Vijay Kumar Bhat

Minimal prime ideals of skew polynomial rings and near pseudo-valuation rings

Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 4, 1049–1056

Persistent URL: `http://dml.cz/dmlcz/143616`

MINIMAL PRIME IDEALS OF SKEW POLYNOMIAL RINGS AND NEAR PSEUDO-VALUATION RINGS

Vijay Kumar Bhat, Katra

(Received September 6, 2012)

Abstract. Let R be a ring. We recall that R is called a near pseudo-valuation ring if every minimal prime ideal of R is strongly prime.

Let now σ be an automorphism of R and δ a σ-derivation of R. Then R is said to be an almost δ-divided ring if every minimal prime ideal of R is δ-divided.

Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$ ($\mathbb{Q}$ is the field of rational numbers). Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further, if for any strongly prime ideal U of R with $\sigma(U) = U$ and $\delta(U) \subseteq \delta$, $U[x;\sigma,\delta]$ is a strongly prime ideal of $R[x;\sigma,\delta]$, then we prove the following:

(1) R is a near pseudo valuation ring if and only if the Ore extension $R[x;\sigma,\delta]$ is a near pseudo valuation ring.

(2) R is an almost δ-divided ring if and only if $R[x;\sigma,\delta]$ is an almost δ-divided ring.

Keywords: Ore extension; automorphism; derivation; minimal prime; pseudo-valuation ring; near pseudo-valuation ring

MSC 2010: 16N40, 16P40, 16S36

Introduction

In this paper we generalize Theorems 4.3 and 4.4 of [13], and thus answer (partially) the following question:

Question A (Question 1 of [13]). Let R be a near pseudo-valuation ring (NPVR), σ an automorphism of R and δ a σ-derivation of R. Is the Ore extension $O(R) = R[x;\sigma,\delta]$ a near pseudo-valuation ring (NPVR) (even if R is commutative Noetherian)?

The work was supported by UGC Grant F. No. 40-484/2011(SR)

All the notation is the same as in Bhat and Kumari [13], but to make the paper self contained, we give the following introduction.

All rings are associative with identity. Throughout the paper R denotes a ring with identity $1 \neq 0$. The set of all nilpotent elements of R and the prime radical of R are denoted by $N(R)$ and $P(R)$ respectively. The set of prime ideals of R is denoted by $\mathrm{Spec}(R)$ and the set of minimal prime ideals of R is denoted by $\mathrm{Min\,Spec}(R)$. The center of R is denoted by $Z(R)$. The field of rational numbers and the ring of integers are denoted by $\mathbb{Q}$ and $\mathbb{Z}$ respectively unless otherwise stated. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J.

Skew polynomial rings: This article concerns the study of skew polynomial rings over pseudo valuation rings. Therefore, we discuss these notions one by one.

Let R be a ring, σ an automorphism of R and δ a σ-derivation of R ($\delta\colon R \to R$ is an additive map with $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$ for all $a, b \in R$).

For example, let σ be an automorphism of a ring R and $\delta\colon R \to R$ any map.

Let $\varphi\colon R \to M_2(R)$ be defined by

$$\varphi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix} \quad \text{for all } r \in R.$$

Then δ is a σ-derivation of R if and only if φ is a homomorphism.

We denote the Ore extension $R[x;\sigma,\delta]$ by $O(R)$. If I is an ideal of R such that I is σ-stable, i.e., $\sigma(I) = I$ and I is δ-invariant, i.e., $\delta(I) \subseteq I$, then we denote $I[x;\sigma,\delta]$ by $O(I)$. We would like to mention that $R[x;\sigma,\delta]$ is the usual set of polynomials with coefficients in R, i.e., $\left\{\sum\limits_{i=0}^{n} x^i a_i,\ a_i \in R\right\}$ with the usual addition of polynomials and multiplication subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We take coefficients of polynomials on the left as in McConnell and Robson [19].

In case δ is the zero map, we denote the skew polynomial ring $R[x;\sigma]$ by $S(R)$ and for any ideal I of R with $\sigma(I) = I$, we denote $I[x;\sigma]$ by $S(I)$.

In case σ is the identity map, we denote the differential operator ring $R[x;\delta]$ by $D(R)$ and for any ideal J of R with $\delta(J) \subseteq J$, we denote $J[x;\delta]$ by $D(J)$.

Ore-extensions (skew-polynomial rings and differential operator rings) have been of interest to many authors. For example, see [12], [11], [14], [10], [15], [18], [19].

Pseudo-valuation rings (PVRs):

We recall that as in Hedstrom and Houston [16], an integral domain R with quotient field F is called a pseudo-valuation domain (PVD) if each prime ideal P of R is strongly prime ($ab \in P$, $a \in F$, $b \in F$ implies that either $a \in P$ or $b \in P$). Later on, Badawi and Houston in [8] showed that the definition of a strongly prime ideal is equivalent to a prime ideal being powerful.

For example, let $F = \mathbb{Q}(\sqrt{2})$. Set $V = F + xF[[x]] = F[[x]]$. Then V is a pseudo-valuation domain. We also note that $S = \mathbb{Q} + \mathbb{Q}x + x^2V$ is not a pseudo-valuation domain (Badawi [6]). For more details on pseudo-valuation rings and almost-pseudo valuation rings, the reader is referred to Badawi [6].

In Badawi, Anderson and Dobbs [7], the study of pseudo-valuation domains was generalized to arbitrary rings in the following way:

A prime ideal P of R is said to be strongly prime if aP and bR are comparable (under inclusion, i.e., $aP \subseteq bR$ or $bR \subseteq aP$) for all $a, b \in R$. A ring R is said to be a pseudo-valuation ring (PVR) if each prime ideal P of R is strongly prime. We note that a PVR is quasilocal by Lemma 1 (b) of Badawi, Anderson and Dobbs [7].

An integral domain is a PVR if and only if it is a PVD by Proposition 3.1 of Anderson [1], Proposition 4.2 of Anderson [2] and Proposition 3 of Badawi [4]. We denote the set of strongly prime ideals of R by $\mathrm{SSpec}(R)$.

In Badawi [5], another generalization of PVDs is given in the following way:

For a ring R with a total quotient ring Q such that $N(R)$ is a divided prime ideal of R, let $\varphi\colon Q \to R_{N(R)}$ be such that $\varphi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then φ is a ring homomorphism from Q into $R_{N(R)}$, and φ restricted to R is also a ring homomorphism from R into $R_{N(R)}$ given by $\varphi(r) = r/1$ for every $r \in R$. Denote $R_{N(R)}$ by T. A prime ideal P of $\varphi(R)$ is called a T-strongly prime ideal if $xy \in P$, $x \in T$, $y \in T$ implies that either $x \in P$ or $y \in P$. A ring $\varphi(R)$ is said to be a T-pseudo-valuation ring (T-PVR) if each prime ideal of $\varphi(R)$ is T-strongly prime. A prime ideal S of R is called a φ-strongly prime ideal if $\varphi(S)$ is a T-strongly prime ideal of $\varphi(R)$. If each prime ideal of R is φ-strongly prime, then R is called a φ-pseudo-valuation ring (φ-PVR).

Near pseudo-valuation rings (NPVRs):

Definition 0.1 (Definition 1.1 of Bhat [11]). A ring R is said to be a near pseudo-valuation ring (NPVR) if each minimal prime ideal P of R is strongly prime.

For example, a reduced ring is NPVR.

Here the term near may not be interpreted as near ring (Bell and Mason [9]). We note that a near pseudo-valuation ring (NPVR) is a pseudo-valuation ring (PVR), but the converse is not true. For example, a reduced ring is a NPVR, but need not be a PVR.

We recall that a prime ideal P of R is said to be divided if it is comparable (under inclusion) to every ideal of R. A ring R is called a divided ring if every prime ideal of R is divided (Badawi [3]). It is known (Lemma 1 of Badawi, Anderson and Dobbs [7]) that a pseudo-valuation ring is a divided ring.

Recall that in Bhat [11] an almost divided ring has been defined in the following way:

Let R be a ring, σ an automorphism of R and δ a σ derivation of R. An ideal I of R is called σ stable if $\sigma(I) = I$ and is called δ-invariant if $\delta(I) \subseteq I$.

Definition 0.2 (Definition 1.2 of Bhat [11])**.** Let R be a ring. Then R is said to be an almost divided ring if every minimal prime ideal of R is divided.

We also recall that a prime ideal P of R is σ-divided if it is comparable (under inclusion) to every σ-stable ideal I of R. A ring R is called a σ-divided ring if every prime ideal of R is σ-divided (see Bhat [12]).

Recall that an almost σ-divided ring and an almost δ-divided ring has been defined in Bhat [11] in the following way:

Definition 0.3 (Definition 1.3 of Bhat [11])**.** Let R be a ring. Then R is said to be an almost σ-divided ring if every minimal prime ideal of R is σ-divided.

Recall that a prime ideal P of R is δ-divided if it is comparable (under inclusion) to every σ-stable and δ-invariant ideal I of R. A ring R is called a δ-divided ring if every prime ideal of R is δ-divided.

Definition 0.4 (Definition 1.4 of Bhat [11])**.** Let R be a ring. Then R is said to be an almost δ-divided ring if every minimal prime ideal of R is δ-divided.

It is clear that every divided ring is an almost divided ring.

$\sigma(*)$ **rings:** Recall that in Krempa [17], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies that $a = 0$ for $a \in R$.

We also recall that in [18], Kwak defines a $\sigma(*)$-ring R to be a ring in which $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$, and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring.

Example 0.5. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is a field. Then $P(R) = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. Let $\sigma\colon R \to R$ be defined by $\sigma\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$. Then it can be seen that σ is an endomorphism of R and R is a $\sigma(*)$-ring.

Main result. Let R be a Noetherian ring which is an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then

(1) $P \in \operatorname{Min Spec}(O(R))$ implies that $P \cap R \in \operatorname{Min Spec}(R)$, and conversely $P_1 \in \operatorname{Min Spec}(R)$ implies that $O(P_1) \in \operatorname{Min Spec}(O(R))$.

Further, if for any $U \in \operatorname{SSpec}(R)$ with $\sigma(U) = U$ and $\delta(U) \subseteq \delta$, $O(U) = U[x; \sigma, \delta] \in \operatorname{SSpec}(R)$, then

(2) R is a near pseudo-valuation ring if and only if $O(R) = R[x;\sigma,\delta]$ is a near pseudo-valuation ring;
(3) R is an almost δ-divided ring if and only if $O(R) = R[x;\sigma,\delta]$ is an almost δ-divided ring.

These results are proved in Theorems 1.3, 1.8 and 1.9 respectively.

1. Minimal prime ideals and near pseudo-valuation rings

Theorem 1.1. *Let R be a Noetherian ring and σ an automorphism of R. Then R is a $\sigma(*)$-ring if and only if for each minimal prime U of R, $\sigma(U) = U$ and U is a completely prime ideal of R.*

Proof. See Theorem 2.4 of [14]. □

Proposition 1.2. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then $P \in \operatorname{Min\,Spec}(R)$ implies $\delta(P) \subseteq P$.*

Proof. See Proposition 3.3 of [13]. □

Theorem 1.3. *Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R. Then $P \in \operatorname{Min\,Spec}(O(R))$ implies that $P \cap R \in \operatorname{Min\,Spec}(R)$, and conversely $P_1 \in \operatorname{Min\,Spec}(R)$ implies that $O(P_1) \in \operatorname{Min\,Spec}(O(R))$.*

Proof. Let $P_1 \in \operatorname{Min\,Spec}(R)$. Then $\sigma(P_1) = P_1$ by Theorem 1.1 and $\delta(P_1) \subseteq P_1$ by Proposition 1.2. Now it can be seen that $O(P_1) \in \operatorname{Spec}(O(R))$. Suppose $O(P_1) \notin \operatorname{Min\,Spec}(O(R))$ and let $P_2 \subset O(P_1)$ be a minimal prime ideal of $O(R)$. Then $P_2 = O(P_2) \cap R \subset O(P_1) \in \operatorname{Min\,Spec}(O(R))$. Therefore $P_2 \cap R \subset P_1$, which is a contradiction, as $P_2 \cap R \in \operatorname{Spec}(R)$. Hence $O(P_1) \in \operatorname{Min\,Spec}\,(O(R))$.

Conversely suppose that $P \in \operatorname{Min\,Spec}(R)$, then it can be seen that $P \cap R \in \operatorname{Spec}(R)$, and $O(P \cap R) \in \operatorname{Spec}(O(R))$. Therefore, $O(P \cap R) = P$. We now show that $P \cap R \in \operatorname{Min\,Spec}(R)$. Suppose $P_1 \subset P \cap R$ is a minimal prime ideal of R. Then $O(P_1) \subset O(P \cap R)$ and as in the first paragraph $O(P_1) \in \operatorname{Spec}(O(R))$, which is a contradiction. Hence $P \cap R \in \operatorname{Min\,Spec}(R)$. □

Remark 1.4. Let R be a Noetherian ring which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R and δ a σ-derivation such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Then if $P \in \operatorname{Min\,Spec}(O(R))$, then $P \cap R \in \operatorname{Min\,Spec}(R)$ with $\sigma(P \cap R) = P \cap R$ and $\delta(P \cap R) \subseteq P \cap R$, and if $P_1 \in \operatorname{Min\,Spec}(R)$ such that $\sigma(P_1) = P_1$, and $\delta(P_1) \subseteq P_1$, then $O(P_1) \in \operatorname{Min\,Spec}(O(R))$.

P r o o f. The proof follows from Theorem 1.3 above. □

Theorem 1.5 (Hilbert Basis Theorem)**.** *Let R be a right/left Noetherian ring. Let σ and δ be as usual. Then the Ore extension $O(R) = R[x; \sigma, \delta]$ is right/left Noetherian.*

P r o o f. See Theorem 1.12 of Goodearl and Warfield [15]. □

The following example shows that the extension of a strongly prime ideal need not be a strongly prime ideal:

Example 1.6 (Example 3.1 of [10])**.** Let $R = \mathbb{Q}[t] = (t^2)$. Let $\sigma = \mathrm{id}$ and $\delta = 0$. For all $p(t) \in Q[t]$, we denote by $\overline{p(t)}$ the image of $p(t)$ under the natural projection $\mathbb{Q}[t] \to R$.

Now $P = \bar{t}R$ is a strongly prime ideal of R. Let $a = 1$ and $b = x$ and $J = PR[x] = \bar{t}R[x]$. Then neither $aJ \subseteq bR[x]$ nor $bR[x] \subseteq aJ$. Therefore, J is not a strongly prime ideal of $R[x]$.

Example 1.7 (Example 3.2 of [10])**.** $R = \mathbb{Z}_{(p)}$. This is in fact a discrete valuation domain, and therefore, its maximal ideal $P = pR$ is strongly prime. But $pR[x]$ is not strongly prime in $R[x]$ because it is not comparable with $xR[x]$ (so the condition of being strongly prime in $R[x]$ fails for $a = 1$ and $b = x$).

In view of Examples 1.6, 1.7 we are not able to answer Question A completely and moreover, in answering it partially we impose some conditions as given in the statements of Theorems 1.8, 1.9 below:

Theorem 1.8. *Let R be a Noetherian ring which is an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further, let $U \in \mathrm{SSpec}(R)$ with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ imply $O(U) \in \mathrm{SSpec}(R)$. Then R is a near pseudo-valuation if and only if $O(R)$ is a near pseudo-valuation ring.*

P r o o f. Let R be a near pseudo-valuation ring which is also an algebra over $\mathbb{Q}$. Now $O(R)$ is Noetherian by Theorem 1.5. Let $J \in \mathrm{Min\,Spec}(O(R))$. Then by Theorem 1.3, $J \cap R \in \mathrm{Min\,Spec}(R)$. Since R is a $\sigma(*)$-ring, $\sigma(J \cap R) = J \cap R$ and $\delta(J \cap R) \subseteq J \cap R$ by virtue of Theorem 1.1 and Proposition 1.2. Now R is a Noetherian near pseudo-valuation $\mathbb{Q}$-algebra, therefore $J \cap R \in \mathrm{SSpec}(R)$. Now by hypothesis $O(J \cap R) \in \mathrm{SSpec}(O(R))$. Now it is easy to see that $O(J \cap R) = J$. Therefore $J \in \mathrm{SSpec}(O(R))$. Hence $O(R)$ is a Noetherian near pseudo-valuation ring.

Conversely, let $O(R)$ be a near pseudo-valuation ring. Let $U \in \mathrm{Min\,Spec}(R)$ and $a, b \in R$. Then $O(U) \in \mathrm{Min\,Spec}(O(R))$, by virtue of Theorem 1.3. Since $O(R)$

is a near pseudo-valuation ring, so $O(U) \in \mathrm{SSpec}(O(R))$. Therefore $a(O(U))$ and $b(O(R))$ are comparable (say $a(O(U)) \subseteq b(O(R))$). So $a(O(U)) \cap R \subseteq b(O(R)) \cap R$, i.e., $aU \subseteq bR$. Hence R is a near pseudo-valuation ring. $\square$

Theorem 1.9. *Let R be a Noetherian ring which is an algebra over $\mathbb{Q}$ and let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$ for all $a \in R$. Further, let $U \in \mathrm{SSpec}(R)$ with $\sigma(U) \subseteq U$ and $\delta(U) \subseteq U$ imply $O(U) \in \mathrm{SSpec}(R)$. Then R is an almost δ-divided ring if and only if $O(R)$ is a Noetherian almost δ-divided ring.*

Proof. Let R be an almost δ-divided ring which is also an algebra over $\mathbb{Q}$. Hence $O(R)$ is Noetherian by Theorem 1.5. Let $J \in \mathrm{Min\,Spec}(O(R))$. Since R is a $\sigma(*)$-ring, we have $\sigma(J \cap R) = J \cap R$ and $\delta(J \cap R) \subseteq J \cap R$ by Theorem 1.1 and Proposition 1.2. Let K be a proper ideal of $O(R)$ such that $\sigma(K) = K$ and $\delta(K) \subseteq K$. Now by Theorem 1.3, $J \cap R \in \mathrm{Min\,Spec}(R)$. Also $K \cap R$ is an ideal of R with $\sigma(K \cap R) = K \cap R$ and $\delta(K \cap R) \subseteq K \cap R$. Now R is almost δ-divided, therefore $J \cap R$ and $K \cap R$ are comparable under inclusion. Say $J \cap R \subseteq K \cap R$. Therefore, $O(J \cap R) \subseteq O(K \cap R)$. Thus $J \subseteq K$. Hence $O(R)$ is a Noetherian almost δ-divided ring.

Conversely, suppose that $O(R)$ is an almost δ-divided ring. Let $U \in \mathrm{Min\,Spec}(R)$. Since R is a $\sigma(*)$-ring, we have $\sigma(U) = U$ and $\delta(U) \subseteq U$, using Theorem 1.1 and Proposition 1.2. Let V be an ideal of R such that $\sigma(V) = V$ and $\delta(V) \subseteq V$. Theorem 1.3 implies that $O(U) \in \mathrm{Min\,Spec}(O(R))$. Now $O(R)$ is an almost δ-divided ring implies that $O(U)$ and $O(V)$ are comparable under inclusion, i.e., $O(U) \subseteq O(V)$ (say). This implies that $O(U) \cap R \subseteq O(V) \cap R$, i.e., $U \subseteq V$. Hence R is an almost δ-divided ring. $\square$

Question 1.10. Let R be an NPVR. Let σ be an automorphism of R and δ a σ-derivation of R. Is $O(R) = R[x; \sigma, \delta]$ an NPVR (even if R is commutative Noetherian)?

Acknowledgement. The author would like to express his sincere thanks to the referee for his/her suggestions.

References

[1] *D. F. Anderson*: Comparability of ideals and valuation overrings. Houston J. Math. *5* (1979), 451–463.
[2] *D. F. Anderson*: When the dual of an ideal is a ring. Houston J. Math. *9* (1983), 325–332.
[3] *A. Badawi*: On divided commutative rings. Commun. Algebra *27* (1999), 1465–1474.
[4] *A. Badawi*: On domains which have prime ideals that are linearly ordered. Commun. Algebra *23* (1995), 4365–4373.

[5] *A. Badawi*: On φ-pseudo-valuation Rings. Advances in Commutative Ring Theory (D. E. Dobbs et al., eds.). Proceedings of the 3rd International Conference, Fez, Morocco Lect. Notes Pure Appl. Math. 205, Marcel Dekker, New York, 1999, pp. 101–110.
[6] *A. Badawi*: On pseudo-almost valuation domains. Commun. Algebra *35* (2007), 1167–1181.
[7] *A. Badawi, D. F. Anderson, D. E. Dobbs*: Pseudo-valuation Rings. Commutative Ring Theory (P.-J. Cahen et al., eds.). Proceedings of the 2nd International Conference, Fes, Morocco, June 5–10, 1995. Lect. Notes Pure Appl. Math. 185, Marcel Dekker, New York, 1997, pp. 57–67.
[8] *A. Badawi, E. Houston*: Powerful ideals, strongly primary ideals, almost pseudo-valuation domains, and conducive domains. Commun. Algebra *30* (2002), 1591–1606.
[9] *H. E. Bell, G. Mason*: On derivations in near-rings and rings. Math. J. Okayama Univ. *34* (1992), 135–144.
[10] *V. K. Bhat*: A note on completely prime ideals of Ore extensions. Int. J. Algebra Comput. *20* (2010), 457–463.
[11] *V. K. Bhat*: On near pseudo valuation rings and their extensions. Int. Electron. J. Algebra (electronic only) *5* (2009), 70–77.
[12] *V. K. Bhat*: Polynomial rings over pseudovaluation rings. Int. J. Math. Math. Sci. *2007* (2007), Article ID 20138, 6 pages.
[13] *V. K. Bhat, N. Kumari*: On Ore extensions over near pseudo valuation rings. Int. J. Math. Game Theory Algebra *20* (2011), 69–77.
[14] *V. K. Bhat, N. Kumari*: Transparency of $\sigma(*)$-rings and their extensions. Int. J. Algebra *2* (2008), 919–924.
[15] *K. R. Goodearl, R. B. Warfield Jr.*: An Introduction to Noncommutative Noetherian Rings. London Mathematical Society Student Texts 16, Cambridge University Press, Cambridge, 1989.
[16] *J. R. Hedstrom, E. G. Houston*: Pseudo-valuation domains. Pac. J. Math. *75* (1978), 137–147.
[17] *J. Krempa*: Some examples of reduced rings. Algebra Colloq. *3* (1996), 289–300.
[18] *T. K. Kwak*: Prime radicals of skew polynomial rings. Int. J. Math. Sci. *2* (2003), 219–227.
[19] *J. C. McConnell, J. C. Robson*: Noncommutative Noetherian Rings. With the cooperation of L. W. Small. Reprinted with corrections from the 1987 original. Graduate Studies in Mathematics 30, American Mathematical Society, Providence, 2001.

Author's address: V. K. Bhat, School of Mathematics, SMVD University, P/o Kakryal, Katra, J and K, India 182301, e-mail: `vijaykumarbhat2000@yahoo.com`.