geq 0 \\ &= (x_1x_2, x_1x_2)^T && \text{if} \quad x_1 < 0, x_2 < 0 \\ &= (x_1x_2, -x_1x_2)^T && \text{if} \quad x_1 \geq 0, x_2 < 0. \end{aligned} \tag{23}$$

The equation (18) with coefficients b^* and σ^* given by (23) and (22) defines a diffusion in K with reflecting boundary S. A simulation of this diffusion is shown on the right in Figure 3.

The functions f considered in Example 3 and Example 4 are not in $\mathscr{C}^2$, hence not in a competence of the Lemmas 3 and 4. Their corresponding suitable localizations read as follows:

Denote S^2 the set of points $x \in S$ such that there exists en open neighborhood $U_x \ni x$ in which the function f is $\mathscr{C}^2$ and $S^1 = S \setminus S^2$.

Lemma 5 *Consider that equation (1) has unique strong solution X and assume that for all $x \in S^2$ there is an open neighborhood $U_x \ni x$ such that (5) and (6) hold for all $y \in U_x \cap K^e$. Moreover suppose that $Lf(x) < 0$ is true for all $x \in S^2$ and $b(x) = \sigma(x) = 0$ hold for all $x \in S^1$.*

Then S^2 is a reflecting boundary, which means that outside a P-null set N, there is no pair $0 \leq u < v < \infty$ such that $X_s \in S^2$ for all $s \in (u, v)$ and all points $x \in S^1$ are absorbing points for X.

Lemma 6 *Consider an equation (1) that has a unique strong solution such that*

$$\tau := \inf\{t \geq 0 : X_t \in S\} < \infty \quad \textit{almost surely if} \quad X_0 \in K.$$

Moreover, assume that there is an equation

$$dX_t = b^*(X_t)dt + \sigma^*(X_t)dB_t \tag{24}$$

where b^* and σ^* are Lipschitz continuous in an open neighborhood $G \supset S$ and for all $x \in S^2$ there exists an open neighborhood $U_x \ni x$ such that for all $y \in U_x$ (8) and (9) hold. Further assume that

$$b^*(x) = b(x), \quad \sigma^*(x) = \sigma(x) \quad \textit{holds for all} \quad x \in S,$$

and $b^*(x) = \sigma^*(x) = 0$ for all $x \in S^1$.

Then S is an absorbing boundary for X.

The points $x \in S^1$ are absorbing points for X due to uniqueness of the solution X. The proof of Lemma 5 and Lemma 6, respectively, for $x \in S^2$ is analogous to the proof of Lemma 3 and Lemma 4, respectively.

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