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SOME PROPERTIES OF THE FAMILY Γ OF MODULAR LIE SUPERALGEBRAS

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Abstract. In this paper, we continue to investigate some properties of the family Γ of finite-dimensional simple modular Lie superalgebras which were constructed by X. N. Xu, Y. Z. Zhang, L. Y. Chen (2010). For each algebra in the family, a filtration is defined and proved to be invariant under the automorphism group. Then an intrinsic property is proved by the invariance of the filtration; that is, the integer parameters in the definition of Lie superalgebras Γ are intrinsic. Thereby, we classify these Lie superalgebras in the sense of isomorphism. Finally, we study the associative forms and Killing forms of these Lie superalgebras and determine which superalgebras in the family are restrictable.

Keywords: modular Lie superalgebra; restricted Lie superalgebra; filtration

MSC 2010: 17B50

1. INTRODUCTION

It is well known that filtration structures play an important role both in the classification of modular Lie algebras (i.e., Lie algebras over a field of prime characteristic) (see [1], [7], [19], [21], [26]) and Lie superalgebras (i.e., Lie superalgebras over a field of characteristic zero) (see [9], [10], [16]). Similarly, filtration structures will provide useful tools in the research of modular Lie superalgebras (i.e., Lie superalgebras over a field of prime characteristic). The filtrations of modular Lie algebras of Cartan type and Lie superalgebras were proved to be invariant in papers [20], [17] and [8], respectively. The same results for modular Lie superalgebras W and S were obtained

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by using ad-nilpotent elements in paper [30] and for modular Lie superalgebras H and K they were obtained by means of minimal dimension of image spaces in papers [31], [32]. The invariance of the nontrivial transitive filtrations of modular Lie superalgebras HO was discussed in paper [25].

The research on modular Lie superalgebras just began in recent years (see [11], [15]). The complete classification of the finite-dimensional simple modular Lie superalgebras remains an open problem [12]. So constructing finite-dimensional simple modular Lie superalgebras and studying their natural properties is necessary at present stage (see [27], [33]). Many important results for modular Lie superalgebras have been obtained (see [2], [4], [13], [14], [22]–[33]). The study of graded Lie superalgebras also have got several deep results in recent years (see [3], [5]).

This paper is devoted to investigating the filtration structures of the family Γ of modular Lie superalgebras by the method of minimal dimension of image spaces and then some properties are discussed. This paper is organized as follows: In Section 2, we recall some necessary definitions and useful results of the Lie superalgebras Γ. In Section 3, we establish some technical lemmas which will be employed to determine the invariance of the filtrations. Then the filtrations of the Lie superalgebras Γ are proved to be invariant under automorphisms. Therefore, we are able to obtain an intrinsic characterization of these Lie superalgebras. In Section 4, we discuss the associative forms and Killing forms of the Lie superalgebras Γ and find the conditions for the restrictability of these Lie superalgebras.

2. Preliminaries

Throughout this article, $\mathbb{F}$ denotes an algebraically closed field of characteristic $p > 3$ and $\mathbb{F}$ is not equal to its prime field Π. For $m > 0$, let $\mathbb{E} = \{z_1, \ldots, z_m\}$ be a subset of $\mathbb{F}$ that is linearly independent over the prime field Π, and let H be the additive subgroup generated by $\mathbb{E}$. If $\lambda \in H$, then we let $\lambda = \sum\limits_{i=1}^{m} \lambda_i z_i$ and $y^\lambda = y_1^{\lambda_1} \ldots y_m^{\lambda_m}$, where $0 \leqslant \lambda_i < p$. We use the notation $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ for the set of non-negative integers. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the ring of integers modulo 2.

Given $n \in \mathbb{N}$ and $r = 2n$, we put $M = \{0, 1, \ldots, r\}$. Suppose that $\mu_0, \ldots, \mu_r \in \mathbb{F}$ such that $\mu_0 = 0$ and $\mu_j + \mu_{n+j} = 1$ for $j = 1, \ldots, n$. Let $k_i \in \mathbb{N}_0$ for $i \in M$, then k_i can be uniquely expressed in p-adic form $k_i = \sum\limits_{v=0}^{s_i} \varepsilon_v(k_i) p^v$, where $0 \leqslant \varepsilon_v(k_i) < p$. Let $\underline{s} = (s_0 + 1, \ldots, s_r + 1) \in \mathbb{N}^{r+1}$. We define the truncated polynomial algebras

$$A = \mathbb{F}[x_{00}, x_{01}, \ldots, x_{0s_0}, \ldots, x_{r0}, x_{r1}, \ldots, x_{rs_r}, y_1, \ldots, y_m]$$

such that

$$x_{ij}^p = 0, \quad \forall\, i \in M,\ j = 0, 1, \ldots, s_i;\ y_i^p = 1,\ i = 1, \ldots, m.$$

Let $Q = \{(k_0, \ldots, k_r);\ 0 \leqslant k_i \leqslant \pi_i,\ \pi_i = p^{s_i+1} - 1,\ i \in M\}$. If $k = (k_0, \ldots, k_r) \in Q$, we write $x^k = x_0^{k_0} \ldots x_r^{k_r}$, where ${x_i}^{k_i} = \prod\limits_{v=0}^{s_i} x_{iv}^{\varepsilon_v(k_i)}$ for $i \in M$. For $0 \leqslant k_i, k_i' \leqslant \pi_i$, it is easy to see that

$$\text{(2.1)} \qquad {x_i}^{k_i} {x_i}^{k_i'} = {x_i}^{k_i + k_i'} \neq 0 \Leftrightarrow \varepsilon_v(k_i) + \varepsilon_v(k_i') < p, \quad v = 0, 1, \ldots, s_i,\ i \in M.$$

Let $\Lambda(q)$ be the Grassmann superalgebras over $\mathbb{F}$ in q variables $\xi_{r+1}, \ldots, \xi_{r+q}$ with $q \in \mathbb{N}$ and $q > 1$. Denote the tensor product by $\widetilde{\Omega} := A \otimes_{\mathbb{F}} \Lambda(q)$. Obviously, $\widetilde{\Omega}$ are associative superalgebras with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of A and the natural $\mathbb{Z}_2$-gradation of $\Lambda(q)$:

$$\widetilde{\Omega}_{\bar{0}} = A \otimes_{\mathbb{F}} \Lambda(q)_{\bar{0}}, \quad \widetilde{\Omega}_{\bar{1}} = A \otimes_{\mathbb{F}} \Lambda(q)_{\bar{1}}.$$

For $f \in A$ and $g \in \Lambda(q)$, we abbreviate $f \otimes g$ to fg. For $k \in \{1, \ldots, q\}$, we set

$$\mathbb{B}_k = \{(i_1, i_2, \ldots, i_k);\ r + 1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant r + q\}$$

and $\mathbb{B}(q) = \bigcup\limits_{k=0}^{q} \mathbb{B}_k$, where $\mathbb{B}_0 = \emptyset$. If $u = (i_1, \ldots, i_k) \in \mathbb{B}_k$, we let $|u| = k$, $\{u\} = \{i_1, \ldots, i_k\}$ and $\xi^u = \xi_{i_1} \ldots \xi_{i_k}$. Put $|\emptyset| = 0$ and $\xi^{\emptyset} = 1$. Then $\{x^k y^\lambda \xi^u;\ k \in Q, \lambda \in H, u \in \mathbb{B}(q)\}$ is an $\mathbb{F}$-basis of $\widetilde{\Omega}$.

If L is a Lie superalgebra, then $h(L)$ denotes the set of all $\mathbb{Z}_2$-homogeneous elements of L, i.e., $h(L) = L_{\bar{0}} \cup L_{\bar{1}}$. If $|x|$ appears in some expression in this paper, we always regard x as a $\mathbb{Z}_2$-homogeneous element and $|x|$ as its $\mathbb{Z}_2$-degree.

Set $s = r + q$, $T = \{r + 1, \ldots, s\}$ and $R = M \cup T$. Put $M_1 = \{1, \ldots, r\}$. Define $\tilde{i} = \bar{0}$ if $i \in M_1$, and $\tilde{i} = \bar{1}$ if $i \in T$. Let

$$i' = \begin{cases} i + n, & 1 \leqslant i \leqslant n, \\ i - n, & n + 1 \leqslant i \leqslant r, \\ i, & r + 1 \leqslant i \leqslant s, \end{cases} \qquad [i] = \begin{cases} 1, & 1 \leqslant i \leqslant n, \\ -1, & n + 1 \leqslant i \leqslant r, \\ 1, & r + 1 \leqslant i \leqslant s. \end{cases}$$

For $e_i = (\delta_{i0}, \ldots, \delta_{ir})$, $i \in M$, we abbreviate x^{e_i} to x_i. Let D_i, $i \in R$, be the linear transformations of $\widetilde{\Omega}$ such that

$$D_i(x^k y^\lambda \xi^u) = \begin{cases} k_i^* x^{k - e_i} y^\lambda \xi^u, & i \in M, \\ x^k y^\lambda \cdot \partial \xi^u / \partial \xi_i, & i \in T, \end{cases}$$

where k_i^* is the first nonzero number of $\varepsilon_0(k_i), \varepsilon_1(k_i), \ldots, \varepsilon_{s_i}(k_i)$. Then $D_i \in \operatorname{Der} \widetilde{\Omega}$. Set

$$\overline{\partial} = I - \sum_{j \in M_1} \mu_j x_{j0} \frac{\partial}{\partial x_{j0}} - \sum_{j=1}^{m} z_j y_j \frac{\partial}{\partial y_j} - 2^{-1} \sum_{j \in T} \xi_j \frac{\partial}{\partial \xi_j},$$

where I is the identity mapping of $\widetilde{\Omega}$. For $f \in h(\widetilde{\Omega})$, $g \in \widetilde{\Omega}$, we define a bilinear operation $[\ ,\ ]$ in $\widetilde{\Omega}$ such that

$$[f, g] = D_0(f)\overline{\partial}(g) - \overline{\partial}(f)D_0(g) + \sum_{i \in M_1 \cup T} [i](-1)^{\tilde{i}|f|} D_i(f) D_{i'}(g).$$

Then $\widetilde{\Omega}$ are Lie superalgebras for the operation $[\ ,\ ]$ defined above (see [33]). Note that $\widetilde{\Omega} = \bigoplus_{\alpha \in \mathbb{Z}_2} \widetilde{\Omega}_\alpha$, where

$$\widetilde{\Omega}_\alpha = \operatorname{span}_{\mathbb{F}} \{x^k y^\lambda \xi^u;\ k \in Q,\ \lambda \in H,\ u \in \mathbb{B}(q),\ \alpha = |\bar{u}|\}.$$

If $1 \in H$, then we put $H' = H \setminus \{1\}$ and $y = y^1$. By computation, we obtain that $\langle y \rangle := \{\alpha y;\ \alpha \in \mathbb{F}\}$ is the center of $\widetilde{\Omega}$ and the commutator subalgebra:

$$[\widetilde{\Omega}, \widetilde{\Omega}] = \operatorname{span}_{\mathbb{F}} \{x^k y^\lambda \xi^u;\ (k, \lambda, u) \neq (\pi, n + 2 - 2^{-1}q, \omega)\},$$

where $\pi = (\pi_0, \ldots, \pi_r) \in Q$ and $\omega = (r+1, \ldots, s) \in \mathbb{B}(q)$. Define $\Gamma(r, H, q, \underline{s}) := [\widetilde{\Omega}, \widetilde{\Omega}]/\langle y \rangle$. Then $\Gamma(r, H, q, \underline{s})$ become simple Lie superalgebras (see [27]).

If $1 \notin H$, then $\Omega := [\widetilde{\Omega}, \widetilde{\Omega}]$ are simple Lie superalgebras. The case $1 \notin H$ is a different family (Ω rather than Γ) and is not treated in this paper because it has been studied in [33].

For simplicity, we sometimes write Γ instead of $\Gamma(r, H, q, \underline{s})$. The derivations D_i of $\widetilde{\Omega}$ induce the derivations of Γ by $D_i(f + \langle y \rangle) = D_i(f) + \langle y \rangle$. We write any element $f + \langle y \rangle$ of Γ as f for simplicity. By the convention, we see that $\alpha y = 0$ in Γ for all $\alpha \in \mathbb{F}$.

Note that $\Gamma = \bigoplus_{j \in X} \Gamma_j$ are $\mathbb{Z}$-gradation Lie superalgebras, where

$$\Gamma_j = \operatorname{span}_{\mathbb{F}} \left\{ x^k y^\lambda \xi^u;\ \sum_{i \in M_1} k_i + 2k_0 + |u| - 2 = j \right\}, \tag{2.2}$$

and $X = \{-2, -1, \ldots, \tau\}$, $\tau = \sum_{i \in M_1} \pi_i + 2\pi_0 + q - 2$. Let $f \in \Gamma$. If $f \in \Gamma_j$, then f is called a $\mathbb{Z}$-homogeneous element and j is the $\mathbb{Z}$-degree of f which is denoted by $\operatorname{zd}(f)$.

Let $\Delta = \{\theta\colon H \to \mathbb{F};\ \theta(\lambda+\eta) = \theta(\lambda)+\theta(\eta),\ \forall\, \lambda, \eta \in H\}$. For $\theta \in \Delta$, we define a linear transformation D_θ of Γ such that $D_\theta(x^k y^\lambda \xi^u) = \theta(\lambda) x^k y^\lambda \xi^u$. Clearly $D_\theta \in \operatorname{Der}\Gamma$.

Put $W_1 = \{D_\theta;\ \theta \in \Delta\}$. Then W_1 is an m-dimensional linear space. Set $W_2 = \operatorname{span}_{\mathbb{F}}\{D_i^{p^{v_i}};\ 0 < v_i \leqslant s_i,\ i \in M\}$. Denote by $\operatorname{Der}\Gamma$ the derivation superalgebras of Γ.

Lemma 2.1 ([27])**.** *$\operatorname{Der}\Gamma = \operatorname{ad}\overline{L} \oplus \operatorname{span}_{\mathbb{F}}\{yD_0\} \oplus W_1 \oplus W_2$, where*

$$\begin{aligned}\overline{L} &= \widehat{L} \oplus \operatorname{span}_{\mathbb{F}}\{y x_i^{\pi_i+1};\ i \in M\}\\ &= \Gamma \oplus \operatorname{span}_{\mathbb{F}}\{x^\pi y^\delta \xi^\omega;\ \delta = n+2-2^{-1}q\} \oplus \operatorname{span}_{\mathbb{F}}\{y x_i^{\pi_i+1};\ i \in M\}.\end{aligned}$$

Lemma 2.2 ([27])**.** *If $D_i(f) = 0$ for all $i \in R$, then $f = \sum_{j\in M} \alpha_j x_j y + \sum_{j \in T} \beta_j \xi_j y + z(y)$, where $\alpha_j,\ \beta_j \in \mathbb{F}$ and $z(y) = \sum_{\lambda \in H'} a_\lambda y^\lambda \in \Gamma_{-2}$ with $a_\lambda \in \mathbb{F}$.*

3. Filtration

Put $I(\varphi) = \dim(\operatorname{Im}\varphi)$, where $\varphi \in \operatorname{Der}\Gamma$. Let Θ be a set of $\operatorname{Der}\Gamma$ and $I(\Theta) := \min\{I(\varphi);\ 0 \neq \varphi \in \Theta\}$. Set

$$b = x^\pi \xi^\omega \chi(y),\ B = \operatorname{ad} b\big|_\Gamma, \text{ where } \chi(y) = \sum_{\eta \in H} y^\eta.$$

If $\alpha := \{\alpha_\lambda;\ \lambda \in H\}$ is a subset of $\mathbb{F}$, then we let $\alpha(y) = \sum_{\lambda\in H} \alpha_\lambda y^\lambda$.

Lemma 3.1. *$I(B) = s+2$, where $s = r+q$ and*

$$\begin{aligned}\mathfrak{C} := \ker B = P &\oplus \operatorname{span}_{\mathbb{F}}\Big\{x^k \xi^u \alpha(y);\ \sum_{i\in M} k_i + |u| = 1,\ \sum_{\lambda \in H}\alpha_\lambda = 0\Big\}\\ &\oplus \operatorname{span}_{\mathbb{F}}\Big\{\alpha(y);\ \sum_{\lambda\in H}(1-\lambda)\alpha_\lambda = 0\Big\},\end{aligned}$$

where $P = \operatorname{span}_{\mathbb{F}}\Big\{x^k \xi^u y^\lambda;\ \sum_{i\in M} k_i + |u| \geqslant 2,\ \lambda \in H\Big\}$.

P r o o f. Clearly $B(z)=0$ for all $z\in P$. Note that $\chi(y)y^\lambda=\chi(y)$ for all $\lambda\in H$. If $\sum_{\lambda\in H}\alpha_\lambda=0$, then we obtain

$$\begin{aligned}
B(x_0\alpha(y)) &= \Big[x^\pi\xi^\omega\sum_{\eta\in H}y^\eta, x_0\sum_{\lambda\in H}\alpha_\lambda y^\lambda\Big]=\Big(\sum_{\lambda\in H}\alpha_\lambda\Big)\sum_{\eta\in H}[x^\pi\xi^\omega y^\eta, x_0y^\lambda]\\
&=\Big(\sum_{\lambda\in H}\alpha_\lambda\Big)\sum_{\eta\in H}((p-1)(1-\lambda)x^\pi\xi^\omega y^{\eta+\lambda}-(1+n-\eta-2^{-1}q)x^\pi\xi^\omega y^{\eta+\lambda})\\
&=\Big(\sum_{\lambda\in H}\alpha_\lambda\Big)\Big(x^\pi\xi^\omega\Big(\sum_{\eta\in H}\eta y^\eta\Big)-(n+2-2^{-1}q)b\Big)=0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
B(x_i\alpha(y)) &= -[i']\Big(\sum_{\lambda\in H}\alpha_\lambda\Big)x^{\pi-e_{i'}}\xi^\omega\chi(y)=0, \quad \forall\, i\in M_1,\\
B(\xi_j\alpha(y)) &= (-1)^{|q|}\Big(\sum_{\lambda\in H}\alpha_\lambda\Big)x^\pi\xi^{\omega-(j)}\chi(y)=0, \quad \forall\, j\in T.
\end{aligned}$$

If $\sum_{\lambda\in H}(1-\lambda)\alpha_\lambda=0$, then we have

$$B(\alpha(y))=\Big(\sum_{\lambda\in H}(\lambda-1)\alpha_\lambda\Big)x^{\pi-e_0}\xi^\omega\chi(y)=0.$$

We see that

$$\begin{aligned}
B(x_0y) &= x^\pi\xi^\omega\sum_{\eta\in H}(\eta+2^{-1}q-n-1)y^{\eta+1}=x^\pi\xi^\omega\sum_{\eta\in H}(\eta+2^{-1}q-n-2)y^\eta\neq 0,\\
B(x_0y^\lambda) &= x^\pi\xi^\omega\Big(\sum_{\eta\in H}\eta y^\eta\Big)-(n+2-2^{-1}q)b\neq 0,
\end{aligned}$$

which is independent of λ for all $\lambda\in H$.

Similarly, by a direct computation we get

$$\begin{aligned}
B(x_iy^\lambda) &= -[i']x^{\pi-e_{i'}}\xi^\omega\chi(y)\neq 0, \quad \forall\, i\in M_1,\ \lambda\in H,\\
B(\xi_jy^\lambda) &= (-1)^{|q|}x^\pi\xi^{\omega-(j)}\chi(y)\neq 0, \quad \forall\, j\in T,\ \lambda\in H,\\
B(y^\lambda) &= (\lambda-1)x^{\pi-e_1}\xi^\omega\chi(y)\neq 0, \quad \forall\ \lambda\in H'.
\end{aligned}$$

Let $\mathfrak{N}=\mathrm{span}_{\mathbb{F}}\{1,x_i,\xi_j;\ i\in M_1,\ j\in T\}$. Then $\Omega=\mathfrak{C}\oplus\mathfrak{N}$. It is easily seen that $B(1)$, $B(x_i)$ and $B(\xi_j)$ are linearly independent for all $i\in M$ and $j\in T$. Hence $I(B)=r+q+2=s+2$, as desired. □

Lemma 3.2. *If* $0 \neq f \in h(\Gamma)$ *and* $f \notin \operatorname{span}_{\mathbb{F}}\{x^\pi \xi^\omega \alpha(y)\}$, *then there exist two basis elements* f_1 *and* f_2 *such that* $[f, f_1]$ *and* $[f, f_2]$ *are linearly independent with* $\operatorname{zd}(f_i) \geqslant 0$ *for* $i = 1, 2$.

Proof. (1) If f does not contain any ξ_j for all $j \in T$, then every term of f can be expressed in the $\alpha_{k\lambda} x^k y^\lambda$ form with $\alpha_{k\lambda} \in \mathbb{F}$, and two cases arise:

Case 1. $\operatorname{zd}(f) = \sum\limits_{i \in M_1} \pi_i + 2\pi_0 - 2$. Then we can suppose $f = \sum\limits_{\lambda \in S} \alpha_{\pi\lambda} x^\pi y^\lambda$, where $0 \neq \alpha_{\pi\lambda} \in \mathbb{F}$ and $S \subseteq H$. So we get

$$[f, x_i \xi_j] = -[i'] \sum_{\lambda \in S} \alpha_{\pi\lambda} x^{\pi - e_{i'}} y^\lambda \xi_j \neq 0,$$
$$[f, x_{i'} \xi_j] = -[i] \sum_{\lambda \in S} \alpha_{\pi\lambda} x^{\pi - e_i} y^\lambda \xi_j \neq 0,$$

and they are linearly independent.

Case 2. $\operatorname{zd}(f) < \sum\limits_{i \in M_1} \pi_i + 2\pi_0 - 2$. Then we may assume that $f = \sum\limits_{k \in \Delta, \lambda \in S} \alpha_{k\lambda} x^k y^\lambda$, where $\Delta \subseteq Q$, $S \subseteq H$ and $0 \neq \alpha_{k\lambda} \in \mathbb{F}$. Put $\beta_{k\lambda} = 1 - \lambda - \sum\limits_{i \in M_1} k_i \mu_i$. For $i, j \in T$ with $i \neq j$, we have

$$z_1 := \Big[\sum_{k \in \Delta, \lambda \in S} \alpha_{k\lambda} x^k y^\lambda, x_0\Big] = \sum_{k \in \Delta, \lambda \in S} \alpha_{k\lambda} (k_0^* x^{k - e_0} x_0 y^\lambda - \beta_{k\lambda} x^k y^\lambda),$$
$$z_2 := \Big[\sum_{k \in \Delta, \lambda \in S} \alpha_{k\lambda} x^k y^\lambda, x_0 \xi_i\Big] = \sum_{k \in \Delta, \lambda \in S} \alpha_{k\lambda} (2^{-1} k_0^* x^{k - e_0} x_0 y^\lambda \xi_i - \beta_{k\lambda} x^k y^\lambda \xi_i),$$
$$z_3 := \Big[\sum_{k \in \Delta, \lambda \in S} \alpha_{k\lambda} x^k y^\lambda, x_0 \xi_i \xi_j\Big] = \sum_{k \in \Delta, \lambda \in S} \alpha_{k\lambda} (-\beta_{k\lambda} x^k y^\lambda \xi_i \xi_j).$$

If there is a $k \in \Delta$ such that $\varepsilon_0(k_0) \neq 0$, then $\varepsilon_v(k_0 - 1) + \varepsilon_v(1) < p$ for any $v \geqslant 0$. Equality (2.1) ensures that $x^{k - e_0} x_0 = x^k$. Similarly, $\varepsilon_0(k_0) = 0$ implies that $\varepsilon_0(k_0 - 1) + \varepsilon_0(1) = p$ and thereby $x^{k - e_0} x_0 = 0$. Put $W = \{k \in \Delta;\ \varepsilon_0(k_0) \neq 0\}$. Thus

$$z_1 = \sum_{k \in W, \lambda \in S} \alpha_{k\lambda} (k_0^* - \beta_{k\lambda}) x^k y^\lambda + \sum_{k \in \Delta \setminus W, \lambda \in S} \alpha_{k\lambda} (-\beta_{k\lambda} x^k y^\lambda),$$
$$z_2 = \sum_{k \in W, \lambda \in S} \alpha_{k\lambda} (2^{-1} k_0^* - \beta_{k\lambda}) x^k y^\lambda \xi_i + \sum_{k \in \Delta \setminus W, \lambda \in S} \alpha_{k\lambda} (-\beta_{k\lambda} x^k y^\lambda \xi_i),$$
$$z_3 = \sum_{k \in \Delta, \lambda \in S} \alpha_{k\lambda} (-\beta_{k\lambda} x^k y^\lambda \xi_i \xi_j).$$

If there is a 2-tuple (k, λ), $k \in \Delta$, $\lambda \in S$, such that $\beta_{k\lambda} \not\equiv 0 \pmod p$, then at least two of two elements z_1, z_2, z_3 are nonzero and our assertion is affirmed. Otherwise, z_1 and z_2 are linearly independent.

If $\varepsilon_0(k_0) = 0$ for all $k \in \Delta$, then $x^{k-e_0}x_0 = 0$ ensures that

$$z_1 = \sum_{k\in\Delta,\lambda\in S} \alpha_{k\lambda}(-\beta_{k\lambda}x^k y^\lambda),$$
$$z_2 = \sum_{k\in\Delta,\lambda\in S} \alpha_{k\lambda}(-\beta_{k\lambda}x^k y^\lambda \xi_i),$$
$$z_3 = \sum_{k\in\Delta,\lambda\in S} \alpha_{k\lambda}(-\beta_{k\lambda}x^k y^\lambda \xi_i\xi_j).$$

If there exists a 2-tuple (k, λ), $k \in \Delta$, $\lambda \in S$, such that $\beta_{k\lambda} \not\equiv 0 \pmod p$, then all z_1, z_2 and z_3 are nonzero elements. Considering the basic elements $x^k y^\lambda$, $x^k y^\lambda \xi_i$ and $x^k y^\lambda \xi_i \xi_j$ on the right-hand side of the equalities above, we know that any two of the elements z_1, z_2, z_3 are linearly independent. If $\beta_{k\lambda} \equiv 0 \pmod p$ for all $k \in \Delta$ and $\lambda \in S$, then for any $k \in \Delta$ there is an $i \in M_1$ such that $k_i \neq 0$. For $j \in T$, we have

$$[f, x_0x_{i'}] = [i]\alpha_{k\lambda}k_i^* x^{k-e_i}y^\lambda x_0 + \ldots \neq 0, \quad [f, x_{i'}\xi_j] = [i]\alpha_{k\lambda}k_i^* x^{k-e_i}y^\lambda \xi_j + \ldots \neq 0.$$

Since their $\mathbb{Z}$-degrees are unequal, $[f, x_0x_{i'}]$ and $[f, x_{i'}\xi_j]$ are linearly independent.

(2) If f contains some ξ_l, where $l \in T$ and $D_l(f) \neq 0$, then f has only two possibilities.

(a) f contains x^π. Since $f \notin \mathrm{span}_{\mathbb{F}}\{x^\pi \xi^\omega \alpha(y)\}$, there exists a $j \in T$ such that ξ_j does not occur in f. So we can suppose that $f = x^\pi y^\lambda \xi^u + \ldots$, where $u \neq \emptyset$ and $j \notin \{u\}$. Then

$$z_1 := [f, x_i\xi_j] = -[i']x^{\pi-e_{i'}}y^\lambda \xi^u \xi_j + \ldots \neq 0,$$
$$z_2 := [f, x_{i'}\xi_j] = -[i]x^{\pi-e_i}y^\lambda \xi^u \xi_j + \ldots \neq 0.$$

It is easy to see that z_1 and z_2 are linearly independent.

(b) There is some $i \in M$ such that $x_i^{\pi_i}$ does not appear in f. If ξ^ω occurs in f, then we may assume that $f = x^k y^\lambda \xi^\omega + \ldots$, where $k_i \neq \pi_i$ for some $i \in M$. Hence there exists a t $(0 \leqslant t \leqslant s_i)$ such that $x^k x^{p^t e_i} \neq 0$. Then

$$z_1 := [f, x^{p^t e_i}\xi_j] = (-1)^{|q|}x^k x^{p^t e_i}y^\lambda \xi^{\omega-(j)} + \ldots \neq 0,$$
$$z_2 := [f, x^{p^t e_i}\xi_{j+1}] = (-1)^{|q|}x^k x^{p^t e_i}y^\lambda \xi^{\omega-(j+1)} + \ldots \neq 0,$$

and they are linearly independent.

If ξ_j does not arise in f for some $j \in T$, then we let

$$f = x^k y^\lambda \xi^u + \sum_{l,\eta,v} a_{l\eta v}x^l y^\eta \xi^v,$$

where $a_{l\eta v} \in \mathbb{F}$ and $u \neq \emptyset$. By the assumption, we see that $j \notin \{u\}$, $j \notin \{v\}$, $k_i < \pi_i$ and $l_i < \pi_i$. Now let $\iota \in \{u\}$. Then

$$z_1 := [f, \xi_\iota \xi_j] = (-1)^{|u|} x^k y^\lambda \xi^{u-(\iota)} \xi_j + \ldots \neq 0.$$

By virtue of $k_i < \pi_i$, there is a $t \in \{0, 1, \ldots, s_i\}$ such that $x^k x^{p^t e_i} \neq 0$. Then

$$z_2 := [f, x^{p^t e_i} \xi_\iota] = (-1)^{|u|} x^k x^{p^t e_i} y^\lambda \xi^{u-(\iota)} + \ldots \neq 0,$$

and our assertion follows.

(3) If f contains some ξ_l, where $l \in T$ and $D_l(f) = 0$, then $f = \xi_l y + \ldots$. We see that

$$[f, x_0 x_i] = -2^{-1} x_i y \xi_l + \ldots \neq 0, \quad [f, x_0 x_i'] = -2^{-1} x_{i'} y \xi_l + \ldots \neq 0,$$

and they are linearly independent. $\square$

Let L be a finite-dimensional $\mathbb{Z}$-graded Lie superalgebra. We denote by $\varepsilon(f)$ the nonzero $\mathbb{Z}$-homogeneous component of $f \in L$ with the least $\mathbb{Z}$-degree.

Lemma 3.3. *Let $f_1, \ldots, f_t \in L \backslash \{0\}$. If $\{f_i;\ i = 1, \ldots, t\}$ are linearly dependent, then $\{\varepsilon(f_i);\ i = 1, \ldots, t\}$ are linearly dependent.*

Lemma 3.4. *Let $f \in h(\Gamma)$ and $f \notin \operatorname{span}_{\mathbb{F}}\{x^\pi \xi^\omega \alpha(y)\}$. Then $I(\operatorname{ad} f) > s + 2$.*

Proof. According to Lemma 3.3, we can suppose that f is a $\mathbb{Z}$-homogeneous element. We shall proceed in two steps.

(i) $[f, y^\lambda] = 0$ for $\lambda \in H' \setminus \{0\}$. Then f does not contain x_0. Let

$$\begin{aligned} R_1 &= \{i \in M_1;\ [f, x_i y^\lambda] = 0,\ \lambda \in H' \setminus \{0\}\}, \\ R_2 &= \{j \in T;\ [f, \xi_j y^\lambda] = 0,\ \lambda \in H' \setminus \{0\}\}. \end{aligned}$$

(a) If $R_1 \cup R_2 = M_1 \cup T$, then neither x_i nor ξ_j occur in f for all $i \in M$ and $j \in T$. Thus we may assume that $f = y^\lambda$, $\lambda \in H'$. Then

$$[f, x^k \xi^u] = [y^\lambda, x^k \xi^u] = k_0^*(\lambda - 1) x^{k-e_0} y^\lambda \xi^u.$$

Hence $I(\operatorname{ad} f) \geqslant (p^{s_0+1} - 1) p^{\sum_{i \in M_1}(s_i+1)} 2^q \geqslant (p-1) p^r 2^q > r + q + 2 = s + 2$.

(b) Let $R_2 = \emptyset$, $|R_1| \leqslant 1$. If $|R_1| = 0$, i.e., $R_1 = \emptyset$, then $\{[f, x_i y^\lambda],\ [f, \xi_j y^\lambda];\ i \in M_1, j \in T, y \in H' \setminus \{0\}\}$ are linearly independent. If $|R_1| = 1$, we suppose $R_1 = \{l\}$.

We see that $\{[f, x_i y^\lambda], [f, \xi_j y^\lambda];\ i \in M_1 \setminus \{l\},\ j \in T,\ y \in H' \setminus \{0\}\}$ are linearly independent. Thus

$$I(\operatorname{ad} f) \geqslant (r+q-1)p^m \geqslant (r+q-1)p > s+2.$$

(c) Let $\emptyset \neq R_1 \cup R_2 \neq M_1 \cup T$. Set $J' = \{i \in R_1;\ i' \in R_1\}$. So we may assume that $J' = \{i_1, i'_1, \dots, i_u, i'_u\}$. Put $J_1 = R_1 \setminus J' = \{i_{u+1}, \dots, i_{u+t}\}$ and $R_2 = \{j_1, \dots, j_h\}$. Let $J_2 = \{i'_{u+1}, \dots, i'_{u+t}\}$ and $\overline{J} = (M_1 \cup T) \setminus (R_1 \cup R_2 \cup J_2)$. Put

$$x^\gamma = \prod_{k \in J'} x^{\gamma_k e_k}, \quad \gamma_k = 0, 1, \dots, \pi_k,\ \xi^v = \prod_{j \in R_2} \xi_j^{v_j},\ v_j = 0, 1.$$

For any $l' \in J_2$ and $\beta_{l'} \in \{1, 2, \dots, p-1\}$, we see that

$$[f, x^\gamma x^{\beta_{l'} e_{l'}} \xi^v] = [l] \beta_{l'} D_l(f) x^\gamma x^{\beta_{l'} e_{l'} - e_{l'}} \xi^v. \tag{3.1}$$

For all $j \in \overline{J}$ we obtain

$$[f, x^\gamma x_j \xi^v] = [j'] D_{j'}(f) x^\gamma \xi^v, \tag{3.2}$$

$$[f, x^\gamma \xi^v \xi_j] = (-1)^{|f|} D_j(f) x^\gamma \xi^v. \tag{3.3}$$

Since $l' \in J_2$, $D_l(f) y^\lambda \neq 0$. As f does not contain x_i for all $i \in J'$, we have $D_l(f)$, $D_l(f) x^\gamma \neq 0$. By a similar argument we obtain $D_l(f) x^\gamma \xi^v \neq 0$ and then $D_l(f) x^\gamma \xi^v x^{\beta_{l'} e_{l'} - e_{l'}} \neq 0$. Similarly, $D_{j'}(f) x^\gamma \xi^v \neq 0$ and $D_j(f) x^\gamma \xi^v \neq 0$. It is easy to see that the nonzero elements on the right-hand side of equalities (3.1), (3.2) and (3.3) are linearly independent. Therefore,

$$\begin{aligned}
I(\operatorname{ad} f) &\geqslant p^{\sum_{i \in J'} (s_i+1)} 2^h (p-1)t + p^{\sum_{i \in J'} (s_i+1)} 2^h (s - 2u - 2t - h) \\
&\geqslant p^{2u} 2^h (p-1)t + p^{2u} 2^h (s - 2u - 2t - h) \\
&= p^{2u} 2^h (s - 2u - h + (p-3)t).
\end{aligned}$$

Let $2u + h > 0$. If $t > 0$, by $s = r + q \geqslant 2n + 2 \geqslant 4$ we have

$$\begin{aligned}
I(\operatorname{ad} f) &\geqslant 2^{2u+h} \big(s - (2u+h) + (p-3)t\big) \\
&= 2^{2u+h} \big(s - (2u+h)\big) + 2^{2u+h} (p-3)t \\
&\geqslant 2(s-1) > s+2.
\end{aligned}$$

If $t = 0$, then $s \geqslant 4$ implies that

$$\begin{aligned}
I(\operatorname{ad} f) &\geqslant p^{2u} 2^h (s - (2u+h)) \\
&= ((p-2)+2)^{2u} 2^h (s - (2u+h)) \\
&\geqslant (p-2)^{2u} 2^h (s - (2u+h)) + 2^{2u+h} (s - (2u+h)) \\
&\geqslant 2(2^{2u+h}(s - (2u+h))) \geqslant 2(2(s-1)) = s + (3s-4) > s+2.
\end{aligned}$$

Let $2u+h=0$. Then $u=h=0$. As $R_1\cup R_2\neq\emptyset$, $t>0$. If $t>1$, then $I(\operatorname{ad} f)\geqslant s+(p-3)t\geqslant s+4>s+2$. If $t=1$, we see that $R_2=\emptyset$ and $|R_1|=1$. Part (b) then yields $I(\operatorname{ad} f)>s+2$.

(ii) $[f,1]\neq 0$. If there exists a $j\in T$ such that $[f,\xi_j]=0$, then

$$0\neq [f,1]=-[f,[\xi_j,\xi_j]]=-[[f,\xi_j],\xi_j]-(-1)^{|f|}[\xi_j,[f,\xi_j]]=0,$$

a contradiction. So $[f,\xi_j]\neq 0$ for all $j\in T$.

(a) Set $R_3=\{i\in M_1;\ [f,x_i]=0\}$. Then $R_3\neq\emptyset$. If $i\in R_3$, then $i'\in R_3$. Otherwise,

$$[i][f,y^{2\lambda}]=[f,[x_iy^\lambda,x_{i'}y^\lambda]]=[[f,x_iy^\lambda],x_{i'}]+[x_i,[f,x_{i'}y^\lambda]]=0,$$

contradicting $[f,1]\neq 0$. Thus we may assume that $R_1=\{1,\ldots,t\}$. Put $J=\{i,i';\ i=1,\ldots,t\}$ and $\widetilde{J}=(M_1\cup T)\setminus J$. Set

$$P=\{k_1e_{1'}+\ldots+k_te_{t'};\ 0\leqslant k_i\leqslant p-1,\ i=1,\ldots,t\}.$$

For all $g\in\operatorname{span}_{\mathbb{F}}\{x^k;\ k\in P\}$, we will show that if $[f,g]=0$, then $g=0$. Otherwise, if $g\neq 0$, we choose $g\in\operatorname{span}_{\mathbb{F}}\{x^k;\ k\in P\}$ with the least $\mathbb{Z}$-degree satisfying $[f,g]=0$. If $\operatorname{zd}(g)=-2$, we let $g=1$. Then $[f,1]=0$, a contradiction. Let $\operatorname{zd}(g)>-2$, then there is an $i\in\{2,\ldots,t\}$ such that $D_{i'}(g)\neq 0$. Hence $[x_i,[f,g]]=[[x_i,f],g]+[f,[x_i,g]]=[f,[x_i,g]]=[i][f,D_{i'}(g)]=0$. This contradicts the choice of g with the least $\mathbb{Z}$-degree and our assertion is true. It is easy to see that $[f,x_j]\neq 0$ and $[f,\xi_j]\neq 0$ for all $j\in\widetilde{J}$. Because $|P|=p^t$, $|\widetilde{J}|=s-2t$ and $t>0$, we have

$$I(\operatorname{ad} f)\geqslant p^t+s-2t\geqslant 1+t(p-1)+(s-2t)=s+1+t(p-3)>s+2.$$

(b) $R_3=\emptyset$. Then $[f,x_i]\neq 0$ for all $i\in M_1$. Moreover, $[f,\xi_j]\neq 0$ for all $j\in T$. According to Lemma 3.2, there exist two basis elements f_1 and f_2 with $\operatorname{zd}(f_j)\geqslant 0,\ j=1,2$, such that $[f,f_1]$ and $[f,f_2]$ are linearly independent. Therefore $\{[f,1],[f,x_i],[f,\xi_i],[f,f_j];\ i\in R,\ j=1,2\}$ are linearly independent. Thus $I(\operatorname{ad} f)>s+2$.

(iii) $[f,1]=0$ and $[f,y^\lambda]\neq 0$ for $\lambda\in H'\setminus\{0\}$. Then we may assume that $f=x_0y$. Put $S=\{i\in M_1; mu_i=0\}$. Clearly, if $i\in S$, then $i'\notin S$. Thus $[f,x_i]\neq 0$ for $i\in S$. By computation, we see that $\{[f,x_\varepsilon],\ [f,x_ix_{i'}\xi^u],\ [f,\xi_j],\ [f,x_{\varepsilon'}\xi_j\xi_l];\ \varepsilon\in S,\ i\in M_1,j,l\in T,\ u\in\mathbb{B}(q)\setminus\mathbb{B}_0\}$ are linearly independent. Hence

$$I(\operatorname{ad} f)\geqslant n+n(2^q-1)+q+nq(q-1)>2n+q+2=s+2,$$

as desired. $\square$

Lemma 3.5. *Let $f_i = g_i + h_i$, where $f_i, g_i, h_i \in L$, $i = 1, 2, \ldots, t$. If $\{g_i;\ i = 1, 2, \ldots, t\}$ are linearly independent and $\operatorname{span}_{\mathbb{F}}\{g_i;\ i = 1, 2, \ldots, t\} \cap \operatorname{span}_{\mathbb{F}}\{h_i;\ i = 1, 2, \ldots, t\} = 0$, then $\{f_i;\ i = 1, 2, \ldots, t\}$ are linearly independent.*

Lemma 3.6. *$I\Big(\operatorname{ad}\Big(\sum\limits_{i\in M} yx_i^{\pi_i+1}\Big)\Big) > s+2$ and $I(yD_0) > s+2$.*

Proof. Set $V_i = \{x^k\xi^u y^\eta;\ k_0 = k_i = 0,\ 2 \leqslant k_t \leqslant \pi_t,\ k \in Q,\ u \in \mathbb{B}(q),\ \eta \in H,\ t \in M_1 \setminus \{i\}\}$ for $i \in M_1$. By computation, we see that

$$\operatorname{ad}\left(\sum_{i\in M} yx_i^{\pi_i+1}\right)(z) = yx_0^{\pi_0}\overline{\partial}(z) + yx_i^{\pi_i}D_{i'}(z) \neq 0,\ \forall\, z \in V_i.$$

Clearly $\operatorname{span}_{\mathbb{F}}\{yx_0^{\pi_0}\overline{\partial}(z);\ z \in V_i\} \cap \operatorname{span}_{\mathbb{F}}\{yx_i^{\pi_i}D_{i'}(z); z \in V_i\} = 0$. Since $\{yx_i^{\pi_i}D_{i'}(z);\ z \in V_i\}$ are linearly independent, it follows from Lemma 3.5 that $\{\operatorname{ad}(\sum\limits_{i\in M} yx_i^{\pi_i+1})(z);\ z \in V_i\}$ are linearly independent. Hence

$$I\left(\operatorname{ad}\left(\sum_{i\in M} yx_i^{\pi_i+1}\right)\right) \geqslant \prod_{j\in M_1\setminus\{i\}} (p^{s_j+1}-2)2^q p^m \geqslant p(p-2)^{r-1}2^q > s+2.$$

As $yD_0(x^k y^\lambda \xi^u) = x^{k-e_0}y^\lambda\xi^u \neq 0$ for $1 \leqslant k_0 \leqslant \pi_0$, we have

$$I(yD_0) \geqslant (p^{s_0+1}-1)p^{\sum_{i\in M_1}(s_i+1)+m}2^q \geqslant (p-1)p^{r+1}2^q > s+2.$$

□

Theorem 3.1. *$I(\operatorname{Der}(\Gamma)) = s+2$. If $\varphi \in h(\operatorname{Der}(\Gamma))$, then $I(\varphi) = s+2$ if and only if $0 \neq \varphi \in \operatorname{span}_{\mathbb{F}}\{B\}$.*

Proof. Lemma 3.1 implies that $I(h(\operatorname{Der}(\Gamma))) \leqslant s+2$. Let $\varphi \in h(\operatorname{Der}(\Gamma))$. Then $I(\varphi) \leqslant s+2$. By virtue of Lemma 2.1, we suppose that

$$\varphi = \operatorname{ad} f + \sum_{i\in M} \beta_i \operatorname{ad}(yx_i^{\pi_i+1}) + \gamma yD_0 + \sum_{i\in M}\sum_{v=1}^{s_i} \alpha_{iv} D_i^{p^v} + D_\theta,$$

where $f \in \widehat{L}$, β_i, γ, $\alpha_{iv} \in \mathbb{F}$. We will prove that $\beta_i = \gamma = \alpha_{iv} = 0$ and $\theta = 0$.

Suppose that there is an $l \in M$ such that $\alpha_{lv} \neq 0$. Put $t = \max\{v; \alpha_{lv} \neq 0\}$. Let

$$U = \{k \in Q;\ k_l = p^t,\ p^{s_i} \leqslant k_i \leqslant \pi_i,\ \forall\, i \in M \setminus \{l\}\}.$$

For any $k \in U$, we have

$$\varphi(x^k y^\lambda \xi^u) = \alpha x^{k-p^t e_l} y^\lambda \xi^u + g,$$

where $\alpha \in \mathbb{F}$ and g is indeed a $\mathbb{F}$-linear combination of some elements of $\{x^{k'}y^{\eta}\xi^{v};\ k'_l \neq 0\}$. It follows from Lemma 3.5 that

$$\{\alpha x^{k-p^t e_l} y^{\lambda}\xi^{u} + g;\ k \in U,\ \lambda \in H,\ u \in \mathbb{B}(q)\}$$

are linearly independent. Then $I(\varphi) \geqslant (p-1)^r p^m 2^q > s+2$, contradicting $I(\varphi) \leqslant s+2$. So $\alpha_{iv} = 0$.

Now let $\varphi = \operatorname{ad} f + \sum_{i\in M} \beta_i \operatorname{ad}(yx_i^{\pi_i+1}) + \gamma y D_0 + D_\theta$. Put $\varepsilon(f) = h$. Assume $\gamma \neq 0$. Set $W = \{x^k\xi^u;\ 1 \leqslant k_0 \leqslant \pi_0\}$. If $\mathrm{zd}(h) = -2$, then $\varepsilon(\varphi(z)) = \operatorname{ad} h(z) + \gamma y D_0(z)$ for $z \in W$. Since $h \neq y$, we have $\operatorname{span}_{\mathbb{F}}\{\operatorname{ad} h(z);\ z \in W\} \cap \operatorname{span}_{\mathbb{F}}\{\gamma y D_0(z);\ z \in W\} = 0$. As $\{\gamma y D_0(z);\ z \in W\}$ are linearly independent, $\{\varepsilon(\varphi(z));\ z \in W\}$ are linearly independent by Lemma 3.5. It follows from Lemma 3.6 that $I(yD_0) > s+2$. Thus $I(\varphi) > s+2$, a contradiction. So $\mathrm{zd}(h) \neq -2$. Let $\mathrm{zd}(h) \geqslant -1$. Then $\varepsilon(\varphi(z)) = \gamma y D_0(z)$. Lemma 3.6 means that $I(yD_0) > s+2$, a contradiction. Thus $\gamma = 0$.

Now let $\varphi = \operatorname{ad} f + \sum_{i\in M} \beta_i \operatorname{ad}(yx_i^{\pi_i+1}) + D_\theta$. If $\mathrm{zd}(h) = -1$, then $\varepsilon(\varphi(z)) = \operatorname{ad} h(z)$ for $z \in \Gamma$. As $I(\operatorname{ad}(h)) > s+2$, we have $I(\varphi) > s+2$, a contradiction. Hence $\mathrm{zd}(h) \geqslant 0$. Suppose that $\theta \neq 0$. Then there is an $\eta \in H$ such that $\theta(\eta) \neq 0$. If $\mathrm{zd}(h) \geqslant 1$, we set

$$U_1 = \left\{x^k y^{\eta}\xi^u;\ 2k_0 + \sum_{i\in M_1} k_i + |u| = 2,\ \theta(\eta) \neq 0\right\}.$$

Then $\varepsilon(\varphi(z)) = D_\theta(z) = \theta(\eta)z$ for all $z \in U_1$. So $\{\varepsilon(\varphi(z));\ z \in U_1\}$ are linearly independent. Thus $I(\varphi) > s+2$, a contradiction. Let $\mathrm{zd}(h) = 0$. Set

$$h = \left(\sum_{i,j\in M_1} a_{ij}x_ix_j + \sum_{i\in M_1, j\in T} b_{ij}x_i\xi_j + \sum_{i,j\in T} c_{ij}\xi_i\xi_j + \mu x_0\right)y^{\lambda},$$

where $a_{ij}, b_{ij}, c_{ij}, \mu \in \mathbb{F}$. Put

$$U_2 = \left\{\prod_{j=1}^{t} \xi_{r+j} y^{\eta};\ t = 1, \dots, q\right\} \cup \{x^{te_i + te_{i'}} y^{\eta}\xi^{\omega};\ i = 1, \dots, n,\ t = 1, \dots, 5\}.$$

By direct computation , we have

$$\varepsilon(\varphi(z))) = (\operatorname{ad} h + D_\theta)(z) \neq 0, \quad \forall\, z \in U_2.$$

Considering the $\mathbb{Z}$-degree of $\varepsilon(\varphi(z))$, we obtain that $\{\varepsilon(\varphi(z));\ z \in U_2\}$ are linearly independent. So $I(\varphi) \geqslant 5n + q > s+2$, a contradiction; that is, $\theta = 0$.

Now $\varphi = \operatorname{ad} f + \sum\limits_{i\in M} \beta_i \operatorname{ad}(yx_i^{\pi_i+1})$. If $\operatorname{zd}(h) < \pi_i - 1$, then $\varepsilon(\varphi(z)) = \operatorname{ad} h(z)$ for all $z \in \Gamma$. As $I(\operatorname{ad}(h)) > s+2$, we have $I(\varphi) > s+2$, a contradiction. Suppose $\operatorname{zd}(h) = \pi_i - 1$ and $\beta_i \neq 0$. For $z \in V_i$ in Lemma 3.6, we have $\varepsilon(\varphi(z)) = \operatorname{ad} h(z) + \beta_i \operatorname{ad}(yx_i^{\pi_i+1})(z) \neq 0$. Considering the $\mathbb{Z}$-degree of $\varepsilon(\varphi(z))$, we obtain that $\{\varepsilon(\varphi(z));\ z \in V_i\}$ are linearly independent. Hence $I(\varphi) > s+2$, a contradiction; that is, $\beta_i = 0$ for all $i \in M$. Let $\operatorname{zd}(h) > \pi_i - 1$. Then $\varepsilon(\varphi(z)) = \operatorname{ad}\Big(\sum\limits_{i\in M} \beta_i y x_i^{\pi_i+1}\Big)(z)$ for all $z \in \Gamma$. It follows from Lemma 3.6 that $I\Big(\operatorname{ad}\Big(\sum\limits_{i\in M} \beta_i y x_i^{\pi_i+1}\Big)\Big) > s+2$. Then $I(\varphi) > s+2$, a contradiction. Thus $\beta_i = 0$.

Now let $\varphi = \operatorname{ad} f$. Lemma 3.4 implies that $I(\operatorname{Der}(\Omega)) = s+2$ and if $I(\varphi) = s+2$, then $\varphi = \operatorname{ad} x^{\pi}\xi^{\omega}\alpha(y)$. Assume that $\alpha(y) \notin \operatorname{span}_{\mathbb{F}}\{\chi(y)\}$. Since $\operatorname{span}_{\mathbb{F}}\{\chi(y)\}$ is the only one-dimensional ideal of $\mathbb{F}[y]$ (see [18]), there is a $\nu \in H$ such that $\alpha(y)$ and $\alpha(y)y^{\nu}$ are linearly independent. Now $\varphi(y^{\nu}) = [x^{\pi}\xi^{\omega}\alpha(y), y^{\nu}] = (\nu - 1) \times x^{\pi - e_0}\xi^{\omega}\alpha(y)y^{\nu}$ implies that the images of the s+1 elements 1, y^{ν}, x_i, ξ_j are linearly independent for all $i \in M$ and $j \in T$. So $I(\varphi) > s+2$. This contradicts the fact that $I(\varphi) = s+2$. Therefore $\alpha(y) \in \operatorname{span}_{\mathbb{F}}\{\chi(y)\}$ and $\varphi \in \operatorname{span}_{\mathbb{F}}\{B\}$. $\square$

Let ϱ be the induced representation of $\mathfrak{C}$ on $\Gamma/\mathfrak{C}$, i.e.,

$$\begin{aligned} \varrho(f)\colon\quad & \Gamma/\mathfrak{C} \to \Gamma/\mathfrak{C} \\ & (g+\mathfrak{C}) \mapsto [f,g] + \mathfrak{C}, \quad \text{where } f \in \mathfrak{C},\ g \in \Gamma. \end{aligned}$$

Lemma 3.7. *$\mathfrak{C}$ is an invariant maximal subalgebra of Γ.*

P r o o f. First we will show that ϱ is irreducible.

For all $f \in \Gamma$, the element $f + \mathfrak{C} \in \Gamma/\mathfrak{C}$ will be denoted by $\overline{f}$. Assume that V is a nonzero submodule of $\Gamma/\mathfrak{C}$ and

$$0 \neq \overline{f} = \gamma\bar{1} + \delta\overline{x_0} + \sum_{i\in M_1} \alpha_i \overline{x_i} + \sum_{j\in T} \beta_j \overline{\xi_j} \in V,$$

where γ, δ, α_i, $\beta_j \in \mathbb{F}$. If there is an $i \in M_1$ (or $j \in T$) such that $\alpha_i \neq 0$ (or $\beta_j \neq 0$), then

$$\varrho(x_i x_{i'})\overline{f} = \sum \Big[x_i x_{i'}, \sum_{i\in M_1} \alpha_i x_i\Big] + \mathfrak{C} = [i']\alpha_i \overline{x_i} \in V$$

$$\Big(\text{or } \varrho(\xi_i\xi_j)\overline{f} = \Big[\xi_i\xi_j, \sum_{j\in T} \beta_j \xi_j\Big] + \mathfrak{C} = \beta_j \overline{\xi_i} \in V\Big).$$

If $\alpha_i = \beta_j = 0$ for all $i \in M_1$ and $j \in T$, when $\gamma \neq 0$, we obtain

$$\varrho(x_0x_i)\overline{f} = [x_0x_i, \gamma] + [x_0x_i, \delta x_0] + \mathfrak{C} = \gamma\overline{x_i} \in V \ \ (\text{or } \varrho(x_0\xi_j)\overline{f} = \gamma\overline{\xi_j} \in V).$$

If $\gamma = 0$, we let $\delta \neq 0$. Then for $\lambda \in H$ we have

$$\begin{aligned}\varrho(x_i(1-y^\lambda))\overline{f} &= [x_i(1-y^\lambda), \delta x_0] + \mathfrak{C}\\ &= -\delta((1-\mu_i)-(1-\mu_i)y^\lambda)x_i - \delta\lambda x_i y^\lambda + \mathfrak{C} = -\delta\lambda\overline{x_i y^\lambda} \in V;\end{aligned}$$

that is, $\overline{x_i} = \overline{x_i y^\lambda} \in V$. Similarly, $\overline{\xi_j} = \overline{\xi_j y^\lambda} \in V$. In all cases we have $\overline{x_i} \in V$ (or $\overline{\xi_j} \in V$) for some $i \in M_1$ (for some $j \in T$). So

$$\begin{aligned}&[i']\varrho\big(\lambda^{-1}(1-y^\lambda)x_{i'}\big)\overline{x_i} = [i']\lambda^{-1}[(1-y^\lambda)x_{i'}, x_i] + \mathfrak{C}\\ &\quad = \lambda^{-1}(1-y^\lambda) + \mathfrak{C} \equiv 1 + \mathfrak{C} = \bar{1} \in V \text{ (or } -\varrho(\lambda^{-1}(1-y^\lambda)\xi_j)\overline{\xi_j} = \bar{1} \in V).\end{aligned}$$

Thus $\overline{x_0} = \varrho(2^{-1}x_0^2)(\bar{1}) \in V$, $\overline{x_i} = \varrho(x_0 x_i)(\bar{1}) \in V$ and $\overline{\xi_j} = \varrho(x_0\xi_j)(\bar{1}) \in V$ for all $i \in M_1$ and $j \in T$. It follows that $V = \Gamma/\mathfrak{C}$.

$\mathfrak{C}$ is invariant according to Lemma 3.1 and Theorem 3.1. Let L be any subalgebra containing $\mathfrak{C}$, then $L/\mathfrak{C}$ is a submodule of $\Gamma/\mathfrak{C}$. By the proof above, $L = \Gamma$ or $L = \mathfrak{C}$ and thereby $\mathfrak{C}$ is maximal. □

Let $\Gamma = \Gamma(r, H, q, \underline{s})$ and $\Gamma' = \Gamma(r', H', q', \underline{s}')$ be two Lie superalgebras. Let $\Gamma_{(-1)} = \Gamma$, $\Gamma_{(0)} = \mathfrak{C}$ and define

$$\Gamma_{(i)} = \{f \in \Gamma_{(i-1)};\ [f, \Gamma_{(-1)}] \subseteq \Gamma_{(i-1)}\},\ \forall\, i \geqslant 1. \tag{3.4}$$

Then we obtain a descending filtration of Γ: $\{\Gamma_{(i)};\ i \geqslant -1\}$. Similarly, Γ' possesses a filtration: $\{\Gamma'_{(i)};\ i \geqslant -1\}$ imitating the definition above with $\mathfrak{C}' = \Gamma'_{(0)}$. Set $\mathfrak{B} = \operatorname{span}_{\mathbb{F}}\{x^\pi \xi^\omega \chi(y)\}$ and $\mathfrak{B}' = \operatorname{span}_{\mathbb{F}}\{x^{\pi'}\xi^{\omega'}\chi'(y)\}$, where $\pi' = (\pi'_0, \ldots, \pi'_r)$ and $\omega' = \langle r'+1, \ldots, r'+q'\rangle$,

Lemma 3.8. *If σ is an isomorphism of Γ onto Γ', then $\sigma(\Gamma_{(0)}) = \Gamma'_{(0)}$.*

Proof. From Lemmas 3.4 and 3.1, we see that $\sigma(\mathfrak{B}) = \mathfrak{B}'$. As

$$[f, \mathfrak{B}] = 0 \iff [\sigma(f), \sigma(\mathfrak{B})] = 0, \quad \forall\, f \in \Gamma,$$

we have

$$\begin{aligned}\sigma(\Gamma_{(0)}) &= \sigma(\mathfrak{C}) = \sigma\{f \in \Gamma;\ [f, \mathfrak{B}] = 0\} = \{\sigma(f) \in \Gamma';\ [f, \mathfrak{B}] = 0\}\\ &= \{\sigma(f) \in \Gamma';\ [\sigma(f), \sigma(\mathfrak{B})] = 0\} = \big\{g \in \Gamma';\ [g, \mathfrak{B}'] = 0\big\} = \mathfrak{C}' = \Gamma'_{(0)}.\end{aligned}$$

□

By virtue of equality (3.4) and Lemma 3.8, we obtain the following theorem.

Theorem 3.2. *Let σ be an isomorphism of Γ onto Γ'. Then $\sigma(\Gamma_{(i)}) = \Gamma'_{(i)}$ for all $i \geqslant -1$.*

Corollary 3.1. *The filtration of Γ is invariant under the automorphism group of Γ.*

P r o o f. This is a direct consequence of Theorem 3.2. $\square$

Corollary 3.2. $\Gamma(r, H, q, \underline{s}) \cong \Gamma(r', H', q', \underline{s}') \iff r = r',\ m = m',\ q = q',\ s_0 = s'_0$ *and*

$$\{\{s_1, s_{1'}\}, \ldots, \{s_n, s_{n'}\}\} = \{\{s'_1, s'_{1'}\}, \ldots, \{s'_n, s'_{n'}\}\}. \tag{3.5}$$

P r o o f. We only need to prove the necessary condition. Since $\dim \Gamma = \dim \Gamma'$, i.e., $2^q p^{\sum_{i\in M}(s_i+1)+m} = 2^{q'} p^{\sum_{i\in M'}(s'_i+1)+m'}$, we have $q = q'$. If σ is an isomorphism of Γ onto Γ' and $D \in \operatorname{Der}\Gamma$, then the mapping $D \mapsto \sigma D \sigma^{-1}$ is an isomorphism of $\operatorname{Der}\Gamma$ onto $\operatorname{Der}\Gamma'$, i.e., $\operatorname{Der}\Gamma \cong \operatorname{Der}\Gamma'$. Hence $I(\operatorname{Der}\Gamma) = I(\operatorname{Der}\Gamma')$; that is, $r + q = r' + q'$. Thus $r = r'$. Furthermore, since the outer derivation subspace has the same dimension and the outer derivation D_θ is not ad-nilpotent, $m = m'$.

Note that $\Gamma = \mathfrak{C} \oplus \mathfrak{N}$ and $\Gamma' = \mathfrak{C}' \oplus \mathfrak{N}'$. One may easily verify that $\sigma(\mathfrak{N}) = \mathfrak{N}'$ by Lemma 3.8. Recall that $\sigma(\Gamma_\alpha) = \Gamma'_\alpha$, where $\alpha \in \mathbb{Z}_2$. Put

$$V_i = \{f \in \Gamma_{(i)} \cap \Gamma_{\bar 0};\ \operatorname{ad} f(\mathfrak{N} \cap \Gamma_{\bar 1}) = 0\}, \quad i \geqslant -1, \tag{3.6}$$

$$V'_i = \{g \in \Gamma'_{(i)} \cap \Gamma'_{\bar 0};\ \operatorname{ad} g(\mathfrak{N}' \cap \Gamma'_{\bar 1}) = 0\}, \quad i \geqslant -1. \tag{3.7}$$

Then $V_i = \Gamma(r, H, \underline{s})_{(i)}$ and $V'_i = \Gamma(r, H', \underline{s}')_{(i)}$. Let $V = \bigcup\limits_{i\geqslant -1} V_i$ and $V' = \bigcup\limits_{i\geqslant -1} V'_i$. It is easy to show that $V = \Gamma(r, H, \underline{s})$ and $V' = \Gamma(r, H', \underline{s}')$. It follows from (3.6) and (3.7) that $\sigma(V_i) = V'_i$ for all $i \geqslant -1$. Hence $\sigma(V) = V'$. Therefore $\Gamma(r, H, \underline{s}) \cong \Gamma(r, H', \underline{s}')$. By the consequence of Lie algebra (see [6]), we obtain $s_0 = s'_0$ and equality (3.5) holds. $\square$

4. Properties

In this section, $k \not\leqslant \pi$ denotes that there exists an $i \in M$ such that $k_i > \pi_i$. We adopt the convention that if $k_i < 0$ or $k_i > \pi_i$, then $x_i^{k_i} = 0$ for $i \in M$. It is easily seen that if $0 < k_i, k'_i < p$, then $x_i^{k_i} x_i^{k'_i - 1} = x_i^{k_i - 1} x_i^{k'_i}$.

The following lemma is easy:

Lemma 4.1. *Let $\alpha \in \mathbb{F}$ and $\varsigma \in \Pi$. Then* $\prod\limits_{j=0}^{p-1}(\alpha - j\varsigma) = \alpha^p - \alpha\varsigma^{p-1}$.

Let $L = \bigoplus\limits_{i=-r}^{s} L_i$ be a finite-dimensional simple $\mathbb{Z}$-graded Lie superalgebra. Put $L^- := \bigoplus\limits_{i=-r}^{-1} L_i$ and $L^+ := \bigoplus\limits_{i=1}^{s} L$. Then $L = L^- \oplus L_0 \oplus L^+$.

The proofs of Lemmas 4.2 and 4.4 are given in reference [32] in Chinese. For the convenience of the reader, their proofs in English will be given in Appendix.

Lemma 4.2 ([32])**.** *Let $L = \bigoplus\limits_{i=-r}^{s} L_i$ be a finite-dimensional simple $\mathbb{Z}$-graded Lie superalgebra. Suppose that $\lambda \neq 0$ is an associative form on L. Then the following statements hold.*

(1) $\lambda(L_i, L_j) = 0$ *if* $i + j \neq s - r$.

(2) $\lambda\big|_{L_i \times L_{s-r-i}}$ *is nondegenerate and* $\dim_{\mathbb{F}} L_i = \dim_{\mathbb{F}} L_{s-r-i}$, *where* $-r \leqslant i \leqslant s$.

Lemma 4.3 ([24])**.** *Suppose that $\lambda\colon L \times L \to \mathbb{F}$ is a supersymmetric bilinear form such that*

(1) λ *is L^--invariant, i.e.,* $\lambda([x, y], z) = \lambda(x, [y, z]),\ \forall\, x, z \in L,\ y \in L^-$;

(2) $\lambda\big|_{L_i \times L_s} = 0$ *for* $i > -r$;

(3) $\lambda\big|_{L_{-r} \times L_s}$ *is L_0-invariant, i.e.,* $\lambda([x, y], z) = \lambda(x, [y, z]),\ \forall\, x \in L_{-r},\ y \in L_0,\ z \in L_s$.

Then λ is an associative form on L.

Lemma 4.4 ([32])**.** *Let $L_0 \cap L_{\bar{0}} \neq 0$. If L has a nondegenerate trace form, then $r = s$.*

Theorem 4.1. *The algebra $\Gamma(r, H, q, \underline{s})$ admits a nondegenerate associative form if and only if* $3 + n - 2^{-1}q \equiv 0 \pmod{p}$.

P r o o f. Let λ be a nondegenerate associative form on Γ. By Lemma 4.2 we see that $\lambda\big|_{\Gamma_\tau \times \Gamma_{-2}}$ is nondegenerate. Then $\lambda(1, x^\pi\xi^\omega) \neq 0$. As λ is associative, $\lambda([1, x_0], x^\pi\xi^\omega) = \lambda(1, [x_0, x^\pi\xi^\omega])$. By computation, we get $-\lambda(1, x^\pi\xi^\omega) = (2 + n - 2^{-1}q)\lambda(1, x^\pi\xi^\omega)$. Since $\lambda(1, x^\pi\xi^\omega) \neq 0$, we have $3 + n - 2^{-1}q \equiv 0 \pmod{p}$.

Conversely, suppose $3 + n - 2^{-1}q \equiv 0 \pmod{p}$. Define $\sigma_{\pi\omega}\colon \Gamma \to \mathbb{F}$ such that

$$\sigma_{\pi\omega}\left(\sum_{k,\eta,u} \alpha_{k\eta u} x^k y^\eta \xi^u\right) = \alpha_{\pi 0\omega},$$

where $\alpha_{k\eta u} \in \mathbb{F}$. Clearly, $\sigma_{\pi\omega}$ is a linear mapping. We define

$$\lambda\colon \Gamma \times \Gamma \to \mathbb{F}, \quad \lambda(f, g) = \sigma_{\pi\omega}(fg).$$

It is easy to see that λ is a super-symmetric bilinear form.

For the basis elements $f = x^k y^\eta \xi^u$ and $g = x^l y^\varsigma \xi^v$ with $\varsigma \in H$, we will prove Lemma 4.3 (1) holds:

$$\lambda([y^\delta, f], g) + \lambda(f, [y^\delta, g]) = (\delta - 1)\big(k_0^* \sigma_{\pi\omega}(x^{k-e_0} x^l y^{\delta+\eta+\theta} \xi^u \xi^v) + l_0^* \sigma_{\pi\omega}(x^k x^{l-e_0} y^{\delta+\eta+\theta} \xi^u \xi^v)\big). \tag{4.1}$$

If $(k+l-e_0, \delta+\eta+\theta, \{u\} \cup \{v\}) \neq (\pi, 0, \{\omega\})$, by the definition of $\sigma_{\pi\omega}$, we see that the right hand side of equality (4.1) equals zero.

If $(k+l-e_0, \delta+\eta+\theta, \{u\} \cup \{v\}) = (\pi, 0, \{\omega\})$, then $k_0 + l_0 - 1 = \pi_0$ and $k_i + l_i = \pi_i$ for all $i \in M_1$. Thereby the right hand side of equality (4.1) equals $(\delta - 1)(k_0^* + l_0^*)$. As $k_0 + l_0 - 1 = \pi_0$, $k_0^* + l_0^* = p$. Thus the right hand side of equality (4.1) equals zero.

Similarly, for $i \in M_1$ we have

$$\begin{aligned}\lambda([x_i y^\delta, f], g) + \lambda(f, [x_i y^\delta, g]) &= (\mu_i + \delta - 1)\big(k_0^* \sigma_{\pi\omega}(x^{k-e_0} x_i x^l y^{\delta+\eta+\theta} \xi^u \xi^v) \\ &\quad + l_0^* \sigma_{\pi\omega}(x^k x^{l-e_0} x_i y^{\delta+\eta+\theta} \xi^u \xi^v)\big) + [i](k_{i'}^* \sigma_{\pi\omega}(x^{k-e_{i'}} x^l y^{\delta+\eta+\theta} \xi^u \xi^v) \\ &\quad + l_{i'}^* \sigma_{\pi\omega}(x^k x^{l-e_{i'}} y^{\delta+\eta+\theta} \xi^u \xi^v)).\end{aligned} \tag{4.2}$$

Note that $k + l - e_0 + e_{i'}$ and $k + l - e_{i'}$ cannot equal π in the mean time. If neither of them is equivalent to π, then both the sum of the first two terms and the sum of the last two terms on the right-hand side of equality (4.2) equal zero. If $(k+l-e_0+e_{i'}, \delta+\eta+\theta, \{u\} \cup \{v\}) = (\pi, 0, \{\omega\})$, then the sum of the last two terms on the right-hand side of equality (4.2) equals zero. Since $k + l - e_0 + e_{i'} = \pi$ so that $k_0^* + l_0^* = p$, the sum of the first two terms on the right-hand side of equality (4.2) equals $(\mu_i + \delta - 1)(k_0^* + l_0^*) = 0$. If $(k + l - e_{i'}, \delta + \eta + \theta, \{u\} \cup \{v\}) = (\pi, 0, \{\omega\})$, then the sum of the first two terms on the right-hand side of equality (4.2) equals zero. As $k + l - e_{i'} = \pi$ so that $k_{i'}^* + l_{i'}^* = p$, the sum of the last two terms on the right-hand side of equality (4.2) equals $[i](k_{i'}^* + l_{i'}^*) = 0$.

For $j \in T$ we obtain

$$\begin{aligned}\lambda([f, \xi_j y^\delta], g) - \lambda(f, [\xi_j y^\delta, g]) &= (2^{-1} - \delta)\big(k_0^* \sigma_{\pi\omega}(x^{k-e_0} x^l y^{\delta+\eta+\theta} \xi^u \xi^v \xi^j) \\ &\quad + l_0^* \sigma_{\pi\omega}(x^k x^{l-e_0} y^{\delta+\eta+\theta} \xi^u \xi^v \xi^j)\big) + ((-1)^{|u|} \sigma_{\pi\omega}(x^k x^l y^{\delta+\eta+\theta} \xi^{u-(j)} \xi^v) \\ &\quad + \sigma_{\pi\omega}(x^k x^l y^{\delta+\eta+\theta} \xi^u \xi^{v-(j)})).\end{aligned} \tag{4.3}$$

Similarly, if $(k + l - e_0, \delta + \eta + \theta, \{u\} \cup \{v\} \cup \{j\}) = (\pi, 0, \{\omega\})$, then both the sum of the first two terms and the sum of the last two terms on the right-hand side of equality (4.3) equal 0. If $(k + l, \delta + \eta + \theta, \{u\} \cup \{v\} \setminus \{j\}) = (\pi, 0, \{\omega\})$, then the sum of the first two terms on the right-hand side of equality (4.3) equals 0. Since $\xi^{u-(j)} \xi^v = -(-1)^{|u|} \xi^u \xi^{v-(j)}$, the sum of the last two terms on the right-hand side of equality (4.3) equals 0. Thus λ is Γ^--invariant.

Now we show that Lemma 4.3 (3) holds. If $\delta+\theta+\eta=0$, by $3+n-2^{-1}q\equiv 0 \pmod p$, we get

$$\lambda([x_0y^\delta, y^\theta], x^\pi y^\eta \xi^\omega) + \lambda\big(y^\theta, [x_0y^\delta, x^\pi y^\eta\xi^\omega]\big) = (3+n-2^{-1}q) - (\delta+\theta+\eta) = 0.$$

Thus $\lambda\big|_{\Gamma_{-2}\times\Gamma_\tau}$ is $\mathbb{F}x_0y^\delta-$invariant. Similarly, one may easily prove that $\lambda\big|_{\Gamma_{-2}\times\Gamma_\tau}$ is Γ_0-invariant. Finally, by equality (2.2) and the definition of λ, Lemma 4.3 (2) holds. It follows that λ is an associative form on Γ. As $\lambda\neq 0$ and Γ is simple, it is nondegenerate. □

Theorem 4.2. *The Killing form of each algebra in the family* Γ *is degenerate.*

Proof. As $\Gamma = \bigoplus\limits_{i=-2}^{\tau}\Gamma_i$, where $\tau = \sum\limits_{i\in M_1}\pi_i + 2\pi_0 + q - 2$, we see that $\Gamma_0\cap\Gamma_{\bar 0}\neq 0$ and $\tau\neq 2$. By Lemma 4.4, every trace form of Γ is degenerate. Since the trace form of the adjoint representation is the Killing form, the Killing form of Γ is degenerate. □

Lemma 4.5. *Let* $s_0 = s_i = 0$ *for* $i\in M_1$ *and* $\eta\in H'$. *Suppose* $f = x^k y^\nu \xi^u \in \Gamma_{\bar 0}$, *where* $|u|=0$ *or* $|u|$ *is an even number and* $\nu\in H$. *The following statements hold.*

(1) *If* $|u|\neq 0$, *then* $(\operatorname{ad} f)^p y^\eta = 0$.

(2) *If* $|u| = 0$, *then* $(\operatorname{ad} f)^p y^\eta = 0$ *or* $\alpha_p y^\eta$, *where* $\alpha_p\in\mathbb{F}$. *In particular, if* $(\operatorname{ad} f)^p y^\eta\neq 0$, *then we have* $f = x_0y^\nu$.

Proof. (1) As $|u|\geqslant 2$, the assertion holds by direct computation.

(2) We will prove by induction on m that

$$(\operatorname{ad} f)^m y^\eta = 0 \text{ or } \alpha_m x^{mk-me_0} y^{m\nu+\eta} \text{ with } \alpha_m\in\mathbb{F}. \tag{4.4}$$

For the case $m=1$, we have $(\operatorname{ad} f)y^\eta = 0$ or $k_0^*(1-\eta)x^{k-e_0}y^{\nu+\eta}$. Suppose the assertion is true for m. Then

$$\begin{aligned}(\operatorname{ad} f)^{m+1}y^\eta &= (\operatorname{ad} f)((\operatorname{ad} f)^m y^\eta) = [x^k y^\nu, \alpha_m x^{mk-me_0}y^{m\nu+\eta}]\\ &= (\beta_1 g - \beta_2 h)y^{(m+1)\nu+\eta} + \sum_{i\in M_1}[i]\alpha_m(mk_i^*k_{i'}^*)(g_i - h_i)y^{(m+1)\nu+\eta},\end{aligned}$$

where β_1, $\beta_2\in\mathbb{F}$ and

$$\begin{aligned} g &= x^{k-e_0}x^{mk-me_0}, \ h = x^k x^{mk-(m+1)e_0};\\ g_i &= x^{k-e_i}x^{mk-me_0-e_{i'}}, \ h_i = x^{k-e_{i'}}x^{mk-me_0-e_i}.\end{aligned}$$

By equality (2.1), we obtain $g_i = h_i = \{0, x^{(m+1)k-me_0-e_i-e_{i'}}\}$; that is, $g_i - h_i = 0$. Also by equality (2.1), we get $g, h \in \{0, x^{(m+1)k-(m+1)e_0}\}$. It follows that

$$(\operatorname{ad} f)^{m+1} y^{\eta} = 0 \text{ or } \alpha_{m+1} x^{(m+1)k-(m+1)e_0} y^{(m+1)\nu+\eta}.$$

Put $m = p$. If $pk - pe_0 \nleqslant \pi$, then $(\operatorname{ad} f)^p y^{\eta} = 0$. Let $pk - pe_0 \leqslant \pi$. As $s_0 = s_i = 0$, we have $k_0 = 0$ or 1, and $k_i = 0$ for all $i \in M_1$. If $k_0 = 0$, then $(\operatorname{ad} f)^p y^{\eta} = (\operatorname{ad} f)^{p-1}[y^{\nu}, y^{\eta}] = 0$. If $k_0 = 1$, then $(\operatorname{ad} f)^p y^{\eta} = 0$ or $(\operatorname{ad} f)^p y^{\eta} = \alpha_p y^{p\nu+\eta} = \alpha_p y^{\eta}$ by $k_i = 0$ and equality (4.4).

In particular, if $(\operatorname{ad} f)^p y^{\eta} \neq 0$, then $k_0 = 1$ and $k_i = 0$ for all $i \in M_1$, i.e., $f = x_0 y^{\nu}$. □

Lemma 4.6. *Let $s_0 = s_i = 0$ for $i \in M_1$. Let $f = x^k y^{\vartheta} \xi^u \in \Gamma_{\bar{1}}$, where $|u|$ is an odd number and $\vartheta \in H$. Then $(\operatorname{ad} f)^{2p} y^{\eta} = 0$ for $\eta \in H'$.*

P r o o f. If $|u| > 1$, a simple computation shows that

$$\begin{aligned}(\operatorname{ad} f) y^{\eta} &= [x^k y^{\vartheta} \xi^u, y^{\eta}] = k_0^*(1-\eta) x^{k-e_0} y^{\vartheta+\eta} \xi^u, \\ (\operatorname{ad} f)^2 y^{\eta} &= (\operatorname{ad} f)((\operatorname{ad} f) y^{\eta}) = k_0^*(1-\eta)[x^k y^{\vartheta} \xi^u, x^{k-e_0} y^{\vartheta+\eta} \xi^u] = 0.\end{aligned}$$

It follows that $(\operatorname{ad} f)^{2p} y^{\eta} = 0$.

If $|u| = 1$, we let $f = x^k y^{\vartheta} \xi_j$. First, we show that $(\operatorname{ad} f)^{2m} y^{\eta} = 0$ or $\alpha_{2m} x^{2mk-me_0} y^{2m\vartheta+\eta}$ with $\alpha_{2m} \in \mathbb{F}$ by induction on m. If $m = 1$, then

$$\begin{aligned}(\operatorname{ad} f) y^{\eta} &= [x^k y^{\vartheta} \xi_j, y^{\eta}] = k_0^*(1-\eta) x^{k-e_0} y^{\vartheta+\eta} \xi_j, \\ (\operatorname{ad} f)^2 y^{\eta} &= k_0^*(1-\eta)[x^k y^{\vartheta} \xi_j, x^{k-e_0} y^{\vartheta+\eta} \xi_j] = k_0^*(\eta-1) x^k x^{k-e_0} y^{2\vartheta+\eta}.\end{aligned}$$

Clearly, $x^k x^{k-e_0} = 0$ or x^{2k-e_0}. Thus $(\operatorname{ad} f)^2 y^{\eta} = 0$ or $\alpha_2 x^{2k-e_0} y^{2\vartheta+\eta}$, where $\alpha_2 = k_0^*(\eta-1) \in \mathbb{F}$. Suppose that the assertion is true for m. Then we have

$$\begin{aligned}(\operatorname{ad} f)^{2m+1} y^{\eta} &= (\operatorname{ad} f)((\operatorname{ad} f)^{2m} y^{\eta}) = [x^k y^{\vartheta} \xi_j, \alpha_{2m} x^{2mk-me_0} y^{2m\vartheta+\eta}] \\ &= (\beta_1 g - \beta_2 h) y^{(2m+1)\vartheta+\eta} \xi_j + \alpha_{2m} \sum_{i=1}^{n} (2mk_i^* k_{i'}^*)(g_i - h_i) y^{(2m+1)\vartheta+\eta} \xi_j,\end{aligned}$$

where $\beta_1, \beta_2 \in \mathbb{F}$,

$$\begin{aligned}g &= x^{k-e_0} x^{2mk-me_0}, \quad h = x^k x^{2mk-(m+1)e_0}; \\ g_i &= x^{k-e_i} x^{2mk-me_0-e_{i'}}, \quad h_i = x^{k-e_{i'}} x^{2mk-me_0-e_i}.\end{aligned}$$

From equality (2.1), we obtain $g_i = h_i = \{0, x^{(2m+1)k-me_0-e_i-e_{i'}}\}$, i.e., $g_i - h_i = 0$, and $g, h \in \{0, x^{(2m+1)k-(m+1)e_0}\}$. It follows that

$$(\operatorname{ad} f)^{2m+1} y^\eta = \gamma x^{(2m+1)k-(m+1)e_0} y^{(2m+1)\vartheta+\eta} \xi_j, \ \gamma \in \mathbb{F}.$$

Moreover,

$$\begin{aligned}(\operatorname{ad} f)^{2(m+1)} y^\eta &= [x^k y^\vartheta \xi_j, \gamma x^{(2m+1)k-(m+1)e_0} y^{(2m+1)\vartheta+\eta} \xi_j] \\ &= \alpha_{2(m+1)} x^k x^{(2m+1)k-(m+1)e_0} y^{2(m+1)\vartheta+\eta},\end{aligned}$$

where $\alpha_{2(m+1)} \in \mathbb{F}$. As $x^k x^{(2m+1)k-(m+1)e_0} = 0$ or $x^{2(m+1)k-(m+1)e_0}$, our assertion is true for $m+1$. The induction is complete.

Set $m = p$. It is easy to see that $(\operatorname{ad} f)^{2p} y^\eta = 0$ or $\alpha_{2p} x^{2pk-pe_0} y^\eta$. If $2pk - pe_0 \nleqslant \pi$, then $(\operatorname{ad} f)^{2p} y^\eta = 0$. Let $2pk - pe_0 \leqslant \pi$. As $s_0 = s_i = 0$, we have $k_0 = k_i = 0$ for all $i \in M_1$. Thus $f = y^\vartheta \xi_j$. By computation, we have $[y^\vartheta \xi_j, y^\eta] = 0$. Hence $(\operatorname{ad} f)^{2p} y^\eta = 0$. □

A Lie superalgebra $L = L_{\bar 0} \oplus L_{\bar 1}$ is called restricted if $L_{\bar 0}$ is a restricted Lie algebra and if $L_{\bar 1}$ is a restricted $L_{\bar 0}$-module (see [15], [22]). Let $p(f) = p$ if $f \in L_{\bar 0}$, and $p(f) = 2p$ if $f \in L_{\bar 1}$.

Theorem 4.3. *The algebra $\Gamma(r, H, q, \underline{s})$ is a restricted Lie superalgebra if and only if $H = \Pi$ and $s_0 = s_i = 0$ for all $i \in M_1$.*

Proof. Suppose Γ is a restricted Lie superalgebra in the family. We see that $(\operatorname{ad} 1)^p$ is an inner derivation of degree $-2p$. Lemma 2.1 implies $(\operatorname{ad} 1)^p = 0$. It follows that $0 = (\operatorname{ad} 1)^p x^\pi = (p-1)! x^{\pi - pe_0}$. Thus $s_0 = 0$. Similarly, $(\operatorname{ad} x_i)^p$ is an inner derivation of degree $-p$ and $(\operatorname{ad} x_i)^p = 0$. Then we have $0 = (\operatorname{ad} x_i)^p x^{\pi - \pi_0 e_0} = (p-1)!\, x^{\pi - \pi_0 e_0 - pe_i}$. Hence $s_i = 0$ for all $i \in M_1$.

Assume $H \neq \Pi$. Put $\theta(\eta) = \eta - \eta^p$ for all $\eta \in H$. As $H \neq \Pi$, θ is a nonzero additional mapping from H into $\mathbb{F}$. Then there exists an $\eta \in H$ such that $\theta(\eta) \neq 0$; that is, D_θ is a nonzero derivation of Γ. Clearly, D_θ is also a nonzero derivation of $\Gamma_{\bar 0}$. Lemma 2.1 implies that D_θ is not an inner derivation. In addition, $\Gamma_{\bar 0}$ is a restricted Lie algebra, whose every derivation is an inner derivation, a contradiction. Consequently, $H = \Pi$.

Now we prove the sufficient condition. By the result in [22], we only need to prove that there is $g \in \Gamma$ such that $(\operatorname{ad} f)^{p(f)} = \operatorname{ad} g$ for every basis element $f = x^k y^\eta \xi^u$. By Lemma 2.1, we suppose

$$(\operatorname{ad} f)^{p(f)} = \operatorname{ad} g + D_\theta + \alpha y D_0 + \beta \operatorname{ad}(x^\pi y^\delta \xi^\omega) + \sum_{i \in M} \gamma_i \operatorname{ad}(y x_i^{\pi_i + 1}),$$

where $g \in \Gamma$, α, β and $\gamma_i \in \mathbb{F}$. According to Lemmas 4.5 and 4.6, we have

$$(\operatorname{ad} f)^{p(f)}(1) = D_0(g) + \beta x^{\pi - e_0} y^\delta \xi^\omega + \gamma_0 y x_0^{\pi_0} = 0 \text{ or } \alpha_p.$$

Clearly, $\beta = \gamma_0 = 0$ and $D_0(g) = 0$ or α_p. Then

$$(\operatorname{ad} f)^{p(f)} = \operatorname{ad} g + D_\theta + \alpha y D_1 + \sum_{i \in M_1} \gamma_i \operatorname{ad}(y x_i^{\pi_i + 1}). \tag{4.5}$$

Let f_0 and g_0 be $\mathbb{Z}$-homogeneous components of f and g of degree 0, respectively. Acting on $x_0 x_{i'}$ by equality (4.5) we have

$$(\operatorname{ad} f_0)^{p(f)}(x_0 x_{i'}) = \operatorname{ad}\ g_0(x_0 x_{i'}) + \alpha x_{i'} y + \gamma_i y x_i^{\pi_i} x_0.$$

Considering the $\mathbb{Z}$-degree of every term, we get $\alpha = \gamma_i = 0$ for all $i \in M_1$. Then

$$(\operatorname{ad} f)^{p(f)} = \operatorname{ad}\ g + D_\theta. \tag{4.6}$$

(1) $f \in \Gamma_{\bar{0}}$. If $|u| \neq 0$, by (1) of Lemma 4.5 we have $0 = (\operatorname{ad} f)^p(1) = \operatorname{ad} g(1) = D_0 g$, i.e., $D_0 g = 0$ or $D_0 g = \iota y$, where $\iota \in \mathbb{F}$. Then $0 = (\operatorname{ad} f)^p y^\eta = \iota(1-\eta) y^{\eta+1} + \theta(\eta) y^\eta$ for all $\eta \in H' \setminus \{0\}$. It follows that $\theta(\eta) = 0$ and $\iota = 0$. In particular, $0 = (\operatorname{ad} f)^p(x_0 y) = \operatorname{ad} g(x_0 y) + D_\theta(x_0 y) = \overline{\partial}(g) y + \theta(1) x_0 y$. Since x_0 does not occur in g, $\theta(1) = 0$. Thus $\theta = 0$.

Let $|u| = 0$ and $f = x^k y^\nu$. Suppose $\theta \neq 0$ and $\theta(\eta) \neq 0$ for $\eta \in H'$. By Lemma 4.5 and equality (4.6), we have $(\operatorname{ad} f)^p y^2 = \operatorname{ad} g(y^2) + D_\theta(y^2) = -D_0(g) y^2 + \theta(2) y^2 = 0$ or $\alpha_p y^2$. Then g does not contain x_0 or $D_0(g) \in \mathbb{F} \setminus \{0\}$. If g does not contain x_0, by equality (4.6) we have $(\operatorname{ad} f)^p y^\eta = \theta(\eta) y^\eta \neq 0$. Lemma 4.5 implies $f = x_0 y^\nu$ with $\nu \in H = \Pi$. It follows from equality (4.6) that $(\operatorname{ad} x_0 y^\nu)^p x_i = \operatorname{ad} g(x_i)$ for all $i \in M_1$. Thus $D_{i'} g = [i'] \alpha_i x_i$, where $\alpha_i = \prod_{j=0}^{p-1} (1 - \mu_i - j\nu)$. Then we may assume that $g = \sum_{i \in M_1} [i'] \alpha_i x_i x_{i'} + h$, where $h \in \Gamma$ does not contain x_0 and $D_{i'} h = 0$ for all $i \in M_1$. Comparing the coefficient of $(\operatorname{ad} x_0 y^\nu)^p x_t^2 = \operatorname{ad} g(x_t^2)$, we obtain $\prod_{j=0}^{p-1} (1 - 2\mu_t - j\nu) = 2\alpha_t$. Since $\nu^{p-1} = 1$, we have $\mu_t = 0$ by Lemma 4.1. Similarly, $\mu_{t'} = 0$, contradicting $\mu_t + \mu_{t'} = 1$. Now let $D_0 g = \varepsilon \neq 0$, where $\varepsilon \in \mathbb{F}$. Then

$$\varepsilon = \operatorname{ad} g(1) = (\operatorname{ad} f)^p(1) = (\operatorname{ad} x_0 y^\nu)^p(1) = \prod_{j=0}^{p-1} (1 - j\nu) = 0,$$

a contradiction. So $\theta(\eta) = 0$ for $\eta \in H'$. As $0 = \theta(2) = \theta(1+1) = \theta(1) + \theta(1)$, $\theta(1) = 0$. Hence $\theta = 0$; that is, $(\operatorname{ad} f)^p = \operatorname{ad} g$.

(2) $f \in \Gamma_{\bar{1}}$. Lemma 4.6 yields $0 = (\operatorname{ad} f)^{2p}(1) = \operatorname{ad} g(1) + D_\theta(1) = (-1)^{|g|}D_0 g$, i.e., $D_0 g = 0$ or ιy, where $\iota \in \mathbb{F}$. Thus $(\operatorname{ad} f)^{2p} y^\eta = \operatorname{ad} g(y^\eta) + D_\theta(y^\eta) = \theta(\eta)y^\eta$ or $\iota(1-\eta)y^{\eta+1} + \theta(\eta)y^\eta$ for $0 \neq \eta \in H'$. As $(\operatorname{ad} f)^{2p} y^\eta = 0$ by Lemma 4.6, we have $\theta(\eta) = 0$ for $\eta \in H'$. Since $0 = \theta(2) = \theta(1+1) = \theta(1) + \theta(1)$, we have $\theta(1) = 0$. It follows that $\theta = 0$; that is, $(\operatorname{ad} f)^{2p} = \operatorname{ad} g$. Consequently, Γ is a restricted Lie superalgebra. □

Appendix

Proof of Lemma 4.2. (1) Clearly λ is nondegenerate. We define $\varphi\colon L \to L^*$ by means of $\varphi(x)(y) = \lambda(x, y)$, for all $x, y \in L$, where L^* denotes the dual space of L. The mapping φ is linear and as $\ker\varphi = 0$, φ is injective. Note that L^* is $\mathbb{Z}$-graded and $\varphi = \sum\limits_{i\in\mathbb{Z}} \varphi_i$, where $\varphi_i \in \operatorname{Hom}_{\mathbb{F}}(L, L^*)_i$. We shall prove that $\ker\varphi_j$ is a right ideal ($\mathbb{Z}_2$-graded is not necessary) of L for $j \in \mathbb{Z}$. We denote by $\operatorname{zh}(L)$ the set of all $\mathbb{Z}$-homogeneous elements of L. By the definition of φ and the invariance of λ, we have $\varphi([x,y])(z) = \varphi(x)([y,z])$ for all $x, y, z \in \operatorname{zh}(L)$. Then $\sum\limits_i \varphi_i([x,y])(z) = \sum\limits_i \varphi_i(x)([y,z])$. It follows that

$$\varphi_j([x,y])(z) = \varphi_j(x)([y,z]), \quad \forall\, j \in \mathbb{Z}.$$

Since φ_j is $\mathbb{Z}$-homogeneous, $\ker\varphi_j$ is a $\mathbb{Z}$-graded subspace of L. Suppose $x \in \operatorname{zh}(\ker\varphi_j)$ in the equality above, then $\varphi_j([x,y])(z) = 0$, for all $x \in \operatorname{zh}(\ker\varphi_j)$, for all $y, z \in \operatorname{zh}(L)$. Thus $\varphi_j([x,y]) = 0$, for all $x \in \operatorname{zh}(\ker\varphi_j)$, $y \in \operatorname{zh}(L)$. Furthermore,

$$\varphi_j([x,b]) = 0, \quad \forall\, x \in \operatorname{zh}(\ker\varphi_j),\ b \in L.$$

For $a \in \ker\varphi_j$, as $\ker\varphi_j$ is a $\mathbb{Z}$-graded subspace of L, we have $a = \sum\limits_i a_i$, where $a_i \in L_i \cap \ker\varphi_j$. By the equality above, we get $\varphi_j([a,b]) = 0$, i.e., $[a,b] \in \ker\varphi_j$, for all $a \in \ker\varphi_j$, $b \in L$. Then $\ker\varphi_j$ is a right ideal of L.

Since φ is an isomorphism of linear spaces, there is an index j such that $\varphi_j \neq 0$. Then $\ker\varphi_j$ is a proper right ideal ($\mathbb{Z}_2$-graded is not necessary) of L. Thus $\ker\varphi_j = 0$ and then φ_j is injective. It follows that $\varphi_j(L_{-r}) \neq 0$ and $\varphi_j(L_s) \neq 0$. As $L^* = \bigoplus\limits_{i=-s}^{r} (L^*)_i$, we have $-s \leqslant j - r,\ j + s \leqslant r$, which implies that $j = r - s$. Hence $\varphi = \varphi_{r-s}$.

For $x \in L_i$, $i \in \mathbb{Z}$, we see that $\varphi(x) = \varphi_{r-s}(x) \in (L^*)_{i+r-s}$. Noting that the $\mathbb{Z}$-gradation of $\mathbb{F}$ is trivial, we have $\lambda(x,y) = \varphi(x)(y) = 0$ for $i + j \neq s - r$, $\forall\, y \in L_j$.

(2) Note that λ is nondegenerate. The assertion follows directly from (1). □

P r o o f of Lemma 4.4. Note that any algebraically closed field is an infinite field. By Lemma 1.4.7 in [20], $L_{\bar{0}} \cap L_0$ has a Cartan subalgebra. Let H be a Cartan subalgebra of $L_{\bar{0}} \cap L_0$. Put

$$\overline{H} = \{x \in L_{\bar{0}};\ \forall h \in H,\ \exists n(h) \in \mathbb{N}\colon\ (\operatorname{ad} h)^{n(h)}(x) = 0\}. \tag{16}$$

Theorem 3.2.3 in [20] implies that $\overline{H}$ is a $\mathbb{Z}$-graded Cartan type subalgebra of $L_{\bar{0}}$; that is, $\overline{H} = \sum\limits_{i=-r}^{s} \overline{H} \cap L_i \cap L_{\bar{0}}$ and $\overline{H}_{\bar{0}} = H$. Let κ_ϱ be the trace form of the representation ϱ of L:

$$k_\varrho\colon\ L \times L \to \mathbb{F}, \quad k_\varrho(x, y) = \operatorname{str}(\varrho(x)\varrho(y)), \quad \forall x, y \in L,$$

where str is the supertrace (see [16]).

Let $L = \bigoplus\limits_{\alpha \in \Delta} L_\alpha$ be the weight space decomposition of L with respect to H. Then $\kappa_\varrho\colon\ L_i \cap L_\alpha \times L_{s-r-i} \cap L_{-\alpha} \to \mathbb{F}$ is nonsingular. Noting that κ_ϱ is a homogeneous linear mapping of degree $\bar{0}$, we obtain

$$\kappa_\varrho\colon\ L_i \cap L_\alpha \cap L_{\bar{0}} \times L_{s-r-i} \cap L_{-\alpha} \cap L_{\bar{0}} \to \mathbb{F}$$

is nonsingular, which yields

$$\dim(L_i \cap L_\alpha \cap L_{\bar{0}}) = \dim(L_{s-r-i} \cap L_{-\alpha} \cap L_{\bar{0}}).$$

Since H is a Cartan subalgebra of $L_{\bar{0}} \cap L_0$, we have $H \neq 0$ and $H = L_\theta \cap L_0 \cap L_{\bar{0}}$ with zero weight θ. Set $i = s - r$ and $\alpha = \theta$ in the equality above. Then $\dim(L_{s-r} \cap L_\theta \cap L_{\bar{0}}) = \dim(L_0 \cap L_\theta \cap L_{\bar{0}}) \neq 0$. It follows from equality (16) that $\overline{H} \supset L_{\bar{0}} \cap L_\theta$. Thus $\overline{H}_{s-r} = \overline{H} \cap L_{s-r} \cap L_{\bar{0}} \supset L_{s-r} \cap L_\theta \cap L_{\bar{0}} \neq 0$. Observing that $\overline{H}_{\bar{0}} = H$, we see that $\overline{H}_{s-r}$ is H-invariant. As $\overline{H}$ is nilpotent, $\operatorname{ad} y$ is a nilpotent linear transformation of $\overline{H}_{s-r}$ for every $y \in H \subset \overline{H}$. According to Engel's Theorem, there exists a $0 \neq x \in \overline{H}_{s-r}$ such that $[y, x] = (\operatorname{ad} y)(x) = 0$, for all $y \in H$. Since L is simple, $y \in L^{(1)}$. Suppose $s \neq r$. Then x is ad-nilpotent by $x \in \overline{H}_{s-r} = \overline{H} \cap L_{s-r} \cap L_{\bar{0}}$. Thus $\kappa_\varrho(x, y) = 0$ for $y \in H$, which indicates that $\kappa_\varrho\colon\ L_{s-r} \cap L_\theta \cap L_0 \times L_0 \cap L_\theta \cap L_{\bar{0}} \to \mathbb{F}$ is singular by $H = L_\theta \cap L_0 \cap L_{\bar{0}}$, a contradiction. Hence $s = r$. $\square$

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