Sebastian Burciu

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SUBGROUPS OF ODD DEPTH—A NECESSARY CONDITION

Sebastian Burciu, Bucharest

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Abstract. This paper gives necessary and sufficient conditions for subgroups with trivial core to be of odd depth. We show that a subgroup with trivial core is an odd depth subgroup if and only if certain induced modules from it are faithful. Algebraically this gives a combinatorial condition that has to be satisfied by the subgroups with trivial core in order to be subgroups of a given odd depth. The condition can be expressed as a certain matrix with $\{0,1\}$-entries to have maximal rank. The entries of the matrix correspond to the sizes of the intersections of the subgroup with any of its conjugate.

Keywords: depth of group algebras; finite group; faithful representation

MSC 2010: 34B16, 34C25

1. Introduction

In a recent paper [5] a characterization of odd depth extensions of semisimple algebras in terms of Bratelli's diagram was given by the authors. If $B \subset A$ is an extension of semisimple algebras then Theorem 3.6 from [5] shows that B is a depth $2m+1$ subalgebra of A if and only if the distance between any two simple B-modules is at most m, inside the Bratelli diagram associated to the inclusion $B \subset A$. For a description of the Bratelli diagram associated to an extension of semisimple algebras one can consult [7].

However, translating this odd depth characterization in terms of the algebra structures of B and A is in general a difficult task. For instance, even in the case of group

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algebra extensions these conditions cannot be translated in only group theoretical terms. Although the depth two subgroups are precisely the normal subgroups [8], a similar description is not yet known for any higher depth. For example, in [4] a sufficient condition for depth three was given but it is not yet known if that condition is necessary. More precisely if $H \subset G$ is an extension of finite groups, it was shown that H is a depth three subgroup of G provided that there is $x \in G$ such that $H^x \cap H = 1$. Here $H^x = x^{-1}Hx$. This result was then generalized to higher odd depth in Theorem 6.9 of [5]. It was shown in Theorem 6.9 of [5] that H is a depth $2m+1$ subgroup of G provided that H intersects trivially m of its conjugate subgroups.

This paper establishes a converse relation for subgroups with trivial core. This converse is given in terms of all the intersections of H and any of its m-conjugate subgroups. It is proven that if H is a subgroup of depth $2m+1$ then the collection of all these intersections has a special property called property (P) in this paper. This property (P) is reformulated as a matrix with $\{0,1\}$-entries to be of maximal rank. The condition is written only in terms of group theoretical concepts.

Describing all the maximal rank $\{0,1\}$-matrices is an old well known and complicated linear algebra problem. Along the years, study of these matrices can be found for instance in [9] and [6] and more recently in [11], [3] and their references. Our paper establishes a relation between odd depth subgroups and $\{0,1\}$-matrices. It would be interesting to investigate what is the exact class of $\{0,1\}$-matrices yielded from the odd depth subgroups of a finite group. To answer this problem a more detailed study of the sets $\mathcal{C}_H^n$ from the last section has to be done.

Suppose now that H is any subgroup of G and $N := \mathrm{core}_G(H)$ is its core. Proposition 6.8 of [5] implies that the depth of H/N inside G/N is less or equal than the depth of H inside G. Since H/N has trivial core inside G/N it follows that our results can also be applied to any subgroup H of G not necessarily with a trivial core. However in this situation the statement of our results become more elaborated. See for example Remark 3.6 for depth three subgroups.

We also should mention that there is not yet known any example of a subgroup of even depth strictly greater than 4. For depth 4 see Example 2.4 from page 137 of [5] namely, $D_8 \subset S_4$ under the standard inclusion.

The paper is organized as follows. In the first section we recall the algebraic notion of depth for extensions of semisimple algebras from [1]. It extends the depth notion introduced in [5]. Its connections with the equivalence relations introduced by Rieffel in [10] are also recalled here. The next section studies the depth three subgroup situation and proves the main result, stated as Corollary 3.5 in this section. The last section generalizes the previous results to higher odd depth subgroups.

2. Preliminaries

Throughout this paper we work over an algebraically closed ground field k. Suppose that $B \subseteq A$ is an inclusion of k-algebras. Set $T_1(B, A) := A$ which can be regarded as an (A, A)-bimodule via left and right multiplication with elements in A. Define inductively other (A, A)-bimodules $T_n(B, A)$ by

$$T_{n+1}(B, A) := A \bigotimes_B T_n(B, A),$$

for all $n \geqslant 1$.

Via restriction $T_n(B, A)$ may be viewed as (B, A)-bimodule and as (A, B)-bimodule, respectively. Denote these restrictions by $T_n^l(B, A)$ and $T_n^r(B, A)$, respectively. Furthermore, let $T_n(B, A)$ viewed as (B, B)-bimodule, be denoted by $T_n'(B, A)$. In addition, define $T_0(B, A)$ to be the (B, B)-bimodule B. With this notation, the ring extension itself $B \subset A$ is said to have left and right depth $2s + 1$ for some $s \geqslant 0$, if there is some $m \in \mathbb{N}$ with $T_{s+1}'(B, A)$; $mT_s'(B, A)$, that is $T_{s+1}'(B, A)$ is isomorphic to a direct summand of a direct sum of m copies of $T'(B, A)$.

Furthermore, the ring extension $B \subset A$ is said to have left depth $2s$ (respectively right depth $2s$), for some $s \geqslant 1$ if $T_{s+1}^r(B, A)$; $mT_s^r(B, A)$ (respectively $T_{s+1}^l(B, A)$; $mT_s^l(B, A)$) for some $m \in \mathbb{N}$.

If $B \subset A$ has both left and right depth d it is said to have depth d. It is easy to observe that if $B \subset A$ has depth d then it also has depth $d + 1$. Let $d(B, A)$ be the minimal depth of the ring extension $B \subset A$. Thus $d(B, A)$ is the smallest integer $d \geqslant 1$ such that $B \subset A$ has depth d provided that such an integer exists; otherwise, let $d(B, A) = \infty$.

It was shown recently in [1] that for any inclusion of finite groups $H \subset G$ the inclusion $kH \subset kG$ is always of finite depth for any field k. It is also not difficult to notice that any extension of semisimple Hopf algebras is of finite depth. It is however still an open question if an arbitrary extension of finite dimensional Hopf algebras has finite depth.

Through the rest of this paper we fix an algebraically closed field k of characteristic zero. We say that a subgroup H is of depth d inside G if the ring extension $kH \subset kG$ has depth d.

2.1. On inclusions of semisimple algebras. In this subsection we recall the characterization of odd depth extensions of semisimple algebras in terms of the Bratelli diagrams from [5]. It will be needed in the next two sections of the paper.

For a semisimple algebra A let $\mathrm{Irr}\,(A)$ be the set of irreducible A-modules. This set can also be identified with the set of all irreducible characters of A.

Suppose that $B \subseteq A$ is an inclusion of semisimple algebras. Label the simple A-modules by $V_1, \ldots, V_s$ and the simple B-modules by $M_1, \ldots, M_r$. Restrict the j-th simple A-module V_j to B and express the result in terms of simple B-modules:

$$V_j\downarrow_B^A \cong \bigoplus_{i=1}^{r} m_{ij} M_i \tag{2.1}$$

Then $M = (m_{ij})$ is an $r \times s$-matrix, and

$$M_i\uparrow_B^A = A \bigotimes_B M_i \cong \bigoplus_{j=1}^{s} m_{ij} V_j \tag{2.2}$$

since $\mathrm{Hom}_A\left(A \bigotimes_B M_i, V_j\right) \cong \mathrm{Hom}_B(M_i, V_j)$ for all i, j. In other words, we have

$$[M_i\uparrow_B^A, V_j] = m_{ij} = [M_i, V_j\downarrow_B^A] \tag{2.3}$$

where $[X, Y] := \dim \mathrm{Hom}_A(X, Y)$ for any two finite-dimensional A-modules X, Y.

2.2. The equivalence relations. We define a relation on $\mathrm{Irr}(B)$ as follows. If M and N are two irreducible B-modules we say $M \sim N$ if $M\uparrow_B^A$ and $N\uparrow_B^A$ have a common irreducible A-constituent. This relation $\sim$ is reflexive and symmetric but not transitive in general. Its transitive closure is an equivalence relation denoted by $\approx$ or d_B^A. Thus $M \approx N$ if and only if there are $M_0, \ldots, M_m \in \mathrm{Irr}(B)$ such that $M = M_0 \sim M_1 \sim \ldots \sim M_m = N$.

Similarly define a relation on $\mathrm{Irr}(A)$ by $W \sim V$ if $W\downarrow_B^A$ and $V\downarrow_B^A$ have a common irreducible constituent. This relation $\sim$ is again reflexive and symmetric but not transitive in general. Its transitive closure is an equivalence relation denoted by $\approx$ or u_B^A. Thus $W \approx V$ if and only if there are $V_0, \ldots, V_r \in \mathrm{Irr}(A)$ such that $W = V_0 \sim V_1 \sim \ldots \sim V_r = V$.

These two equivalence relations were first considered in [10].

2.3. Distance between modules (characters). Let $i, j \in \{1, \ldots, r\}$ be two different indices. We say that the distance $d(M_i, M_j)$ between M_i and M_j is m if m is the smallest number such that there are $m-1$ intermediate simple B-modules with $M_i = M_{i_0} \sim M_{i_1} \sim \ldots \sim M_{i_m} = M_j$. Thus $d(M_i, M_j) = 1$ if and only if $M_i \sim M_j$. We put $d(M_i, M_j) = -\infty$ if M_i and M_j are not d_B^A-equivalent, and $d(M_i, M_i) = 0$ for all $1 \leqslant i \leqslant r$. Note that the distance defined here is half of the *graphical distance* between the black points corresponding to M_i and M_j in the Bratelli diagram [7].

Theorem 3.6 from [5] shows that B is a depth $2m+1$ subalgebra of A if and only if the distance between two any simple B-modules is at most m. In particular B is a depth three subring of A if and only if the relation $\sim$ on $\mathrm{Irr}(B)$ is transitive.

2.4. On characterizing sets $\mathcal{C}$ with property (P).

Definition 2.1. Let $X = \{1, 2, \ldots, n\}$ be a finite set and

$$\mathcal{C} = \{A_1, \ldots, A_r\} \tag{2.4}$$

be a collection of nonempty subsets of X such that $X = \bigcup_{i=1}^{r} A_i$. Let $x_1, x_2, \ldots, x_n$ be n variables. We say that $\mathcal{C}$ has property (P) if the homogeneous linear system in $\{x_i\}_{i=1,n}$ given by the following r equations

$$\sum_{i\in A_1} x_i = 0,\ \sum_{i\in A_2} x_i = 0,\ \ldots,\ \sum_{i\in A_r} x_i = 0 \tag{2.5}$$

has only the trivial solution $x_1 = x_2 = \ldots = x_n = 0$ in k^n.

Remark 2.2. Clearly the property (P) is equivalent to a certain matrix with $\{0, 1\}$-entries to have maximal rank. We will denote this matrix by $M(\mathcal{C})$. The structure of $\{0, 1\}$-matrices of maximal rank is not completely understood. However asymptotic computations for the number of such matrices of a fixed $(m \times n)$-size are done, see for example [3] and the references therein.

For a set X let $\mathcal{P}(X)$ be its power algebra. Recall that this is the Boolean algebra $(\mathcal{P}(X), \cup, \cap)$ with elements subsets of X and the union, intersection and complements as structure operations.

Theorem 2.3. *Let $X = \{1, 2, \ldots, n\}$ and $\mathcal{C} = \{A_1, \ldots, A_r\}$ be as in Definition* 2.1. *Consider the subalgebra $\langle\mathcal{C}\rangle$ of $(\mathcal{P}(X), \cup, \cap)$ generated by the elements A_i of $\mathcal{C}$. If $\mathcal{C}$ has property (P) then any minimal set (under inclusion) of the subalgebra $\langle\mathcal{C}\rangle$ has exactly one element.*

Proof. Suppose that $\mathcal{C}$ has property (P) and let $M \in \langle\mathcal{C}\rangle$ be a minimal set. Suppose moreover that M has at least two elements, let us say, i and j with $i \neq j$. Then one can assign the values 1 and -1 to x_i and x_j respectively, and zero to all the other variables x_l with $l \in X$ and $l \neq i, j$. Note that since M is minimal (under inclusion) in $\langle\mathcal{C}\rangle$ any other set N containing one of the elements i and j also contains the other element. In this situation $\{x_s\}_{s=1,\ldots,n}$ is a non zero solution of the system (2.2) and this contradicts the property (P) of $\mathcal{C}$. $\square$

3. Set theoretical condition for depth three subgroups with trivial core

Consider now an inclusion of finite groups $H \subset G$. For $A = kG$ and $B = kH$ denote the equivalence relations from (2.2) by u_H^G and d_H^G.

Let $N := \mathrm{core}_G(H)$ be the largest normal subgroup of G contained in H. Recall that N is the intersection of all conjugates of G. Thus $\mathrm{core}_G(H) = \bigcap\limits_{g \in G} gHg^{-1}$. The following is Corollary 6.6 from [5].

Corollary 3.1. *Let H be a subgroup of G and $N := \mathrm{core}_G(H)$. The equivalence relation u_H^G is the same as the equivalence relation u_N^G coming from $N \trianglelefteq G$. Thus the equivalence classes of* $\mathrm{Irr}(G)$ *under u_H^G are in natural bijection with the G-orbits on* $\mathrm{Irr}(N)$.

Proposition 3.2. *Suppose $H \subset G$ with $\mathrm{core}_G(H) = 1$. Then H is a depth three subgroup of G if and only if $kG \bigotimes\limits_{kH} M$ is a faithful kH-module for all simple H-modules M.*

P r o o f. Since $\mathrm{core}_G(H) = 1$ by Corollary 3.1 the equivalence relation d_H^G has just one equivalence class. Proposition 3.1 of [5] implies that also u_H^G has just one equivalence class. Then Corollary 3.7 of the same paper implies that H is a depth three subgroup if and only if $T(M) := kG \bigotimes\limits_{kH} M$ is a faithful kH-module for all simple H-modules M. $\square$

3.1. Definition of the set $\mathcal{C}_H$. Let H be a subgroup of a finite group G. Define $\mathcal{C}_H$ to be the collection of all subsets $H \cap xHy^{-1}$ of H where the elements x and y run through all the elements of G.

Let $G = \bigsqcup\limits_{i=1}^{s} g_i H$ be a decomposition of G into right H-cosets. Note that any element of $\mathcal{C}_H$ is of the form $H \cap g_i H {g_j}^{-1}$ for some indices i and j. Indeed, if $x = g_i h$ and $y = g_j h'$ then $xHy^{-1} = g_i H {g_j}^{-1}$. Also note that

$$H = \bigcup_{i,j=1}^{s} (H \cap g_i H {g_j}^{-1}).$$

Let $H_{i,j} := H \cap g_i H {g_j}^{-1}$ and $H_i := H_{i,i}$. It is clear that $H_{i,j} = \{h \in H;\ h g_j \in g_i H\}$.

Proposition 3.3. *Let $G = \bigsqcup_{i=1}^{s} g_i H$ be a decomposition of G as right H-cosets. Then*

(1) $H_i H_{i,j} H_j = H_{i,j}$ *for all* $1 \leqslant i, j \leqslant s$.
(2) *If* $|H_{i,j}| = 1$ *then* $|H_i| = |H_j| = 1$.
(3) *For a fixed* i *it follows that* $H = \bigsqcup_{j=1}^{r} H_{i,j}$.
(4) $H_{i,j}^{-1} = H_{j,i}$.
(5) *One has* $H_{i,j} \neq \emptyset$ *if and only if* $Hg_j \cap g_i H \neq \emptyset$.
(6) *Let* $h, l \in H$. *Then* $H \cap hxHyl = h(H \cap xHy)l$.
(7) *Suppose* $H_{i,j} \neq \emptyset$ *and let* $h \in H_{i,j}$. *Then* $H_{i,j} = hH_j = H_i h$.

Proof. (1) It is easy to check that $H_i H_{i,j} H_j = H_{i,j}$.

(2) The second item follows from the first one.

(3) Let $G = \bigsqcup_{j=1}^{r} g_j H$. Then for a fixed index i one has: $G = \bigsqcup_{j=1}^{r} Hg_j^{-1}$ and $G = g_i G = \bigsqcup_{j=1}^{r} g_i H g_j^{-1}$. Thus

$$H = H \cap G = H \cap \left(\bigsqcup_{j=1}^{r} g_i H g_j^{-1} \right) = \bigsqcup_{j=1}^{r} (H \cap g_i H g_j^{-1}) = \bigsqcup_{j=1}^{r} H_{i,j}.$$

(5) One has to verify that $H_{i,j} \neq \emptyset$ if and only if $Hg_j \cap g_i H \neq \emptyset$. Indeed if $h = g_i l g_j^{-1} \in H_{i,j}$ with $l \in H$ then $hg_j = g_i l \in Hg_j \cap g_i H$. Conversely if $Hg_j \cap g_i H \neq \emptyset$ then there is $h, l \in H$ such that $hg_j = g_i l$ and therefore $h = g_i l g_j^{-1} \in H_{i,j}$.

(6) Straightforward computation.

(7) Suppose that $h \in H_{i,j}$. One has $h = g_i l g_j^{-1}$ with $l \in H$. Thus $g_j = h^{-1} g_i l$ and

$$\begin{aligned} H_{i,j} &= H \cap g_i H g_j^{-1} = H \cap g_i H (h^{-1} g_i l)^{-1} = H \cap g_i H l^{-1} g_i^{-1} h \\ &= H \cap g_i H g_i^{-1} h = (H \cap g_i H g_i^{-1}) h = H_i h. \end{aligned}$$

Here we have used the previous item of this proposition. On the other hand the equality $hg_j = g_i l$ implies $g_i = hg_j l^{-1}$ and

$$H_{i,j} = H \cap g_i H g_j^{-1} = H \cap hg_j l^{-1} H g_j^{-1} = hH_j.$$

$\square$

3.2. Depth three subgroups.

Theorem 3.4. *Suppose $H \subset G$ with $\mathrm{core}_G(H) = 1$. Then $kG \bigotimes_{kH} k$ is a faithful kH-module if and only if $\mathcal{C}_H$ has property (P).*

P r o o f. Note that $\left\{g_i \bigotimes_{kH} 1\right\}_{i=1,s}$ form a basis for the induced module $k\uparrow_H^G$. Thus an element $a = \sum_{h\in H} a_h h$ is in the annihilator of $k\uparrow_H^G\downarrow_H^G$ if and only if:

$$a\Big(g_i \bigotimes_{kH} 1\Big) = 0$$

for all $i = 1, \ldots, s$.

Note that $h\Big(g_i \bigotimes_{kH} 1\Big) = g_j \bigotimes_{kH} 1$ where j is an index such that $hg_i \in g_jH$, i.e. $h \in H_{i,j}$.

Thus $a = \sum_{h\in H} a_h h$ annihilates $kG \bigotimes_{kH} k$ if and only if $\sum_{h\in H_{i,j}} a_h = 0$ for all i and j. This implies that $\mathrm{Ann}_{kH}\Big(kG \bigotimes_{kH} k\Big) = 0$ if and only if $\mathcal{C}_H$ has property (P). □

Corollary 3.5. *Suppose that H is a subgroup of G with trivial core. If H is a depth 3 subgroup then $\mathcal{C}_H$ has property (P).*

P r o o f. By Proposition 3.2 it follows that $k\uparrow_H^G\downarrow_H^G$ is a faithful H-module. By Theorem 3.4 this is equivalent to the fact that the set $\mathcal{C}_H$ has property (P). □

In [5] it was shown that the subgroup H is depth three provided that one of the subsets of $\mathcal{C}_H$ is a one element subset. By Theorem 2.3 the previous corollary can be regarded as a weaker converse of the statement from [5]. If H is depth three inside G then the minimal sets (under inclusion) of $\mathcal{C}_H$ are one element sets.

Remark 3.6. Suppose that H is a depth three subgroup of G and let $N := \mathrm{core}_G(H)$. It follows from Proposition 6.8 of [5] that depth of H/N inside G/N is less or equal than the depth of H inside G. Thus $d(H/N, G/N) \leqslant 3$. On the other hand it is easy to see that H/N has trivial core inside G/N. Thus H/N is not normal inside G/N. This implies that $d(H/N, G/N) = 3$ and therefore the previous theorem can be applied for the inclusion $H/N \subset G/N$.

4. Odd higher depth for coreless subgroups

Let $H \subset G$ be an inclusion of finite groups. In this section we generalize the results from the previous section to higher odd depth subgroups with trivial core. For any $n \geqslant 1$ a set $\mathcal{C}_H^n$ is constructed and it will be shown that depth $2n+1$ implies that the set $\mathcal{C}_H^n$ has property (P).

4.1. The linear operator T. Let $C(H)$ be the character ring of kH, i.e., the space of class functions on H. Recall that the operator $T\colon C(H) \to C(H)$ from [5] was given by $T(\alpha) = \mathrm{Res}_H^G(\mathrm{Ind}_H^G(\alpha))$. Thus one can also write $T(\alpha) = \alpha\uparrow_H^G\downarrow_H^G$.

Proposition 4.1. *Suppose $H \subset G$ with* $\operatorname{core}_G(H) = 1$. *Then H is a depth $2n+1$ subgroup of G if and only if $T^n(M)$ is a faithful kH-module for all simple H-modules M.*

Proof. Since $\operatorname{core}_G(H) = 1$ by Corollary 6.6 of [5] the equivalence relation d_H^G has just one equivalence class. Proposition 3.1 of the same paper implies that also u_H^G has just one equivalence class. Then Theorem 3.9 together with Lemma 5.4 from [5] implies that H is a depth $2n+1$ subgroup if and only if $T^n(M)$ is a faithful kH-module for all simple H-modules M. $\square$

Let $\mathcal{C}_H^{(k)}$ be the collection of subsets of H given by

$$H^{i_1,i_2,\ldots,i_k}_{m_1,m_2,\ldots,m_k} := H_{m_1,i_1} \cap g_{m_1} H_{m_2,i_2} g_{i_1}{}^{-1} \cap \ldots \\ \cap\, g_{m_1} g_{m_2} \ldots g_{m_{k-1}} H_{m_k,i_k} (g_{i_1} g_{i_2} \ldots g_{i_{k-1}})^{-1}$$

where $m_1, \ldots, m_k$ and $i_1, \ldots, i_k$ run through all possible indices.

Proposition 4.2. *Suppose $H \subset G$ with* $\operatorname{core}_G(H) = 1$. *Then $T^n(k)$ is a faithful kH-module if and only if $\mathcal{C}_H^{(n)}$ has property (P).*

Proof. Note that $T^n(k) = kG \bigotimes_{kH} kG \bigotimes_{kH} \ldots kG \bigotimes_{kH} k$ has a basis as vector space given by

$$g_{i_1} \bigotimes_{kH} g_{i_2} \bigotimes_{kH} \cdots g_{i_k} \bigotimes_{kH} 1$$

and

$$h\Big(g_{i_1} \bigotimes_{kH} g_{i_2} \bigotimes_{kH} \cdots g_{i_k} \bigotimes_{kH} 1\Big) = g_{m_1} \bigotimes_{kH} g_{m_2} \bigotimes_{kH} \cdots \bigotimes_{kH} g_{m_k} \bigotimes_{kH} 1$$

if and only if $h \in H^{i_1,i_2,\ldots,i_k}_{m_1,m_2,\ldots,m_k}$. Therefore an element $a = \sum_{h\in H} a_h h$ is in the annihilator of $T^n(k)$ if and only if $\sum_{h \in H^{i_1,i_2,\ldots,i_k}_{m_1,m_2,\ldots,m_k}} a_h = 0$ for all indices i_s and m_s. It follows that $T^n(k)$ is a faithful kH-module if and only if $\mathcal{C}_H^{(n)}$ has property (P). $\square$

Put $H_{m_1,m_2,\ldots,m_k} := H^{m_1,m_2,\ldots,m_k}_{m_1,m_2,\ldots,m_k}$. Note that

$$H_{m_1,m_2,\ldots,m_k} = H \cap g_{m_1} H g_{m_1}^{-1} \cap g_{m_1} g_{m_2} H (g_{m_1} g_{m_2})^{-1} \cap \ldots \\ \cap (g_{m_1} g_{m_2} \ldots g_{m_k}) H (g_{m_1} g_{m_2} \ldots g_{m_k})^{-1}.$$

Lemma 4.3. *With the above notations one has that*

$$H_{m_1,m_2,\ldots,m_k} H^{i_1,i_2,\ldots,i_k}_{m_1,m_2,\ldots,m_k} H_{i_1,i_2,\ldots,i_k} = H^{i_1,i_2,\ldots,i_k}_{m_1,m_2,\ldots,m_k}.$$

Proof. This follows from the first item of Proposition 3.3. $\square$

Corollary 4.4. *Suppose that H is a depth $2m+1$ subgroup of G with trivial core. Then $\mathcal{C}_H^m$ has property (P).*

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Author's address: Sebastian Burciu, Inst. of Math. "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700, Bucharest, Romania, and University of Bucharest, Faculty of Mathematics and Computer Science, 14 Academiei St., Bucharest, Romania, e-mail: sebastian.burciu@imar.ro.