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ESTIMATES FOR THE COMMUTATOR OF BILINEAR FOURIER MULTIPLIER

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Abstract. Let $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ and T_σ be a bilinear Fourier multiplier operator with associated multiplier σ satisfying the Sobolev regularity that $\sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{s_1,s_2}(\mathbb{R}^{2n})} < \infty$ for some $s_1, s_2 \in (n/2, n]$. In this paper, the behavior on $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ $(p_1, p_2 \in (1,\infty))$, on $H^1(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ $(p_2 \in [2,\infty))$, and on $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, is considered for the commutator $T_{\sigma,\vec{b}}$ defined by

$$T_{\sigma,\vec{b}}(f_1,f_2)(x) = b_1(x)T_\sigma(f_1,f_2)(x) - T_\sigma(b_1f_1,f_2)(x) + b_2(x)T_\sigma(f_1,f_2)(x) - T_\sigma(f_1,b_2f_2)(x).$$

By kernel estimates of the bilinear Fourier multiplier operators and employing some techniques in the theory of bilinear singular integral operators, it is proved that these mapping properties are very similar to those of the bilinear Fourier multiplier operator which were established by Miyachi and Tomita.

Keywords: bilinear Fourier multiplier operator; commutator; Hardy space

MSC 2010: 42B15

1. Introduction

In their seminal works [3], [4], Coifman and Meyer considered the mapping properties of the bilinear Fourier multiplier operator. Let $\sigma \in L^\infty(\mathbb{R}^{2n})$. Define the bilinear Fourier multiplier operator T_σ by

$$T_\sigma(f_1,f_2)(x) = \int_{\mathbb{R}^{2n}} \exp\left(2\pi \mathrm{i} x(\xi_1+\xi_2)\right)\sigma(\xi_1,\xi_2)\hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\,\mathrm{d}\vec{\xi}, \tag{1.1}$$

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initially for $f_1,\ f_2 \in \mathscr{S}(\mathbb{R}^n)$, where $\hat{f}$ denotes the Fourier transform of f and $\mathrm{d}\vec{\xi} = \mathrm{d}\xi_1\,\mathrm{d}\xi_2$. Coifman and Meyer [4] proved that if $\sigma \in C^s(\mathbb{R}^{2n} \setminus \{0\})$ satisfies that

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\sigma(\xi_1,\xi_2)| \leqslant C_{\alpha_1,\alpha_2}(|\xi_1|+|\xi_2|)^{-(|\alpha_1|+|\alpha_2|)} \tag{1.2}$$

for all multiindices $\alpha_1,\ \alpha_2$ such that $|\alpha_1|+|\alpha_2| \leqslant s$ with $s \geqslant 4n+1$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, p_2, p < \infty$ with $1/p = 1/p_1 + 1/p_2$. Using the theory of the multilinear Calderón-Zygmund operator, Grafakos and Torres [9], Kenig and Stein [11] improved the results of Coifman and Meyer, proving that if σ satisfies (1.2) for all $|\alpha_1|+|\alpha_2| \leqslant s$ with $s \geqslant 2n+1$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ when $p_1,p_2 \in [1,\infty]$ and $p \in [1/2,\infty)$ with $1/p = 1/p_1+1/p_2$, and is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $p_1,p_2 \in (1,\infty]$ and $p \in (1/2,\infty)$. In recent years, considerable attention has been paid to the mapping properties of T_σ when σ satisfies some conditions of Hörmander-Mihlin type. Let $\Psi \in \mathscr{S}(\mathbb{R}^{2n})$ be such that $\operatorname{supp}\Psi \subset \{(\xi_1,\xi_2)\colon\ 1/2 \leqslant |\xi_1|+|\xi_2| \leqslant 2\}$ and $\sum_{\kappa\in\mathbb{Z}} \Psi(2^{-\kappa}\xi_1, 2^{-\kappa}\xi_2) = 1$ for all $(\xi_1,\xi_2) \in \mathbb{R}^{2n}\setminus\{0\}$. Set

$$\sigma_\kappa(\xi_1,\xi_2) = \Psi(\xi_1,\xi_2)\sigma(2^\kappa\xi_1, 2^\kappa\xi_2). \tag{1.3}$$

Tomita [17] proved that if $\sup_{\kappa\in\mathbb{Z}}\|\sigma_\kappa\|_{W^s(\mathbb{R}^{2n})} < \infty$ for some $s > n$, then T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $p_1,p_2,p \in (1,\infty)$ and $1/p = 1/p_1+1/p_2$. Grafakos and Si [8] considered the mapping properties from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for T_σ when $p \leqslant 1$. Fairly recently, Miyachi and Tomita [15] considered the problem to find the minimal smoothness conditions for the boundedness of T_σ. For $s_1, s_2 \in (0,\infty)$, define

$$W^{s_1,s_2}(\mathbb{R}^{2n}) := \{f \in L^2(\mathbb{R}^{2n})\colon\ \|f\|_{W^{s_1,s_2}(\mathbb{R}^{2n})} < \infty\},$$

with

$$\|f\|^2_{W^{s_1,s_2}(\mathbb{R}^{2n})} = \int_{\mathbb{R}^{2n}} \langle\xi_1\rangle^{2s_1}\langle\xi_2\rangle^{2s_2}|\hat{f}(\xi_1,\xi_2)|^2\,\mathrm{d}\vec{\xi},$$

where $\langle\xi_k\rangle = (1+|\xi_k|^2)^{1/2}$. Miyachi and Tomita [15] proved the following result.

Theorem 1.1. *Let $\sigma \in L^\infty(\mathbb{R}^{2n})$ and let T_σ be the operator defined by* (1.1). *If σ satisfies that*

$$\sup_{\kappa\in\mathbb{Z}}\|\sigma_\kappa\|_{W^{s_1,s_2}(\mathbb{R}^{2n})} < \infty \tag{1.4}$$

for $s_1,s_2 \in (n/2,n]$, then for $p_1,p_2 \in (1,\infty]$, $p \in [2/3,\infty)$ with $1/p = 1/p_1+1/p_2$, T_σ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Moreover,

(i) *if* $p_1 \in (0,1]$ *and* $p_2 \in [2,\infty)$, σ *satisfies* (1.4) *for some* $s_1 \in (n/p_1,\infty)$ *and* $s_2 \in (n/2,n]$, *then* T_σ *is bounded from* $H^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$;

(ii) *if* $p_1, p_2 \in (0,1]$, σ *satisfies* (1.4) *for some* $s_1 \in (n/p_1 - n/2,\infty)$, $s_2 \in (n/p_2 - n/2,\infty)$ *and* $s_1 + s_2 > n/p_1 + n/p_2 - n/2$, *then* T_σ *is bounded from* $H^{p_1}(\mathbb{R}^n) \times H^{p_2}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$.

Now let us consider the commutator of T_σ. For a function $b \in \mathrm{BMO}(\mathbb{R}^n)$, set

$$[b,T_\sigma]^1(f_1,f_2)(x) = b(x)T_\sigma(f_1,f_2)(x) - T_\sigma(bf_1,f_2)(x),$$

and

$$[b,T_\sigma]^2(f_1,f_2)(x) = b(x)T_\sigma(f_1,f_2)(x) - T_\sigma(f_1,bf_2)(x),$$

initially for $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$. Let $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ and $\vec{b} = (b_1,b_2)$. Define the commutator generated by $\vec{b}$ and T_σ by

$$T_{\sigma,\vec{b}}(f_1,f_2)(x) = \sum_{k=1}^{2}[b_k,T_\sigma]^k(f_1,f_2)(x). \tag{1.5}$$

By the result of Lernel et al. [13], we know that if σ satisfies (1.2) for all $|\alpha_1|+|\alpha_2| \leqslant s$ with $s \geqslant 2n+1$, then $T_{\sigma,\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ provided that $p_1, p_2 \in (1,\infty]$ and $p \in (1/2,\infty)$ with $1/p = 1/p_1 + 1/p_2$, and enjoys an endpoint estimate of $L\log L \times L\log L$ type. Anh and Duong [1] considered the weighted estimates with multiple weights for $T_{\sigma,\vec{b}}$ when σ satisfies (1.2) for $n < s \leqslant 2n$.

Our first purpose of this paper is to consider the behavior on the product of Lebesgue spaces for the commutator $T_{\sigma,\vec{b}}$. We will show that the behavior of $T_{\sigma,\vec{b}}$ on $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ is similar to that of T_σ. More precisely, we have

Theorem 1.2. *Let* $\sigma \in L^\infty(\mathbb{R}^{2n})$ *and let* T_σ *be the operator defined by* (1.1). *If*

(i) σ *satisfies* (1.4) *for* $s_1, s_2 \in (n/2,n]$, $p_1, p_2 \in (1,\infty)$, $p \in [2/3,\infty)$ *with* $1/p = 1/p_1 + 1/p_2$,

or

(ii) σ *satisfies* (1.4) *for* $s_1, s_2 \in (n/2,n]$ *and* $s_1 + s_2 > (3/2)n$, $p_1, p_2 \in (1,\infty)$ *and* $p \in (1/2,\infty)$ *with* $1/p = 1/p_1 + 1/p_2$,

then $T_{\sigma,\vec{b}}$ *is bounded from* $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ *to* $L^p(\mathbb{R}^n)$.

Remark 1.1. By Theorem 1.2 and the argument used in [15], it follows that if $p_1, p_2 \in (1,\infty)$, $p \in (1/2, 2/3)$ with $1/p = 1/p_1 + 1/p_2$, and σ satisfies (1.4) for $s_1, s_2 \in (n/2,n]$ such that $s_1 + s_2 > n/p_1 + n/p_2 - n/2$, then $T_{\sigma,\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

We will also consider the endpoint estimates for $T_{\sigma,\vec{b}}$.

Theorem 1.3. *Let $\sigma \in L^\infty(\mathbb{R}^{2n})$ and let T_σ be the operator defined by* (1.1).

(a) *If $p_2 \in [2,\infty)$ and σ satisfies* (1.4) *for $s_1, s_2 \in (n/2, n]$, then $T_{\sigma,\vec{b}}$ is bounded from $H^1(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ with $1/p = 1 + 1/p_2$;*

(b) *if σ satisfies* (1.4) *for $s_1, s_2 \in (n/2, n]$ with $s_1 + s_2 > 3n/2$, then $T_{\sigma,\vec{b}}$ is bounded from $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ to $L^{1/2,\infty}(\mathbb{R}^n)$.*

Remark 1.2. Our proof of Theorem 1.2, which will be given in Section 2, also applies to the iterated commutator of T_σ defined by

$$T^*_{\sigma,\vec{b}}(f_1, f_2)(x) = [b_1, [b_2, T_\sigma]^2]^1(f_1, f_2)(x).$$

However, we do not know if Theorem 1.3 is true for $T^*_{\sigma,\vec{b}}$.

We make some conventions. In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \leqslant CB$. For $x \in \mathbb{R}^n$ and $r > 0$, $B(x,r)$ denotes the ball centered at x and has radius r. For any set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For a ball B in $\mathbb{R}^n$ and $\lambda \in (0,\infty)$, we use λB to denote the ball with the same center as B whose radius is λ times that of B. For any $p \in [1,\infty)$, we use p' to denote the dual exponent of p, namely, $p' = p/(p-1)$. Let M be the Hardy-Littlewood maximal operator. For $r \in (0,\infty)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, the maximal operators M_r and $M_{b,r}$ are defined as

$$\begin{aligned} M_r f(x) &= (M(|f|^r)(x))^{1/r}, \\ M_{b,r} f(x) &= \sup_{B \ni x} \left(\frac{1}{|B|} \int_B |(b(x) - b(y)) f(y)|^r \,\mathrm{d}y \right)^{1/r} \end{aligned}$$

respectively. It is well known that for $t \in (r,\infty)$,

$$\|M_{b,r} f\|_{L^t(\mathbb{R}^n)} \leqslant C \|b\|_{\mathrm{BMO}} \|f\|_{L^t(\mathbb{R}^n)}, \tag{1.6}$$

see [6]. For a locally integrable function f, $M^\sharp f$ denotes the Fefferman-Stein sharp maximal function of f, that is,

$$M^\sharp f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(x) - m_B(f)| \,\mathrm{d}x,$$

where the supremum is taken over all balls containing x, and $m_B(f)$ denotes the mean value of f on B. For $r \in (0,\infty)$, let $M^\sharp_r$ be the maximal operator defined by $M^\sharp_r f(x) = \{M^\sharp(|f|^r)(x)\}^{1/r}$.

2. Proof of Theorem 1.2

Let $\Psi \in \mathscr{S}(\mathbb{R}^{2n})$ and let σ_κ be the same as in (1.3). Define $\tilde{\sigma}_\kappa$ by

$$\tilde{\sigma}_\kappa(\xi_1, \xi_2) = \sigma_\kappa(2^{-\kappa}\xi_1, 2^{-\kappa}\xi_2).$$

It is obvious that

$$\mathcal{F}^{-1}\tilde{\sigma}_\kappa(\xi_1, \xi_2) = 2^{2\kappa n}\mathcal{F}^{-1}\sigma_\kappa(2^\kappa\xi_1, 2^\kappa\xi_2),$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. For $x, y_1, y_2, y_1', y_2', x' \in \mathbb{R}^n$, let

$$\begin{aligned}
W_{0,\kappa}(x, y_1, y_2; x') &= \mathcal{F}^{-1}\tilde{\sigma}_\kappa(x - y_1, x - y_2) - \mathcal{F}^{-1}\tilde{\sigma}_\kappa(x' - y_1, x' - y_2),\\
W_{1,\kappa}(x, y_1, y_2; y_1') &= \mathcal{F}^{-1}\tilde{\sigma}_\kappa(x - y_1, x - y_2) - \mathcal{F}^{-1}\tilde{\sigma}_\kappa(x - y_1', x - y_2),\\
W_{2,\kappa}(x, y_1, y_2; y_2') &= \mathcal{F}^{-1}\tilde{\sigma}_\kappa(x - y_1, x - y_2) - \mathcal{F}^{-1}\tilde{\sigma}_\kappa(x - y_1, x - y_2').
\end{aligned}$$

Lemma 2.1. *Let σ_k be defined by* (1.3), $q_1, q_2 \in [2, \infty)$ *and* $s_1, s_2 \geqslant 0$. *Then*

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}|\hat{\sigma}_\kappa(\xi_1, \xi_2)|^{q_2}\langle\xi_2\rangle^{s_2}\,\mathrm{d}\xi_2\right)^{q_1/q_2}\langle\xi_1\rangle^{s_1}\,\mathrm{d}\xi_1\right)^{1/q_1} \lesssim \|\sigma_\kappa\|_{W^{s_1/q_1, s_2/q_2}(\mathbb{R}^{2n})}.$$

For the proof of Lemma 2.1, see [5].

Lemma 2.2. *Let σ be a bilinear multiplier which satisfies* (1.4) *for some* $s_1, s_2 \in (n/2, n]$, $u_1, u_2 \in (1, 2]$, *B be a ball with radius R and* $x, x' \in (1/4)B$.

(i) *For nonnegative integers j_1, j_2 and an integer κ with $2^\kappa R < 1$,*

$$\begin{aligned}
&\left(\int_{S_{j_1}(B)}\left(\int_{S_{j_2}(B)}|W_{0,\kappa}(x, y_1, y_2; x')|^{u_2'}\,\mathrm{d}y_2\right)^{u_1'/u_2'}\mathrm{d}y_1\right)^{1/u_1'}\\
&\qquad\lesssim 2^\kappa R\frac{2^{-\kappa(s_1+s_2-n/u_1-n/u_2)}}{\prod_{k=1}^{2}(2^{j_k}R)^{s_k}},
\end{aligned}$$

where and in the sequel $S_0(B) = B$ and for positive integer j, $S_j(B) = 2^jB \setminus 2^{j-1}B$;

(ii) *for positive integers j_1, j_2 and an integer κ,*

$$\begin{aligned}
&\left(\int_{S_{j_1}(B)}\left(\int_{S_{j_2}(B)}|\mathcal{F}^{-1}\tilde{\sigma}_\kappa(x - y_1, x - y_2)|^{u_2'}\,\mathrm{d}y_2\right)^{u_1'/u_2'}\mathrm{d}y_1\right)^{1/u_1'}\\
&\qquad\lesssim \frac{2^{-\kappa(s_1+s_2-n/u_1-n/u_2)}}{\prod_{k=1}^{2}(2^{j_k}R)^{s_k}};
\end{aligned}$$

(iii) *if* $u_2 > n/s_2$, *then for any* $\mu \in (1,\infty)$, *integer* κ *with* $2^\kappa R > 1$, *positive integer* j *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\int_{S_j(B)}\int_B |\mathcal{F}^{-1}\tilde{\sigma}_\kappa(x-y_1,x-y_2)|\,|f_2(y_2)|\,\mathrm{d}y_2|f_1(y_1)|\,\mathrm{d}y_1 \lesssim 2^{-\kappa(s_1-n/u_1)}(2^jR)^{-s_1}M_{u_2}f_2(x)\Big(\int_{S_j(B)}|f_1(y_1)|^{u_1}\,\mathrm{d}y_1\Big)^{1/u_1} \tag{2.1}$$

and

$$\int_{S_j(B)}\int_B |\mathcal{F}^{-1}\tilde{\sigma}_\kappa(x-y_1,x-y_2)|\,|b(y_2)-m_B(b)|\,|f_2(y_2)|\,\mathrm{d}y_2|f_1(y_1)|\,\mathrm{d}y_1 \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\frac{\log(2^\kappa R)}{2^{\kappa(s_1-n/u_1)}(2^jR)^{s_1}}M_{\mu u_2}f_2(x)\Big(\int_{S_j(B)}|f_1(y_1)|^{u_1}\,\mathrm{d}y_1\Big)^{1/u_1}. \tag{2.2}$$

Proof. The conclusion (i) is just Lemma 3.5 in [10], and the conclusion (ii) can be proved by an argument similar to the proof of Lemma 3.3 in [10]. For the conclusion (iii), we only consider the estimate (2.2), since the inequality (2.1) can be proved in the same way. A straightforward computation involving the Hölder inequality gives us that

$$\begin{aligned}
&\int_{S_j(B)}\int_B |\mathcal{F}^{-1}\tilde{\sigma}_\kappa(x-y_1,x-y_2)|\,|b(y_2)-m_B(b)|\,|f_2(y_2)|\,\mathrm{d}y_2|f_1(y_1)|\,\mathrm{d}y_1\\
&\lesssim 2^{2\kappa n}\Big(\int_{S_j(B)}\Big(\int_B |\mathcal{F}^{-1}\sigma_\kappa(2^\kappa(x-y_1),2^\kappa(x-y_2))|^{u_2'}\langle 2^\kappa(x-y_2)\rangle^{u_2's_2}\,\mathrm{d}y_2\Big)^{u_1'/u_2'}\\
&\quad\times\langle 2^\kappa(x-y_1)\rangle^{u_1's_1}\,\mathrm{d}y_1\Big)^{1/u_1'}\Big(\int_B\frac{|b(y_2)-m_B(b)|^{u_2\mu'}}{\langle 2^\kappa(x-y)\rangle^{u_2s_2}}\,\mathrm{d}y_2\Big)^{1/(u_2\mu')}\\
&\quad\times(2^\kappa 2^jR)^{-s_1}\Big(\int_B\frac{|f_2(y)|^{u_2\mu}}{\langle 2^\kappa(x-y)\rangle^{u_2s_2}}\,\mathrm{d}y\Big)^{1/(u_2\mu)}\Big(\int_{S_j(B)}|f_1(y_1)|^{u_1}\,\mathrm{d}y_1\Big)^{1/u_1}.
\end{aligned}$$

Let N be the positive integer such that $2^{N-1} < 2^\kappa R \leqslant 2^N$; it follows from the John-Nirenberg inequality that

$$\begin{aligned}
&\int_B\frac{|b(y)-m_B(b)|^{u_2\mu'}}{\langle 2^\kappa(x-y)\rangle^{u_2s_2}}\,\mathrm{d}y\\
&=\int_{|x-y|\leqslant 2^{-\kappa}}|b(y)-m_B(b)|^{u_2\mu'}\,\mathrm{d}y+\sum_{j=1}^{N}\int_{2^j2^{-\kappa}<|x-y|\leqslant 2^{j+1}2^{-\kappa}}\frac{|b(y)-m_B(b)|^{u_2\mu'}}{|2^\kappa(x-y)|^{u_2s_2}}\,\mathrm{d}y
\end{aligned}$$

$$\lesssim 2^{-\kappa n}(1+|m_B(b)-m_{B(x,2^{-\kappa})}(b)|^{u_2\mu'})+2^{-\kappa n}\sum_{j=1}^{N}\frac{|m_B(b)-m_{B(x,2^j2^{-\kappa})}(b)|^{u_2\mu'}}{2^{j(s_2u_2-n)}}$$

$$\lesssim 2^{-\kappa n}\log^{u_2\mu'}(2^\kappa R)\|b\|_{\mathrm{BMO}}^{\mu' u_2}.$$

Note that

$$\left(\int_B \frac{|f_2(y_2)|^{u_2\mu}}{\langle 2^\kappa(x-y_2)\rangle^{u_2 s_2}}\,\mathrm{d}y_2\right)^{1/(u_2\mu)} \lesssim 2^{-\kappa n/\mu u_2} M_{u_2\mu} f_2(x).$$

The estimate (2.2) now follows from Lemma 2.1 and the estimates above. □

Lemma 2.3. *Let σ be a multiplier which satisfies* (1.4) *for some $s_1,s_2\in(n/2,n]$ and $u_1,u_2\in(1,2]$, let B be a ball with radius R and $y_l,y_l'\in\frac{1}{4}B$ with $l=1,2$.*

(i) *For nonnegative integers j_0,j_1,j_2 and an integer κ with $2^\kappa R<1$,*

$$\left(\int_{S_{j_0}(B)}\left(\int_{E_{j_2}^R(x)}|W_{1,\kappa}(x,y_1,y_2;y_1')|^{u_2'}\,\mathrm{d}y_2\right)^{u_1'/u_2'}\mathrm{d}x\right)^{1/u_1'} \lesssim 2^\kappa R\frac{2^{-\kappa(s_1+s_2-n/u_1-n/u_2)}}{(2^{j_0}R)^{s_1}(2^{j_2}R)^{s_2}},$$

and

$$\left(\int_{S_{j_0}(B)}\left(\int_{E_{j_1}^R(x)}|W_{2,\kappa}(x,y_1,y_2;y_2')|^{u_1'}\,\mathrm{d}y_1\right)^{u_2'/u_1'}\mathrm{d}x\right)^{1/u_2'} \lesssim 2^\kappa R\frac{2^{-\kappa(s_1+s_2-n/u_1-n/u_2)}}{(2^{j_0}R)^{s_2}(2^{j_1}R)^{s_1}},$$

where and in the sequel, $E_0^R(x)=B(x,R)$ and for any positive integer j, $E_j^R(x)=2^jB(x,R)\setminus 2^{j-1}B(x,R)$;

(ii) *for each $k=1,2$ and each integer κ, there exists a function $\mathrm{H}_{k,\kappa,B}$ such that for functions f_1,f_2 with* $\operatorname{supp} f_1$, $\operatorname{supp} f_2\subset B$,

$$\int_{\mathbb{R}^{2n}}|W_{k,\kappa}(x,y_1,y_2;y_k')|\prod_{l=1}^{2}f_l(y_l)\,\mathrm{d}\vec{y} \lesssim \int_{\mathbb{R}^n}|f_k(y_k)|\mathrm{H}_{k,\kappa,B}(x,y_k,y_k')\,\mathrm{d}y_k\prod_{1\leqslant l\leqslant 2,l\neq k}M_{r_l}f_l(x),$$

and for any integer $j\geqslant 3$,

$$\left(\int_{S_j(B)}|\mathrm{H}_{k,\kappa,B}(x,y_k,y_k')|^{u_k'}\,\mathrm{d}x\right)^{1/u_k'} \lesssim \frac{R^{(s_k-n/u_k)}}{|2^jB|^{1/s_k}};$$

(iii) *if* $u_2 > n/s_2$, *then for any integer* κ *and positive integer* j, *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$,

$$\int_{S_j(B)} \int_B |\mathcal{F}^{-1}\tilde{\sigma}_\kappa(x-y_1,x-y_2)|\,|b(x)-b(y_2)|\,|f_2(y_2)|\,\mathrm{d}y_2|f_1(y_1)|\,\mathrm{d}y_1$$
$$\lesssim 2^{-\kappa(s_1-n/u_1)}(2^jR)^{-s_1}M_{b,u_2}f_2(x)\bigg(\int_{S_j(B)}|f_1(y_1)|^{u_1}\,\mathrm{d}y_1\bigg)^{1/u_1}.$$

For the conclusions (i) and (ii) of Lemma 2.3, see [10]. The conclusion (iii) can be proved by repeating the proof of (iii) in Lemma 2.2.

Lemma 2.4. *Let* $\theta \in (0,1)$, $0 < p_j, p_{j,k} \leqslant \infty$ *and* $s_{j,k} > n/2$ *where* $j = 1,2$ *and* $k = 1,2$. *Set* $1/p = 1/p_1 + 1/p_2$, $1/p_k = (1-\theta)/p_{1,k} + \theta/p_{2,k}$, *and* $s_k = (1-\theta)s_{1,k} + \theta s_{2,k}$. *Suppose that the commutator* $T_{\sigma,\vec{b}}$ *satisfies*

$$\|T_{\sigma,\vec{b}}\|_{L^{p_{1,1}}(\mathbb{R}^n)\times L^{p_{1,2}}(\mathbb{R}^n)\to L^{p_1}(\mathbb{R}^n)} \lesssim \sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{(s_{1,1},s_{1,2})}(\mathbb{R}^{2n})},$$

and

$$\|T_{\sigma,\vec{b}}\|_{L^{p_{2,1}}(\mathbb{R}^n)\times L^{p_{2,2}}(\mathbb{R}^n)\to L^{p_2}(\mathbb{R}^n)} \lesssim \sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{(s_{2,1},s_{2,2})}(\mathbb{R}^{2n})}.$$

Then

$$\|T_{\sigma,\vec{b}}\|_{L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)\to L^{p}(\mathbb{R}^n)} \lesssim \sup_{\kappa\in\mathbb{Z}} \|\sigma_\kappa\|_{W^{(s_1,s_2)}(\mathbb{R}^{2n})}.$$

This lemma can be proved by repeating the argument used in the proof of Theorem 6.1 in [7]. We omit the details for brevity.

For $\kappa \in \mathbb{Z}$, let $T_{\tilde{\sigma}_\kappa}$ be the operator defined by

$$T_{\tilde{\sigma}_\kappa}(f_1,f_2)(x) = \int_{\mathbb{R}^{2n}} \tilde{\sigma}_\kappa(x-y_1,x-y_2)f_1(y_1)f_2(y_2)\,\mathrm{d}\vec{y}, \tag{2.3}$$

and set

$$T_\sigma^N(f_1,f_2)(x) = \sum_{|\kappa|<N} T_{\tilde{\sigma}_\kappa}(f_1,f_2)(x). \tag{2.4}$$

For $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, we define $T^N_{\sigma,\vec{b}}$, the commutator of T^N_σ, as in (1.5).

Lemma 2.5. *Let $\sigma \in L^\infty(\mathbb{R}^{2n})$ which satisfies (1.4) for some $s_1, s_2 \in (n/2, n]$, let T_σ^N be the operator defined by (2.4), $T_{\sigma,\vec{b}}^N$ the commutator of T_σ^N. Let $t_k = n/s_k$ with $k = 1,2$. Then for any $p_k \in (t_k, \infty)$ $(k = 1,2)$, $1/p = 1/p_1 + 1/p_2$, and $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$,*

$$\|T_{\sigma,\vec{b}}^N(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \leqslant C \sum_{l=1}^2 \|b_l\|_{\mathrm{BMO}} \prod_{k=1}^2 \|f_k\|_{L^{p_k}(\mathbb{R}^n)},$$

with C independent of N.

Proof. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. We first claim that for any $r_k \in (t_k, \infty)$ $(k = 1,2)$, $0 < \delta < \varepsilon < \min\{r/r_1, r/r_2\}$ with $1/r = 1/r_1 + 1/r_2$,

$$M_\delta^\sharp([b, T_\sigma^N]^1(f_1, f_2))(x) \leqslant C\|b\|_{\mathrm{BMO}}\biggl(M_\varepsilon(T_\sigma^N(f_1, f_2))(x) + \prod_{k=1}^2 M_{r_k} f_k(x)\biggr) \tag{2.5}$$

for bounded functions f_1, f_2 with compact supports.

The proof of (2.5) is fairly standard, see [13], [16]. Without loss of generality, we may assume that $\|b\|_{\mathrm{BMO}} = 1$. Let $x \in \mathbb{R}^n$ and let B be a ball containing x. Decompose f_k $(k = 1,2)$ as

$$f_k(y) = f_k(y)\chi_{4B}(y) + f_k(y)\chi_{\mathbb{R}^n \setminus 4B}(y) := f_k^1(y) + f_k^2(y).$$

Let $\Lambda = \{(i_1, i_2)\colon\ i_1, i_2 \in \{1,2\}, (i_1, i_2) \neq (1,1)\}$. For $(i_1, i_2) \in \Lambda$, set

$$\mathrm{L}_{i_1,i_2}(z, z_0) = T_\sigma^N((b - m_{4B}(b))f_1^{i_1}, f_2^{i_2})(z) - T_\sigma^N((b - m_{4B}(b))f_1^{i_1}, f_2^{i_2})(z_0).$$

Let $f_1^B(y) = f_1^1(y)(b(y) - m_{4B}(b))$. Take $s > 1$ such that $\delta s < \varepsilon$. An application of the Hölder inequality then gives that

$$\begin{aligned}
&\biggl(\frac{1}{|B|}\int_B |(b(z) - m_{4B}(b))T_\sigma^N(f_1, f_2)(z)|^\delta \,\mathrm{d}z\biggr)^{1/\delta} \\
&\quad \lesssim \biggl(\frac{1}{|B|}\int_B |b(z) - m_{4B}(b)|^{\delta s'} \,\mathrm{d}z\biggr)^{1/\delta s'} \biggl(\frac{1}{|B|}\int_B |T_\sigma^N(f_1, f_2)(z)|^{\delta s} \,\mathrm{d}z\biggr)^{1/\delta s} \\
&\quad \lesssim M_\varepsilon(T_\sigma^N(f_1, f_2))(x).
\end{aligned}$$

Let $u_k \in (t_k, r_k)$ $(k = 1,2)$ such that $1 + n/u_1 + n/u_2 > s_1 + s_2$. Since T_σ^N is bounded from $L^{u_1}(\mathbb{R}^n) \times L^{u_2}(\mathbb{R}^n)$ to $L^u(\mathbb{R}^n)$ with $1/u = 1/u_1 + 1/u_2$ and the bound is independent of N, see [5], it is easy to verify that

$$\begin{aligned}
&\left(\frac{1}{|B|}\int_B |T_\sigma^N(f_1^B,f_2^1)(z)|^u\,\mathrm{d}z\right)^{1/u}\\
&\quad\lesssim \left(\frac{1}{|B|}\int_{4B}|f_1^B(z)|^{u_1}\,\mathrm{d}z\right)^{1/u_1}\left(\frac{1}{|B|}\int_{4B}|f_2^1(z)|^{u_2}\,\mathrm{d}z\right)^{1/u_2}\\
&\quad\lesssim \prod_{k=1}^m M_{r_k}f_k(x).
\end{aligned}$$

Lemma 2.2, via a trivial computation, tells us that for each $z\in B$ and $z_0\in B$ satisfying $|T_\sigma((b-m_{4B}(b))f_1^1,f_2^2)(z_0)|<\infty$,

$$\begin{aligned}
&|T_\sigma^N((b-m_{4B}(b))f_1^1,f_2^2)(z)-T_\sigma^N((b-m_{4B}(b))f_1^1,f_2^2)(z_0)|\\
&\lesssim \sum_{\{\kappa:\,2^\kappa R\leqslant 1\}}\sum_{j_2=1}^\infty\int_{S_{j_2}(4B)}\int_{4B}|W_{0,\kappa}(z,y_1,y_2;z_0)|\,|f_1^B(y_1)|\,\mathrm{d}y_1|f_2(y_2)|\,\mathrm{d}y_2\\
&\quad+\sum_{\{\kappa:\,2^\kappa R>1\}}\sum_{j_2=1}^\infty\int_{S_{j_2}(4B)}\int_{4B}|\mathcal{F}^{-1}\tilde\sigma_\kappa(z-y_1,z-y_2)|\,|f_1^B(y_1)|\,\mathrm{d}y_1|f_2(y_2)|\,\mathrm{d}y_2\\
&\quad+\sum_{\{\kappa:\,2^\kappa R>1\}}\sum_{j_2=1}^\infty\int_{S_{j_2}(4B)}\int_{4B}|\mathcal{F}^{-1}\tilde\sigma_\kappa(z_0-y_1,z_0-y_2)|\,|f_1^B(y_1)|\,\mathrm{d}y_1|f_2(y_2)|\,\mathrm{d}y_2\\
&\lesssim \sum_{\{\kappa:\,2^\kappa R\leqslant 1\}}\sum_{j_2=1}^\infty R\frac{2^{\kappa(1+n/u_1+n/u_2)}}{2^{\kappa(s_1+s_2)}R^{s_1}(2^{j_2}R)^{s_2}}\left(\int_B|f_1^B(y)|^{u_1}\,\mathrm{d}y\right)^{1/u_1}(2^{j_2}R)^{n/u_2}M_{u_2}f_2(z)\\
&\quad+\sum_{\{\kappa:\,2^\kappa R>1\}}\sum_{j_2=1}^\infty\frac{\log(2^\kappa R)}{2^{\kappa(s_2-n/u_2)}(2^{j_2}R)^{s_2}}M_{\mu u_1}f_1(z)\left(\int_{S_{j_2}(B)}|f_2(y)|^{u_2}\,\mathrm{d}y\right)^{1/u_2}\\
&\quad+\sum_{\{\kappa:\,2^\kappa R>1\}}\sum_{j_2=1}^\infty\frac{\log(2^\kappa R)}{2^{\kappa(s_2-n/u_2)}(2^{j_2}R)^{s_2}}M_{\mu u_1}f_1(z_0)\left(\int_{S_{j_2}(B)}|f_2(y)|^{u_2}\,\mathrm{d}y\right)^{1/u_2}\\
&\lesssim \prod_{k=1}^2(M_{r_k}f_k(z)+M_{r_k}f_k(z_0)),
\end{aligned}$$

if we choose $\mu\in(1,r_1/u_1)$. Similarly, we have that for $(i_1,i_2)=(2,1)$ or $(i_1,i_2)=(2,2)$, each $z\in B$ and $z_0\in B$ satisfying $|T_\sigma^N((b-m_{4B}(b))f_1^{i_1},f_2^{i_2})(z_0)|<\infty$,

$$|\mathrm{L}_{i_1,i_2}(z,z_0)|\lesssim\prod_{k=1}^2(M_{r_k}f_k(z)+M_{r_k}f_k(z_0)).$$

Therefore,

$$\sum_{(i_1,i_2)\in\Lambda}\left(\frac{1}{|B|^2}\int_B\int_B |\mathrm{L}_{i_1,i_2}(z,z_0)|^\delta\,\mathrm{d}z\,\mathrm{d}z_0\right)^{1/\delta}$$
$$\lesssim\left(\frac{1}{|B|}\int_B\left\{\prod_{k=1}^2 M_{r_k}f_k(z)\right\}^\delta\mathrm{d}z\right)^{1/\delta}\lesssim\prod_{k=1}^2 M_{r_k}f_k(x).$$

Note that

$$\left(\frac{1}{|B|}\int_B |[b,T_\sigma^N]^1(f_1,f_2)(z)-c|^\delta\,\mathrm{d}z\right)^{1/\delta}\lesssim\left(\frac{1}{|B|}\int_B |T_\sigma^N(f_1^B,f_2^1)(z)|^\delta\,\mathrm{d}z\right)^{1/\delta}$$
$$+\left(\frac{1}{|B|}\int_B |(b(z)-m_{4B}(b))T_\sigma^N(f_1,f_2)(z)|^\delta\,\mathrm{d}z\right)^{1/\delta}$$
$$+\sum_{(i_1,i_2)\in\Lambda}\left(\frac{1}{|B|^2}\int_B\int_B |\mathrm{L}_{i_1,i_2}(z,z_0)|^\delta\,\mathrm{d}z\,\mathrm{d}z_0\right)^{1/\delta}.$$

The desired estimate (2.5) then follows directly.

We now conclude the proof of Lemma 2.5. It suffices to prove that the commutator $[b,T_\sigma^N]^1$ is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}}$ and C independent of N. Let $p_k\in(t_k,\infty)$ $(k=1,2)$. By Theorem 6.1 in [5], we know that T_σ^N is bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with bound independent of N. So, for $b\in L^\infty(\mathbb{R}^n)$ and bounded functions f_1,f_2 with compact supports, $[b,T_\sigma^N]^1(f_1,f_2)\in L^p(\mathbb{R}^n)$. This, together with (2.5) for $r_k\in(t_k,p_k)$ $(k=1,2)$, implies that

$$\|[b,T_\sigma^N]^1(f_1,f_2)\|_{L^p(\mathbb{R}^n)}\lesssim\|b\|_{\mathrm{BMO}}\left(\|M_\varepsilon(T_\sigma^N(f_1,f_2))\|_{L^p(\mathbb{R}^n)}+\prod_{k=1}^2\|M_{r_k}f_k\|_{L^{p_k}(\mathbb{R}^n)}\right)$$
$$\lesssim\|b\|_{\mathrm{BMO}}\prod_{k=1}^2\|f_k\|_{L^{p_k}(\mathbb{R}^n)},$$

provided that $b\in L^\infty(\mathbb{R}^n)$ and f_1,f_2 are bounded functions with compact supports. A standard argument shows that $[b,T_\sigma^N]^1$ can be extended to a bounded operator from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}}$ and C independent of N. □

Lemma 2.6. *Let $\sigma \in L^\infty(\mathbb{R}^{2n})$ which satisfies (1.4) for some $s_1, s_2 \in (n/2, n]$, T_σ^N be the operator defined by (1.1) and let $T_{\sigma,\vec{b}}^N$ be its commutator. Let $t_k = n/s_k$, $k = 1, 2$. Then for $p_1 \in (1, \infty)$, $p_2 \in (t_2, \infty)$ and $1/p = 1/p_1 + 1/p_2$, $T_{\sigma,\vec{b}}^N$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with bound $C \sum_{k=1}^{2} \|b_k\|_{\rm BMO}$, and C is independent of N.*

Proof. Our aim is to prove that for each fixed $\lambda > 0$,

$$(2.6)\quad |\{x \in \mathbb{R}^n \colon |T_{\sigma,\vec{b}}^N(f_1, f_2)(x)| > \lambda\}| \lesssim \sum_{k=1}^{2} \|b_k\|_{\rm BMO}^p \lambda^{-p} \|f_1\|_{L^{p_1}(\mathbb{R}^n)}^p \|f_2\|_{L^{p_2}(\mathbb{R}^n)}^p.$$

Without loss of generality, we may assume that $\|b_1\|_{\rm BMO} = \|b_2\|_{\rm BMO} = \|f_1\|_{L^{p_1}(\mathbb{R}^n)} = \|f_2\|_{L^{p_2}(\mathbb{R}^n)} = 1$. For each fixed $\lambda > 0$, we apply the Calderón-Zygmund decomposition to $|f_1|^{p_1}$ at level λ^p, and obtain pairwise disjoint cubes $\{Q_1^j\}_j$ satisfying

$$\lambda^p < \frac{1}{|Q_1^j|} \int_{Q_1^j} |f_1(x)|^{p_1}\, {\rm d}x \leqslant 2^n \lambda^p, \quad |f_1(x)| \leqslant C\lambda^{p/p_1} \quad \text{a.e. } x \in \mathbb{R}^n \setminus \bigcup_j Q_1^j.$$

Let

$$g_1(x) = f_1(x)\chi_{\mathbb{R}^n \setminus \bigcup_j Q_1^j}(x) + \sum_j m_{Q_1^j}(f_1)\chi_{Q_1^j}(x),$$

and

$$h_1(x) = f_1(x) - g_1(x) = \sum_j h_1^j(x), \text{ with } h_1^j(x) = (f_1(x) - m_{Q_1^j}(f_1))\chi_{Q_1^j}(x).$$

Observe that $\|g_1\|_{L^\infty(\mathbb{R}^n)} \leqslant C\lambda^{p/p_1}$. Let $\gamma \in (\max\{p_1, t_1\}, \infty)$ and $1/q = 1/\gamma + 1/p_2$. Lemma 2.5 now tells us that

$$|\{x \in \mathbb{R}^n \colon |T_{\sigma,\vec{b}}^N(g_1, f_2)(x)| > \lambda/4\}| \lesssim \lambda^{-q} \|g_1\|_{L^\gamma(\mathbb{R}^n)}^q \|f_2\|_{L^{p_2}(\mathbb{R}^n)}^q \lesssim \lambda^{-p}.$$

Let B_1^j be the smallest ball which contains Q_1^j, and $\Omega = \bigcup_j 4B_1^j$. It is obvious that $|\Omega| \lesssim \lambda^{-p}$. The proof of (2.6) is then reduced to proving that

$$(2.7)\quad |\{x \in \mathbb{R}^n \setminus \Omega \colon |T_{\sigma,\vec{b}}^N(h_1, f_2)(x)| > \tfrac{3}{4}\lambda\}| \lesssim \lambda^{-p}.$$

We now prove (2.7). Let $u_1 \in (t_1, 2]$, $u_2 \in (t_2, \min\{2, p_2\})$ such that $u_1 + u_2 < s_1/n + s_2/n + 1$. For fixed j, let R_1^j and y_1^j be the radius and center of Q_1^j, respectively.

Let

$$\mathrm{L}_1(x) = \left| T_\sigma^N \biggl(\sum_j (b_1(y_1) - m_{B_1^j}(b_1)) h_1^j, f_2 \biggr)(x) \right|,$$

$$\mathrm{L}_2(x) = \sum_j \sum_{|\kappa|<N} |b_1(x) - m_{B_1^j}(b_1)| \int_{\mathbb{R}^{2n}} |W_{1,\kappa}(x, y_1, y_2; y_1^j)| \, |h_1^j(y_1) f_2(y_2)| \, \mathrm{d}\vec{y},$$

and

$$\mathrm{L}_3(x) = \sum_j \sum_{|\kappa|<N} \int_{\mathbb{R}^{2n}} |W_{1,\kappa}(x, y_1, y_2; y_1^j)| \, |b_2(x) - b_2(y_2)| \, |h_1^j(y_1) f_2(y_2)| \, \mathrm{d}\vec{y}.$$

For $x \in \mathbb{R}^n \setminus \Omega$, it follows from the vanishing moment of h_1^j that

$$|T_{\sigma,\vec{b}}^N(h_1, f_2)(x)| \lesssim \sum_{k=1}^3 \mathrm{L}_k(x).$$

The estimate for L_1 is easy. In fact, for $\tilde{p}_1 \in (1, p_1)$ we deduce by the Hölder inequality and the John-Nirenberg inequality that

$$\int_{\mathbb{R}^n} \left| \sum_j (b_1(x) - m_{B_1^j}(b_1)) h_1^j(x) \right|^{\tilde{p}_1} \mathrm{d}x \lesssim \lambda^{p\tilde{p}_1/p_1 - p},$$

which in turn gives us that

$$|\{x \in \mathbb{R}^n \setminus \Omega \colon \mathrm{L}_1(x) > \lambda/4\}| \lesssim \lambda^{-\tilde{p}} \left\| \sum_j (b_1 - m_{B_1^j}(b_1)) h_1^j \right\|_{L^{\tilde{p}_1}(\mathbb{R}^n)}^{\tilde{p}} \lesssim \lambda^{-p},$$

where $1/\tilde{p} = 1/\tilde{p}_1 + 1/p_2$. As for L_2, we have by Lemma 2.3 that

$$\begin{aligned} \mathrm{L}_2(x) \lesssim {} & M_{u_2} f_2(x) \sum_j |b_1(x) - m_{B_1^j}(b_1)| \sum_{\{\kappa \colon 2^\kappa R_1^j \leqslant 1\}} \sum_{l=0}^{\infty} |2^l B_1^j|^{1/u_2} \\ & \times \int_{\mathbb{R}^n} \biggl(\int_{E_l^{R_1^j}(x)} |W_{1,\kappa}(x, y_1, y_2; y_1^j)|^{u_2'} \, \mathrm{d}y_2 \biggr)^{1/u_2'} |h_1^j(y_1)| \, \mathrm{d}y_1 \\ & + M_{u_2} f_2(x) \sum_j |b_1(x) - m_{B_1^j}(b_1)| \\ & \times \sum_{\{\kappa \colon 2^\kappa R_1^j > 1\}} \int_{\mathbb{R}^n} |\mathrm{H}_{1,\kappa B_1^j}(x, y_1, y_1^j)| \, |h_1^j(y_1)| \, \mathrm{d}y_1 \\ := {} & M_{u_2} f_2(x) \mathrm{L}_2^*(x). \end{aligned}$$

By (i) and (ii) of Lemma 2.3, a straightforward computation leads to

$$
\begin{aligned}
\|\mathrm{L}_2^*\|_{L^1(\mathbb{R}^n\setminus\Omega)} &\lesssim \sum_j \sum_{\{\kappa\colon 2^\kappa R_1^j\leqslant 1\}} \sum_{l=3}^\infty \sum_{l=0}^\infty |2^l B_1^j|^{1/u_2} \left(\int_{S_l(B_1^j)} |b_1(x) - m_{B_1^j}(b_1)|^{u_1}\, \mathrm{d}x \right)^{1/u_1} \\
&\quad \times \int_{\mathbb{R}^n} \left(\int_{S_l(B_1^j)} \left(\int_{E_l^{R_1^j}(x)} |W_{1,\kappa}(x,y,z;y_1^j)|^{u_2'} \mathrm{d}z \right)^{u_1'/u_2'} \mathrm{d}x \right)^{1/u_1'} |h_1^j(y)|\, \mathrm{d}y \\
&\quad + \sum_j \sum_{\{\kappa\colon 2^\kappa R_1^j > 1\}} \sum_{l=3}^\infty \int_{\mathbb{R}^n} \int_{S_l(B_1^j)} |b_1(x) - m_{B_1^j}(b_1)| \\
&\quad \times |\mathrm{H}_{1,\kappa,B_1^j}(x,y_1,y_1^j)|\, \mathrm{d}x |h_1^j(y_1)|\, \mathrm{d}y_1 \\
&\lesssim \sum_j \|h_1^j\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

This in turn implies

$$
|\{x\in\mathbb{R}^n\setminus\Omega\colon\ \mathrm{L}_2(x) > \lambda/4\}| \lesssim \lambda^{-p} \|M_{u_2} f_2\|_{L^{p_2}(\mathbb{R}^n)}^{p_2} + \lambda^{-p/p_1} \|\mathrm{L}_2^*\|_{L^1(\mathbb{R}^n\setminus\Omega)} \lesssim \lambda^{-p}.
$$

To consider L_3, we write

$$
\begin{aligned}
\mathrm{L}_3(x) &\lesssim \sum_j \sum_{\{\kappa\colon 2^\kappa R_1^j\leqslant 1\}} \sum_{l=0}^\infty \int_{\mathbb{R}^n} \left(\int_{E_l^{R_1^j}(x)} |W_{1,\kappa}(x,y_1,y_2;y_1^j)|^{u_2'}\, \mathrm{d}y_2 \right)^{1/u_2'} \\
&\quad \times |h_1^j(y_1)|\, \mathrm{d}y_1 M_{b_2,u_2} f_2(x) \\
&\quad + \sum_j \sum_{\{\kappa\colon 2^\kappa R_1^j > 1\}} \int_{\mathbb{R}^n} |\mathrm{H}_{1,\kappa,B_1^j}(x,y_1,y_1^j)|\, |h_1^j(y_1)|\, \mathrm{d}y_1 M_{b_2,u_2} f_2(x) \\
&:= M_{b_2,u_2} f_2(x) \mathrm{L}_3^*(x).
\end{aligned}
$$

As in the estimate for L_2, it follows from (iii) of Lemma 2.3 that

$$
\begin{aligned}
\|\mathrm{L}_3^*\|_{L^1(\mathbb{R}^n\setminus\Omega)} &\lesssim \sum_j \sum_{\{\kappa\colon 2^\kappa R_1^j\leqslant 1\}} (2^\kappa R)^{n/u_1+n/u_2-s_1-s_2+1} \\
&\quad \times \sum_{l=3}^\infty 2^{l(n/u_2-s_2)} \sum_{l=0}^\infty 2^{l(n/u_1-s_1)} \|h_1^j\|_{L^1(\mathbb{R}^n)} \\
&\quad + \sum_j \sum_{\{\kappa\colon 2^\kappa R_1^j > 1\}} (2^\kappa R)^{n/u_1+n/u_2-s_1-s_2} \sum_{l=3}^\infty 2^{l(n/u_2-s_2)} \|h_1^j\|_{L^1(\mathbb{R}^n)} \\
&\lesssim \sum_j \|h_1^j\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

This, along with (1.6), leads to

$$|\{x \in \mathbb{R}^n \setminus \Omega\colon\ \mathrm{L}_3(x) > \lambda/4\}| \lesssim \lambda^{-p},$$

and this yields (2.7). □

P r o o f of Theorem 1.2. We first consider the conclusion (i). Let $s_1, s_2 \in (n/2, n]$. Lemma 2.6 tells us that if $p_1 \in (1,\infty)$ and $p_2 \in (t_2,\infty)$, then $T^N_{\sigma,\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$. Similarly, we can verify that if $p_1 \in (t_1,\infty)$ and $p_2 \in (1,\infty)$, then $T^N_{\sigma,\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$. An application of the complex interpolation theorem then tells us that, when $p_1, p_2 \in (1,\infty)$, $p \in (\beta,\infty)$ such that $1/p = 1/p_1 + 1/p_2$, $T^N_{\sigma,\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$, where $\beta = \max\{t_1/(t_1+1), t_2/(t_2+1)\}$. Note that for fixed $p_1, p_2 \in (1,\infty)$, $p \in [2/3,\infty)$ with $1/p = 1/p_1 + 1/p_2$, we can choose points $A_1 = (1/p_1^1, 1/p_2^1; 1/p^1)$, $A_2 = (1/p_1^2, 1/p_2^2; 1/p^2)$, $A_3 = (1/p_1^3, 1/p_2^3; 1/p^3)$ such that for $i = 1,2,3$, $p_1^i, p_2^i \in (1,\infty)$, $p^i \in (\beta,\infty)$, $1/p^i = 1/p_1^i + 1/p_2^i$, and $(1/p_1, 1/p_2; 1/p)$ is in the open convex hull of A_1, A_2 and A_3. Thus, by the multilinear Marcinkiewicz interpolation theorem, we see that $T^N_{\sigma,\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with bound independent of N. As it was pointed out in [12], for $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$ and $b_1, b_2 \in L^\infty(\mathbb{R}^n)$,

$$\|T_{\sigma,\vec{b}}(f_1,f_2) - T^N_{\sigma,\vec{b}}(f_1,f_2)\|_{L^\infty(\mathbb{R}^n)} \lesssim \|(\sigma - \sum_{|\kappa|<N} \tilde{\sigma}_\kappa)\hat{f}_1\hat{f}_2\|_{L^1(\mathbb{R}^n)} \to 0, \quad N \to \infty.$$

Thus, for $f_1, f_2 \in \mathscr{S}(\mathbb{R}^n)$ and $b_1, b_2 \in L^\infty(\mathbb{R}^n)$, $p_1, p_2 \in (1,\infty)$, $p \in [2/3,\infty)$ with $1/p = 1/p_1 + 1/p_2$,

$$\|T_{\sigma,\vec{b}}(f_1,f_2)\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{k=1}^{2} \|b_k\|_{\mathrm{BMO}} \prod_{l=1}^{2} \|f_l\|_{L^{p_l}(\mathbb{R}^n)}.$$

This, via a standard argument, gives our conclusion (i).

We turn our attention to the conclusion (ii). By Lemma 2.6 and the argument involving the complex interpolation theorem and the multilinear Marcinkiewicz interpolation theorem, we know that for any $p_1, p_2 \in (1,\infty)$, $p \in (1/2,\infty)$ with $1/p = 1/p_1 + 1/p_2$, $T^N_{\sigma,\vec{b}}$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ with bound $C \sum\limits_{k=1}^{2} \|b_k\|_{\mathrm{BMO}}$ and C is independent of N, provided that $s_1 > n/2$, $s_2 = n$, or that $s_1 = n$, $s_2 > n/2$, and so is $T_{\sigma,\vec{b}}$. This, via Lemma 2.4, implies that when $s_1, s_2 \in (n/2, n]$ and $s_1 + s_2 > (3/2)n$, $p_1, p_2 \in (1,\infty)$ $p \in (1/2,\infty)$ with $1/p = 1/p_1 + 1/p_2$, then

$$\|T_{\sigma,\vec{b}}(f_1,f_2)\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{k=1}^{2} \|b_k\|_{\mathrm{BMO}} \prod_{k=1}^{2} \|f_k\|_{L^{p_k}(\mathbb{R}^n)}.$$

This completes the proof of Theorem 1.2. □

3. Proof of Theorem 1.3

We begin with the atomic decomposition of $H^1(\mathbb{R}^n)$.

Definition 3.1. A function $a(x)$ is called a $(1,\infty,0)$-atom if

(i) $a(x)$ is supported in a cube Q and satisfies that $\|a\|_{L^\infty(\mathbb{R}^n)} \leqslant |Q|^{-1}$;

(ii) $\int_{\mathbb{R}^n} a(x)\,\mathrm{d}x = 0$.

Let $H^{1,\infty,0}_{\mathrm{fin}}(\mathbb{R}^n)$ be the set of all finite linear combinations of $(1,\infty,0)$-atoms. For $f \in H^{1,\infty,0}_{\mathrm{fin}}(\mathbb{R}^n)$, define

$$\|f\|_{H^{1,\infty,0}_{\mathrm{fin}}(\mathbb{R}^n)} \equiv \inf\bigg\{\sum_{i=1}^{k} |\lambda_i| \colon f = \sum_{i=1}^{k} \lambda_i a_i,\ k \in \mathbb{N},\ \{a_i\}_{i=1}^{k} \text{ are } (1,\infty,0)\text{-atoms}\bigg\}.$$

Denote by $\mathcal{C}(\mathbb{R}^n)$ the set of all continuous functions. Meda, Sjögren and Vallarino [14] proved that a bounded linear operator on $H^{1,\infty,0}_{\mathrm{fin}}(\mathbb{R}^n) \cap \mathcal{C}(\mathbb{R}^n)$ can be extended to a bounded operator on $H^1(\mathbb{R}^n)$.

Lemma 3.1. *Let t be a positive real number. For any finite collection of dyadic cubes Q and associated positive scalars r_Q, there exists a collection of pairwise disjoint dyadic cubes S such that*

$$\sum |S| \leqslant t^{-1} \sum r_Q, \quad \bigg\| \sum_{Q \not\subseteq \text{ any } S} r_Q |Q|^{-1} \chi_Q \bigg\|_{L^\infty(\mathbb{R}^n)} \leqslant t,$$

and for all S,

$$\sum_{Q \subset S} r_Q \leqslant 8t|S|.$$

For the proof of Lemma 3.1, see [2].

Lemma 3.2. *Let σ satisfy* (1.4) *for $s_1, s_2 \in (n/2, n]$, let T^N_σ be the operator defined by* (2.4).

(i) *For $p_2 \in [2,\infty)$, T^N_σ is bounded from $L^1(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ with $1/p = 1 + 1/p_2$, and the bound is independent of N;*

(ii) *if $s_1, s_2 \in (n/2, n]$ and $s_1 + s_2 > (3/2)n$, then T^N_σ is bounded from $L^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ to $L^{1/2,\infty}(\mathbb{R}^n)$, and the bound is independent of N.*

Proof. Since $s_1 \in (n/2, n]$, we can take $p_1 \in (0,1)$ such that $s_1 > n/p_1 - n/2$. By Theorem 1.1 in [15], we know that

$$\|T^N_\sigma(f_1, f_2)\|_{L^p(\mathbb{R}^n)} \lesssim \|f_1\|_{H^{p_1}(\mathbb{R}^n)} \|f_2\|_{L^{p_2}(\mathbb{R}^n)}, \quad 1/p = 1/p_1 + 1/p_2$$

and

$$\|T_\sigma^N(f_1,f_2)\|_{L^p(\mathbb{R}^n)} \lesssim \|f_1\|_{L^2(\mathbb{R}^n)}\|f_2\|_{L^{p_2}(\mathbb{R}^n)}, \quad 1/p = 1/2 + 1/p_2.$$

Interpolating the last two inequalities leads to the conclusion (i). The conclusion (ii) can be proved in the same way. □

Lemma 3.3. *Let σ satisfy* (1.4) *for $s_1, s_2 \in (n/2, n]$ with $s_2 + s_2 > (3/2)n$, and let T_σ^N be the operator defined by* (2.4). *Let $\alpha_1, \alpha_2 > 1/2$ be such that $\alpha_1 + \alpha_2 = 3/2$, and $s_1 > \alpha_1 n$, $s_2 > \alpha_2 n$. Define β_1, β_2 by $\beta_k/2 = 1 - \alpha_k$ $(k = 1, 2)$. Then for $(1, \infty, 0)$-atoms a_1 supported on cube Q_1 and a_2 supported on cube Q_2, and any $x \in \mathbb{R}^n \setminus 2\sqrt{n}Q_1 \cup 2\sqrt{n}Q_2$,*

$$|T_{\tilde{\sigma}_\kappa}(a_1,a_2)(x)| \lesssim 2^{2\kappa n}\langle 2^\kappa(x-c_1)\rangle^{-s_1}\langle 2^\kappa(x-c_2)\rangle^{-s_2}u_{\kappa,1}(x)u_{\kappa,2}(x),$$

where c_1, c_2 are the center of Q_1 and Q_2 respectively, $u_{\kappa,1}$, $u_{\kappa,2}$ satisfy that

$$\|u_{\kappa,1}\|_{L^{2/\beta_1}} \lesssim \begin{cases} 2^{-\kappa n\beta_1/2}(2^\kappa l(Q_1))^{\beta_1} & \text{if } 2^\kappa l(Q_1) \leqslant 1 \\ 2^{-\kappa n\beta_1/2} & \text{if } 2^\kappa l(Q_1) > 1; \end{cases} \tag{3.1}$$

$$\|u_{\kappa,2}\|_{L^{2/\beta_2}} \lesssim \begin{cases} 2^{-\kappa n\beta_2/2}(2^\kappa l(Q_2))^{\beta_2} & \text{if } 2^\kappa l(Q_2) \leqslant 1 \\ 2^{-\kappa n\beta_2/2} & \text{if } 2^\kappa l(Q_2) > 1, \end{cases} \tag{3.2}$$

where $l(Q_1)$ denotes the side length of Q_1.

For the proof of Lemma 3.3, see [15].

Proof of Theorem 1.3. We first prove conclusion (a). Let $p_2 \geqslant 2$, $f_1 \in H^1(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$ with $\|f_1\|_{H^1(\mathbb{R}^n)} = \|f_2\|_{L^{p_2}(\mathbb{R}^n)} = 1$, $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|b_1\|_{\mathrm{BMO}} = \|b_2\|_{\mathrm{BMO}} = 1$. It suffices to prove that for each fixed $\lambda > 0$,

$$|\{x \in \mathbb{R}^n \colon |T_{\sigma,\vec{b}}^N(f_1,f_2)(x)| > \lambda\}| \leqslant C\lambda^{-p}, \tag{3.3}$$

with $1/p = 1 + 1/p_2$ and C independent of N. We assume that $f_1 = \sum\limits_{j=1}^{N_1} r_j a_1^j$, with N_1 a positive integer and each a_1^j a $(1, \infty, 0)$-atom. As was pointed out in [2], we shall always assume that each scalar r_1^j is positive and $\operatorname{supp} a_1^j \subset Q_1^j$ for some dyadic cubes. Applying Lemma 3.1 to the collection of cubes $\{Q_1^j\}_j$ and scalars $\{r_j\}_j$ with $t = \lambda^p$, we have cubes $\{S_1^j\}_j$ with disjoint interiors. Set

$$f_1(x) = g_1(x) + h_1(x),$$

where

$$g_1(x) = \sum_{\{k\colon Q_k \not\subseteq \text{ any } S_j\}} r_k a_1^k(x) \quad \text{and} \quad h_1(x) = \sum_j \sum_{\{k\colon Q_1^k \subset S_1^j\}} r_k a_1^k(x).$$

Observe that

$$\|g_1\|_{L^\infty(\mathbb{R}^n)} \leqslant \Bigg\| \sum_{\{k\colon\, Q_1^k \not\subseteq \text{ any } S_j\}} r_k |Q_1^k|^{-1} \chi_{Q_1^k} \Bigg\|_{L^\infty(\mathbb{R}^n)} \leqslant \lambda^p,$$

and

$$\|g_1\|^2_{L^2(\mathbb{R}^n)} \lesssim \|g_1\|_{L^\infty(\mathbb{R}^n)} \|g_1\|_{L^1(\mathbb{R}^n)} \lesssim \lambda^p.$$

The conclusion (i) of Theorem 1.2 tells us that

$$|\{x \in \mathbb{R}^n\colon\, |T^N_{\sigma,\vec{b}}(g_1, f_2)(x)| > \lambda/2\}| \lesssim \lambda^{-q} \|g_1\|^q_{L^2(\mathbb{R}^n)} \|f_2\|^q_{L^{p_2}(\mathbb{R}^n)} \lesssim \lambda^{-p},$$

where $1/q = 1/2 + 1/p_2$. Let B_1^j be the smallest ball containing S_1^j and $\Omega = \bigcup_j 4B_1^j$. It is obvious that $|\Omega| \lesssim \lambda^{-p}$. Thus, the proof of (3.3) is reduced to proving that

$$|\{x \in \mathbb{R}^n \setminus \Omega\colon\, |T^N_{\sigma,\vec{b}}(h_1, f_2)(x)| > \lambda/2\}| \lesssim \lambda^{-p}. \tag{3.4}$$

For $x \in \mathbb{R}^n \setminus \Omega$, write

$$\begin{aligned} |T^N_{\sigma,\vec{b}}(h_1, f_2)(x)| \leqslant{}& \Bigg| T^N_\sigma \Bigg(\sum_j \sum_{\{k\colon\, Q_1^k \subset S_1^j\}} (b_1 - m_{Q_1^k}(b_1)) r_k a_1^k, f_2 \Bigg)(x) \Bigg| \\ &+ \sum_j \sum_{\{k\colon\, Q_1^k \subset S_1^j\}} |r_k| |b_1(x) - m_{Q_1^k}(b_1)|\, |T^N_\sigma(a_1^k, f_2)(x)| \\ &+ |[b_2, T^N_\sigma]^2(h_1, f_2)(x)| := \sum_{l=1}^3 \mathrm{U}_l(x). \end{aligned}$$

It follows from Lemma 3.2 that

$$\begin{aligned} |\{x \in \mathbb{R}^n \setminus \Omega\colon\, \mathrm{U}_1(x) > \lambda/6\}| &\lesssim \lambda^{-p} \Bigg\| \sum_j \sum_{\{k\colon\, Q_1^k \subset S_1^j\}} (b_1 - m_{Q_1^k}(b_1)) r_k a_1^k \Bigg\|^p_{L^1(\mathbb{R}^n)} \\ &\lesssim \lambda^{-p}. \end{aligned}$$

Similarly to the estimate for L_2 in the proof of Lemma 2.6, we get that

$$\begin{aligned} |\{x \in \mathbb{R}^n \setminus \Omega\colon\, \mathrm{U}_2(x) > \lambda/6\}| \lesssim{}& \lambda^{-p} \|M_{u_2} f_2\|^{p_2}_{L^{p_2}(\mathbb{R}^n)} \\ &+ \lambda^{-p} \sum_j \sum_{\{k\colon\, Q_1^k \subset S_1^j\}} |r_k| \lesssim \lambda^{-p} \end{aligned}$$

with $u_2 \in (t_2, 2]$. Also, repeating the estimate for L_3, we deduce that

$$|\{x \in \mathbb{R}^n \setminus \Omega\colon\ \mathrm{U}_3(x) > \lambda/6\}| \lesssim \lambda^{-p}\|M_{b_2,u_2} f_2\|_{L^{p_2}(\mathbb{R}^n)}^{p_2} + \lambda^{-p} \sum_j \sum_{\{k\colon\ Q_1^k \subset S_1^j\}} |r_k| \lesssim \lambda^{-p}.$$

Combining the estimates for terms U_k ($k = 1,2,3$) leads to (3.4) and then completes the proof of conclusion (a).

We now prove conclusion (b). Our aim is to prove that if $f_1, f_2 \in H^1(\mathbb{R}^n)$ with $\|f_1\|_{H^1(\mathbb{R}^n)} = \|f_2\|_{H^1(\mathbb{R}^n)} = 1$ and $b_1, b_2 \in \mathrm{BMO}(\mathbb{R}^n)$, then for each fixed $\lambda > 0$,

$$|\{x \in \mathbb{R}^n\colon\ |T_{\sigma,\vec{b}}^N(f_1, f_2)(x)| > \lambda\}| \lesssim C \sum_{k=1}^2 \|b_k\|_{\mathrm{BMO}}^{1/2} \lambda^{-1/2}. \tag{3.5}$$

Again we assume that $\|b_1\|_{\mathrm{BMO}} = \|b_2\|_{\mathrm{BMO}} = 1$ and

$$f_1(x) = \sum_{j=1}^{N_1} r_1^j a_1^j(x),\ \ f_2(x) = \sum_{j=1}^{N_2} r_2^j a_2^j(x),$$

where N_1, N_2 are positive integers and each a_i^j ($i = 1,2$) is a $(1,\infty,0)$-atom, each scalar r_i^j is positive and $\operatorname{supp} a_i^j \subset Q_i^j$ for some dyadic cube Q_i^j ($i = 1,2$). Invoking Lemma 3.1 to each collection of cubes $\{Q_i^j\}_j$ and scalars $\{r_i^j\}_j$ with $t = \lambda^{1/2}$, we obtain two families of cubes $\{S_1^j\}_j$, $\{S_2^j\}_j$ with disjoint interiors. Decompose f_i as

$$f_i(x) = g_i(x) + h_i(x),$$

where

$$g_i(x) = \sum_{k\colon\ Q_i^k \not\subseteq \text{ any } S_i^j} r_i^k a_i^k(x) \quad \text{and} \quad h_i(x) = \sum_j \sum_{\{k\colon\ Q_i^k \subset S_i^j\}} r_i^k a_i^k(x).$$

It is obvious that $\|g_i\|_{L^2(\mathbb{R}^n)}^2 \lesssim \lambda^{1/2}$. By conclusion (a) of Theorem 1.3, $s_1, s_2 \in (n/2, n]$ implies that $T_{\sigma,\vec{b}}^N$ is bounded from $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^{2/3,\infty}(\mathbb{R}^n)$. Therefore,

$$|\{x \in \mathbb{R}^n\colon\ |T_{\sigma,\vec{b}}^N(f_1, g_2)(x)| > \lambda/3\}| \lesssim \lambda^{-2/3}\|f_1\|_{H^1(\mathbb{R}^n)}^{2/3}\|g_2\|_{L^2(\mathbb{R}^n)}^{2/3} \lesssim \lambda^{-1/2},$$

and

$$|\{x \in \mathbb{R}^n\colon\ |T_{\sigma,\vec{b}}^N(g_1, h_2)(x)| > \lambda/3\}| \lesssim \lambda^{-2/3}\|h_2\|_{H^1(\mathbb{R}^n)}^{2/3}\|g_1\|_{L^2(\mathbb{R}^n)}^{2/3} \lesssim \lambda^{-1/2}.$$

Let B_i^j be the smallest ball containing S_i^j, and $\Omega = \bigcup_{i=1}^{2} \bigcup_j B_i^j$, then it is easy to check that

$$|\Omega| \lesssim \sum_{i=1}^{2} \sum_j |S_i^j| \lesssim \lambda^{-1/2}.$$

The proof of (3.5) is now reduced to proving that

$$|\{x \in \mathbb{R}^n \setminus \Omega \colon\ |[b_1, T_\sigma^N]^1(h_1, h_2)(x)| > \lambda/6\}| \lesssim \lambda^{-1/2}, \tag{3.6}$$

and

$$|\{x \in \mathbb{R}^n \setminus \Omega \colon\ |[b_2, T_\sigma^N]^2(h_1, h_2)(x)| > \lambda/6\}| \lesssim \lambda^{-1/2}. \tag{3.7}$$

We only consider (3.6), since the argument equally works for (3.7). For $x \in \mathbb{R}^n \setminus \Omega$, write

$$\begin{aligned}
|[b_1, T_\sigma^N]^1(h_1, h_2)(x)| \lesssim & \sum_i |r_1^i||b_1(x) - m_{Q_1^i}(b_1)| \sum_j |r_1^j|\, |T_\sigma^N(a_1^i, a_2^j)(x)| \\
& + \left| T_\sigma^N\Bigg(\sum_j \sum_{\{k\colon\, Q_1^k \subset S_1^j\}} (b_1(y_1) - m_{Q_1^i}(b_1)) r_1^k a_1^k, h_2 \Bigg)(x) \right| \\
:= & \ \mathrm{D}_1(x) + \mathrm{D}_2(x).
\end{aligned}$$

Recall that

$$\Bigg\| \sum_j \sum_{\{k\colon\, Q_1^k \subset S_1^j\}} (b_1(y_1) - m_{Q_1^i}(b_1)) r_1^k a_1^k \Bigg\|_{L^1(\mathbb{R}^n)} \lesssim 1.$$

It follows from (ii) of Lemma 3.2 that

$$\left| \left\{ x \in \mathbb{R}^n \setminus \Omega \colon\ \mathrm{D}_2(x) > \frac{\lambda}{12} \right\} \right| \lesssim \lambda^{-1/2}. \tag{3.8}$$

As for D_1, we use Lemma 3.3 and get that for $x \in \mathbb{R}^n \setminus \Omega$,

$$\begin{aligned}
\mathrm{D}_1(x) \lesssim & \sum_{|\kappa| < N} 2^{\kappa n} \sum_i |r_1^i||b_1(x) - m_{Q_1^i}(b_1)| \langle 2^\kappa (x - y_1^i) \rangle^{-s_1} u_{\kappa,1}^i(x) \\
& \times \sum_{|\kappa| < N} 2^{\kappa n} \sum_j |r_2^j| \langle 2^\kappa (x - y_2^j) \rangle^{-s_2} u_{\kappa,2}^j(x) \\
:= & \ \mathrm{D}_1^1(x) \mathrm{D}_1^2(x),
\end{aligned}$$

where y_1^i, y_2^j are the centers of Q_1^i and Q_2^j, $u_{\kappa,1}^i$ and $u_{\kappa,2}^j$ satisfy (3.1) and (3.2) respectively. It was proved in [15] that

$$\|\mathrm{D}_1^2\|_{L^1(\mathbb{R}^n \setminus \Omega)} \lesssim 1.$$

Let α_1, β_1 be the same as in Lemma 3.3. A trivial computation involving the John-Nirenberg inequality shows that if $2^\kappa l(Q_1^i) > 1$, then

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_1^i} &\frac{|b_1(x)-m_{Q_1^i}(b_1)|^{1/\alpha_1}}{\langle 2^\kappa(x-c_1^i)\rangle^{s_1/\alpha_1}}\,\mathrm{d}x\\ &\lesssim \sum_{j=1}^{\infty}|2^\kappa 2^j l(Q_1^i)|^{-s_1/\alpha_1}\int_{S_l(B_1^i)}|b_1(x)-m_{Q_1^i}(b_1)|^{1/\alpha_1}\,\mathrm{d}x\\ &\lesssim \sum_{j=1}^{\infty} j|2^\kappa 2^j l(Q_1^i)|^{-s_1/\alpha_1}|2^j Q_1^i| \lesssim 2^{-\kappa n}(2^\kappa l(Q_1^i))^{-s_1/\alpha_1+n},\end{aligned}$$

and when $2^\kappa l(Q_1^i)\leqslant 1$, then

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{|b_1(x)-m_{Q_1^i}(b_1)|^{1/\alpha_1}}{\langle 2^\kappa(x-c_1^i)\rangle^{s_1/\alpha_1}}\,\mathrm{d}x &\lesssim \int_{|x-c_1^i|<2^{-\kappa}}|b_1(x)-m_{Q_1^i}(b_1)|^{1/\alpha_1}\,\mathrm{d}x\\ &\quad+\sum_{l=1}^{\infty}\int_{2^l\leqslant 2^\kappa|x-c_1^i|<2^{l+1}}\frac{|b_1(x)-m_{Q_1^i}(b_1)|^{1/\alpha_1}}{|2^\kappa(x-c_1^i)|^{s_1/\alpha_1}}\,\mathrm{d}x\\ &\lesssim 2^{-\kappa n}\log^{1/\alpha_1}(2^\kappa l(Q_1^i))^{-1}+\sum_{l=1}^{\infty}(l-\log(2^\kappa l(Q_1^i)))^{1/\alpha_1}2^{nl-s_1/\alpha_1}2^{-\kappa n}\\ &\lesssim -2^{-\kappa n}\log^{1/\alpha_1}(2^\kappa l(Q_1^i)).\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathrm{D}_1^1\|_{L^1(\mathbb{R}^n\setminus\Omega)} \lesssim &-\sum_i |r_1^i|\sum_{\{\kappa:\,2^\kappa l(Q_1^i)\leqslant 1\}} 2^{-\kappa n\alpha_1}\log(2^\kappa l(Q_1^i))2^{-\kappa\beta_1 n/2}(2^\kappa l(Q_1^i))^{\beta_1}\\ &+\sum_i |r_1^i|\sum_{\{\kappa:\,2^\kappa l(Q_1^i)>1\}} 2^{-\kappa\alpha_1 n}(2^\kappa l(Q_1^i))^{-s_1+n\alpha_1}2^{-\kappa n\beta_1/2}\lesssim 1.\end{aligned}$$

The estimates for D_1^1 and D_1^2 imply that

$$\|\mathrm{D}_1\|_{L^{1/2}(\mathbb{R}^n\setminus\Omega)}\lesssim 1. \tag{3.9}$$

(3.6) now follows from (3.8) and (3.9). This completes the proof of Theorem 1.3. $\square$

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