Marek Bienias; Szymon Głąb; Robert Rałowski; Szymon Żeberski
Two point sets with additional properties

Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 4, 1019–1037

Persistent URL: `http://dml.cz/dmlcz/143614`

Czechoslovak Mathematical Journal, 63 (138) (2013), 1019–1037

TWO POINT SETS WITH ADDITIONAL PROPERTIES

Marek Bienias, Szymon Głąb, Łódź, Robert Rałowski,
Szymon Żeberski, Wrocław

(Received August 13, 2012)

Abstract. A subset of the plane is called a two point set if it intersects any line in exactly two points. We give constructions of two point sets possessing some additional properties. Among these properties we consider: being a Hamel base, belonging to some σ-ideal, being (completely) nonmeasurable with respect to different σ-ideals, being a κ-covering. We also give examples of properties that are not satisfied by any two point set: being Luzin, Sierpiński and Bernstein set. We also consider natural generalizations of two point sets, namely: partial two point sets and n point sets for $n = 3, 4, \ldots, \aleph_0, \aleph_1$. We obtain consistent results connecting partial two point sets and some combinatorial properties (e.g. being an m.a.d. family).

Keywords: two point set; partial two point set; complete nonmeasurability; Hamel basis; Marczewski measurable set; Marczewski null; s-nonmeasurability; Luzin set; Sierpiński set

MSC 2010: 03E75, 03E35, 28A05, 15A03

1. Introduction

At the beginning of the 20th century Mazurkiewicz in [11] constructed a set in the plane which meets any line in exactly two points. Any such set is called *a two point set.*

Any two point set must be somehow complex, namely Larman in [9] showed that it cannot be F_σ. It is a long standing open problem whether there is a Borel two point set (see [10]). The best known approximation to that problem is due to Miller who, assuming $V = L$, proved that there is a coanalytic two point set [12].

Szymon Głąb has been supported by the Polish Ministry of Science and Higher Education Grant No IP2011 014671 (2012-2014). The work of R. Rałowski and Sz. Żeberski has been partially financed by NCN means granted by decision DEC-2011/01/B/ST1/01439.

The aim of this paper is to construct two point sets which possess some additional properties. First, we focus on their being Hamel base and being completely $\mathbb{I}$-nonmeasurable. (A is completely $\mathbb{I}$-nonmeasurable if the intersection $A \cap B$ does not belong to $\mathbb{I}$ for any Borel set $B \notin \mathbb{I}$; see e.g. [3], [14], [15], [18].) We also construct a two point set which does not belong to the σ-algebra s (of Marczewski measurable sets). In contrast, we prove that there exists a two point set which belongs to the σ-ideal s_0 (of Marczewski null sets). In particular, we generalize a result from [13]. Recently Schmerl proved in [16] that there is a two point set which can be covered by countably many circles. In particular, there is a two point set which is meager and null.

We affirmatively answer the question whether every n point set (for $n = 2, 3, \ldots$) can be represented as a union of n bijections. We also show that no two point set contains an additive function. We construct a two point set which does not contain any measurable function.

We observe that a two point set cannot be any of the following: a Luzin set, a Sierpiński set, or a Bernstein set. However, under CH, we construct a partial two point set which is a strong Luzin set (or a strong Sierpiński set).

We also compare the notion of the κ point set with the notion of the κ-covering and κ-I-covering. (A is a κ-covering if for every subset X of size κ there exists a translation h of $\mathbb{R}^2$ such that $h[X] \subseteq A$; A is a κ-I-covering if for every subset X of size κ there exists an isomorphism h of $\mathbb{R}^2$ such that $h[X] \subseteq A$; see [7].)

We give some consistent examples of partial two point sets which are, in a sense, m.a.d. families, maximal families of eventually different functions.

2. Completely $\mathbb{I}$-nonmeasurable Hamel base

We say that $\mathbb{I}$ is a *σ-ideal* of subsets of $\mathbb{R}^2$ if $\mathbb{I}$ is closed under taking subsets and closed under taking countable unions.

Let $\mathbb{I}$ be a σ-ideal of subsets of $\mathbb{R}^2$ containing all singletons and having a Borel base (i.e. for every $I \in \mathbb{I}$ there is a Borel set $B \in \mathbb{I}$ such that $I \subseteq B$). We recall the notion of completely $\mathbb{I}$-nonmeasurability which was studied e.g. in [3], [7], [14], [15], [18]. This notion is also known as the $\mathbb{I}$-Bernstein set.

Definition 2.1. We say that a set $A \subseteq \mathbb{R}^2$ is *completely $\mathbb{I}$-nonmeasurable* if it intersects all $\mathbb{I}$-positive Borel sets (i.e. sets which are in Borel$\setminus\mathbb{I}$) but does not contain any of them.

When $\mathbb{I} = [\mathbb{R}^2]^{\leqslant\omega}$ then the notion of a completely $\mathbb{I}$-nonmeasurable set coincides with the notion of a Bernstein set.

We will assume that $\mathbb{I}$ is a σ-ideal of subsets of $\mathbb{R}^2$ with the property that for every $\mathbb{I}$-positive Borel set there are $\mathfrak{c}$ many pairwise disjoint lines each of which intersects it in a set of cardinality $\mathfrak{c}$.

Let us observe that the σ-ideal of null sets $\mathcal{N}$ and the σ-ideal of meager sets $\mathcal{M}$ on the real plane (by Fubini Theorem and by Kuratowski-Ulam Theorem) fulfil this condition.

We say that $H \subseteq \mathbb{R}^2$ is a Hamel base if H is a base of $\mathbb{R}^2$ treated as a linear space over $\mathbb{Q}$.

Theorem 2.2. *There exists a two point set $A \subseteq \mathbb{R}^2$ that is a completely $\mathbb{I}$-nonmeasurable Hamel base.*

Proof. Let $\{l_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in the plane $\mathbb{R}^2$, let $\{B_\xi\colon \xi < \mathfrak{c}\}$ be an enumeration of all $\mathbb{I}$-positive Borel sets in the plane $\mathbb{R}^2$ and let $\{h_\xi\colon \xi < \mathfrak{c}\}$ be a Hamel base of $\mathbb{R}^2$. We will define, by induction on $\xi < \mathfrak{c}$, a sequence $\{A_\xi\colon \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ such that for every $\xi < \mathfrak{c}$:

(1) $|A_\xi| < \omega$,

(2) $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta$ does not have three collinear points,

(3) $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta$ contains precisely two points of l_ξ,

(4) $B_\xi \cap \bigcup\limits_{\zeta \leqslant \xi} A_\zeta \neq \emptyset$,

(5) $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta$ is linearly independent over $\mathbb{Q}$,

(6) $h_\xi \in \operatorname{span}_{\mathbb{Q}}\Big(\bigcup\limits_{\zeta \leqslant \xi} A_\zeta\Big)$.

To make an inductive construction assume that for some $\xi < \mathfrak{c}$ we have already defined the sequence $\{A_\zeta\colon \zeta < \xi\}$ which fulfils (1)–(6). Let $A_{<\xi} = \bigcup\limits_{\zeta < \xi} A_\zeta$. Clearly $|A_{<\xi}| < \mathfrak{c}$. Let $\mathcal{L}$ be the family of all lines which meet $A_{<\xi}$ in exactly two points. Then $|\mathcal{L}| \leqslant |A^2_{<\xi}| < \mathfrak{c}$. Moreover, $|\operatorname{span}_{\mathbb{Q}}(A_{<\xi})| < \mathfrak{c}$. We will define A_ξ in three steps. In each step we will focus on one of the desired properties of A_ξ.

Step I (*two point set*). Note that (2) implies that $l_\xi \cap A_{<\xi}$ has at most two points.

If $|l_\xi \cap A_{<\xi}| = 2$, then set $A^{(1)}_\xi = \emptyset$.

Let us focus on $|l_\xi \cap A_{<\xi}| < 2$. Then $|l_\xi \cap l| \leqslant 1$ for any $l \in \mathcal{L}$. Therefore $|l_\xi \setminus \bigcup \mathcal{L}| = \mathfrak{c}$. Choose

$$x^{(1)} \in l_\xi \setminus \operatorname*{span}_{\mathbb{Q}}\Big(A_{<\xi} \cup \bigcup_{l \in \mathcal{L}} (l \cap l_\xi)\Big),$$
$$y^{(1)} \in l_\xi \setminus \operatorname*{span}_{\mathbb{Q}}\Big(A_{<\xi} \cup \{x^{(1)}\} \cup \bigcup_{l \in \mathcal{L}} (l \cap l_\xi)\Big).$$

Set $A^{(1)}_\xi = \{x^{(1)}, y^{(1)}\}$ if $A_{<\xi} \cap l_\xi = \emptyset$ and set $A^{(1)}_\xi = \{x^{(1)}\}$ if $A_{<\xi} \cap l_\xi$ is a singleton.

Step II (*complete* $\mathbb{I}$-*nonmeasurability*). Let $\mathcal{L}'$ be the family of all lines which meet $A_{<\xi} \cup A_\xi^{(1)}$ in exactly two points. Then $|\mathcal{L}'| < \mathfrak{c}$ and $\mathcal{L} \subseteq \mathcal{L}'$. Since B_ξ is an $\mathbb{I}$-positive Borel set, we can find a line l such that $l \cap (A_{<\xi} \cup A_\xi^{(1)}) = \emptyset$ and $|l \cap B_\xi| = \mathfrak{c}$.

Choose

$$x^{(2)} \in (l \cap B_\xi) \setminus \operatorname*{span}_{\mathbb{Q}} \Big(A_{<\xi} \cup A_\xi^{(1)} \cup \bigcup_{l \in \mathcal{L}'} (l \cap l_\xi) \Big).$$

Set $A_\xi^{(2)} = \{x^{(2)}\}$.

Step III (*Hamel base*). Let us focus on the condition (6). If $h_\xi \in \operatorname{span}_{\mathbb{Q}}(A_{<\xi} \cup A_\xi^{(1)} \cup A_\xi^{(2)})$, then set $A_\xi^{(3)} = \emptyset$. Assume now that $h_\xi \notin \operatorname{span}_{\mathbb{Q}}(A_{<\xi} \cup A_\xi^{(1)} \cup A_\xi^{(2)})$. Let $\mathcal{L}''$ be the family of all lines which meet $A_{<\xi} \cup A_\xi^{(1)} \cup A_\xi^{(2)}$ in exactly two points. Then $|\mathcal{L}''| < \mathfrak{c}$ and $\mathcal{L} \subseteq \mathcal{L}' \subseteq \mathcal{L}''$. Choose a line l parallel to h_ξ, with $l \cap (A_{<\xi} \cup A_\xi^{(1)} \cup A_\xi^{(2)}) = \emptyset$. Choose

$$x^{(3)} \in l \setminus \operatorname*{span}_{\mathbb{Q}} \Big(A_{<\xi} \cup A_\xi^{(1)} \cup A_\xi^{(2)} \cup \{h_\xi\} \cup \bigcup_{l \in \mathcal{L}''} (l \cap l_\xi) \Big).$$

Set $y^{(3)} = x^{(3)} + h_\xi$. Then

$$y^{(3)} \in l \setminus \operatorname*{span}_{\mathbb{Q}} \Big(A_{<\xi} \cup A_\xi^{(1)} \cup A_\xi^{(2)} \cup \bigcup_{l \in \mathcal{L}''} (l \cap l_\xi) \Big).$$

Set $A_\xi^{(3)} = \{x^{(3)}, y^{(3)}\}$.

Finally, set $A_\xi = A_\xi^{(1)} \cup A_\xi^{(2)} \cup A_\xi^{(3)}$.

Clearly conditions (1)–(6) are satisfied. So, the inductive construction is completed.

The set $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$ will have the desired properties. Evidently, conditions (2) and (3) imply that the set A is a two point set. Since every $\mathbb{I}$-positive Borel set must have an uncountable section, the set A does not contain any set from $\{B_\xi \colon \xi < \mathfrak{c}\}$ and (4) makes sure it intersects all of them, so the set A is completely $\mathbb{I}$-nonmeasurable. Moreover, conditions (5) and (6) imply that A is a Hamel base of $\mathbb{R}^2$. □

Considering $\mathbb{I} = \mathcal{N}$, we get the following corollary.

Corollary 2.3. *There exists a two point set $A \subseteq \mathbb{R}^2$ that is a Hamel base such that $\lambda_*(A) = \lambda_*(\mathbb{R}^2 \setminus A) = 0$, where λ_* denotes the inner Lebesgue measure on the plane.*

3. Marczewski null and Marczewski nonmeasurable set

In this section we will consider the σ-ideal s_0 and the σ-algebra s of subsets of $\mathbb{R}^2$ that were introduced by Marczewski (see e.g. [17], [6]).

Definition 3.1. We say that a set $A \subseteq \mathbb{R}$

(1) belongs to s_0 if for every perfect set P there exists a perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$.
(2) is s-measurable if for every perfect set P there exists a perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$ or $Q \subseteq A$.
(3) is s-nonmeasurable if A is not s-measurable.

Definition 3.2. We say that a subset $A \subseteq \mathbb{R}^2$ is a *Bernstein set* if for every perfect set $P \subseteq \mathbb{R}^2$

$$A \cap P \neq \emptyset \wedge A^c \cap P \neq \emptyset.$$

Let us recall that every Bernstein set is s-nonmeasurable.

Let us start with the result connected with the σ-ideal s_0 of Marczewski null sets.

Theorem 3.3. *There exists a two point set $A \subseteq \mathbb{R}^2$ that belongs to s_0.*

Proof. Let $\{l_\xi\colon\ \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in the plane $\mathbb{R}^2$. Let $\{Q_\xi\colon\ \xi < \mathfrak{c}\}$ be an enumeration of all perfect sets in $\mathbb{R}^2$ such that every perfect set occurs $\mathfrak{c}$ many times.

We will define, by induction on $\xi < \mathfrak{c}$, sequences $\{A_\xi\colon\ \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ and $\{P_\xi\colon\ \xi < \mathfrak{c}\}$ of perfect or empty sets such that

$(\star)$ for every perfect set Q there is $\xi_0 < \mathfrak{c}$ such that $P_{\xi_0} \neq \emptyset$ and $P_{\xi_0} \subseteq Q$;

and for every $\xi < \mathfrak{c}$,

(1) $|A_\xi| < \omega$,
(2) $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta$ does not contain three collinear points,
(3) $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta$ contains precisely two points of l_ξ,
(4) $P_\xi \subseteq Q_\xi$,
(5) $\bigcup\limits_{\zeta \leqslant \xi} P_\zeta \cap \bigcup\limits_{\zeta \leqslant \xi} A_\zeta = \emptyset$,
(6) $\Big| l_\eta \setminus \bigcup\limits_{\zeta \leqslant \xi} P_\zeta \Big| = \mathfrak{c}$ for every $\eta \geqslant \xi$.

Assume that for some $\xi < \mathfrak{c}$ the sequences $\{A_\zeta\colon\ \zeta < \xi\}$ and $\{P_\zeta\colon\ \zeta < \xi\}$ are already constructed. Set $A_{<\xi} = \bigcup\limits_{\zeta < \xi} A_\zeta$.

Assume first that for every line l in a plane, $|Q_\xi \cap l| < \mathfrak{c}$. Then $|Q_\xi \cap l| \leqslant \omega$. Since $|A_{<\xi}| < \mathfrak{c}$ we can choose a perfect set $P_\xi \subseteq Q_\xi$ such that $P_\xi \cap A_{<\xi} = \emptyset$ and $|P_\xi \cap l| \leqslant \omega$ for every line l. Since the intersection of P_ξ with any line is at most countable hence $\left|l_\eta \setminus \bigcup_{\zeta \leqslant \xi} P_\zeta\right| = \mathfrak{c}$, for every $\eta \geqslant \xi$ and $\bigcup_{\zeta \leqslant \xi} P_\zeta \cap \bigcup_{\zeta < \xi} A_\zeta = \emptyset$.

Assume now that there exists a line l such that $|l \cap Q_\xi| = \mathfrak{c}$. If $l = l_\alpha$ for some $\alpha \geqslant \xi$, then put $P_\xi = \emptyset$. If $l = l_\alpha$ for some $\alpha < \xi$, then $|l \cap A_{<\xi}| = 2$ and since $l \cap Q_\xi$ is closed with $|l \cap Q_\xi| = \mathfrak{c}$ one can choose a perfect set $P_\xi \subseteq Q_\xi \cap l$ disjoint with $A_{<\xi}$. Then $\left|l_\eta \setminus \bigcup_{\zeta \leqslant \xi} P_\zeta\right| = \mathfrak{c}$ for every $\eta \geqslant \xi$ and $\bigcup_{\zeta \leqslant \xi} P_\zeta \cap \bigcup_{\zeta < \xi} A_\zeta = \emptyset$.

As in Theorem 2.3 we can choose a set A_ξ satisfying (1)–(3) outside the set $\bigcup_{\zeta \leqslant \xi} P_\zeta$ and so complete the inductive construction.

Finally, there exist sequences $\{A_\xi \colon \xi < \mathfrak{c}\}$ and $\{P_\xi \colon \xi < \mathfrak{c}\}$, satisfying (1)–(6) and by the construction they fulfil the condition $(\star)$.

Then the set $A = \bigcup_{\xi < \mathfrak{c}} A_\xi$ will have the desired property. $\square$

Let us note here that the unit circle intersects any line in at most two points but cannot be extended to a two point set. In [5] and [4] the authors investigated how small should be a subset of the unit circle to be extendable to a two point set. It turns out that sets of inner positive measure on the unit circle cannot be extended to two point sets. We will show that there is a subset of the unit circle of full outer measure which can be extended to a two point set.

Theorem 3.4. *There exists a two point set $A \subseteq \mathbb{R}^2$ that is s-nonmeasurable. Moreover, A contains a subset of the unit circle of full outer measure.*

P r o o f. Let us observe that if B is a Bernstein set in some uncountable closed set C then B is s-nonmeasurable. Moreover, if a set D is such that $D \cap C = B$ then D is also s-nonmeasurable.

We construct a two point set A such that its intersection with the unit circle is a Bernstein subset of the unit circle. Let $\{l_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in $\mathbb{R}^2$. Let $\{P_\xi \colon \xi < \mathfrak{c}\}$ be an enumeration of all perfect subsets of the unit circle.

We will define inductively a sequence $\{A_\xi \colon \xi < \mathfrak{c}\}$ of subsets of $\mathbb{R}^2$ and a sequence $\{y_\xi \colon \xi < \mathfrak{c}\}$ of points from the unit circle such that for every $\xi < \mathfrak{c}$:

(1) $|A_\xi| < \omega$,

(2) $\bigcup_{\zeta \leqslant \xi} A_\zeta$ does not contain three collinear points,

(3) $\bigcup_{\zeta \leqslant \xi} A_\zeta$ contains precisely two points of l_ξ,

(4) $P_\xi \cap \bigcup_{\zeta \leqslant \xi} A_\zeta \neq \emptyset$,

(5) $y_\xi \in P_\xi$,
(6) $A_\xi \cap \{y_\zeta \colon \zeta \leqslant \xi\} = \emptyset$.

The existence of the sequence $\{A_\xi \colon \xi < \mathfrak{c}\}$ follows in a way similar to that in Theorem 2.3. Here, the key observation is that for each perfect set P_ξ of the unit circle there exist $\mathfrak{c}$ many straight lines passing through P_ξ and the origin.

Setting $A = \bigcup_{\xi<\mathfrak{c}} A_\xi$ we obtain a two point s-nonmeasurable set. Clearly, A is of full outer measure on the unit circle. □

Using the method from the previous section we can strengthen the results in the following way.

Theorem 3.5. *Let $\mathbb{I}$ be a σ-ideal of subsets of $\mathbb{R}^2$ with the property that for every $\mathbb{I}$-positive Borel set there are $\mathfrak{c}$ many pairwise disjoint lines which intersect it on a set of cardinality $\mathfrak{c}$.*

(1) *There exists a two point set $A \subseteq \mathbb{R}^2$ that is a completely $\mathbb{I}$-nonmeasurable, s_0 Hamel base.*
(2) *There exists a two point set $B \subseteq \mathbb{R}^2$ that is a completely $\mathbb{I}$-nonmeasurable, s-nonmeasurable Hamel base.*

To prove it one should combine the ideas of Theorems 2.3, 3.3 and 3.4.

The first part of the above theorem generalizes the result from [13].

4. A union of graphs of functions

In this section we will focus on the question of whether a two point set can be decomposed into a union of two functions having some additional properties.

Let us start with a simple observation.

Proposition 4.1. *Every two point set is a union of two functions.*

P r o o f. Let A be a two point set. In particular, it intersects every vertical line in exactly two points. For $x \in \mathbb{R}$ let us denote $A^x = A \cap (\{x\} \times \mathbb{R})$. Clearly A^x has two elements, so $A^x = \{(x, y_1), (x, y_2)\}$. Define functions $f_1, f_2 \colon \mathbb{R} \to \mathbb{R}$ by $f_1(x) = y_1$, $f_2(x) = y_2$. Then we get that $A = f_1 \cup f_2$. This completes the proof. □

Let us introduce a notion which generalizes in a natural way the notion of a two point set.

Definition 4.2. Let κ be a cardinal number, $\kappa \geqslant 2$. We say that a subset of the plane is a κ *point set* if it meets any line in exactly κ points.

Proposition 4.3. *Let $n \geqslant 2$ be a natural number. For any n point set A there is no additive function $f \subseteq A$.*

Proof. Let A be an n point set and suppose that there is an additive function $f \subseteq A$. Notice that $f(2) = f(1+1) = f(1) + f(1) = 2f(1)$ and, more generally for $k \geqslant 1$, $f(k) = kf(1)$. So points $(1, f(1))$, $(2, 2f(1)), \ldots, (n+1, (n+1)f(1))$ are members of A which lie on the same line. This is a contradiction. $\square$

Now, let us focus on the class of bijections.

We will use the following theorem (see e.g. [1]).

Theorem 4.4 ([Hall]). *Assume that X, Y are infinite sets. Let $R \subseteq X \times Y$ be a relation such that for every $x \in X$ there are at most finitely many $y \in Y$ with $(x, y) \in R$ possessing the following property:*

$$(\forall k \in \mathbb{N})(\forall X' \subseteq X)\ (|X'| = k \Rightarrow |R[X']| \geqslant k),$$

where $R[X'] = \{y\colon\ (\exists x \in X')((x, y) \in R)\}$. Then there exists an injection $h\colon\ X \to Y$ such that $h \subseteq R$.

We will also use the following theorem (see e.g. [6]).

Theorem 4.5 ([Cantor, Bernstein]). *Let X, Y be any sets. Assume that $f\colon\ X \to Y$ and $g\colon\ Y \to X$ are injections. Then there exist $A \subseteq X$ and $B \subseteq Y$ such that $f \restriction A\colon\ A \to Y \setminus B$ and $g \restriction B\colon\ B \to X \setminus A$ are bijections.*

Theorem 4.6. *Fix a natural number n. Let $A \subseteq \mathbb{R}^2$ be such that its intersection with every horizontal and vertical line has exactly n elements. Then there exist n bijections $F_0, \ldots, F_{n-1}\colon\ \mathbb{R} \to \mathbb{R}$ such that $A = F_0 \cup \ldots \cup F_{n-1}$.*

Proof. Let us notice that $A \subseteq \mathbb{R} \times \mathbb{R}$ fulfils the assumptions of Theorem 4.4. So there exists an injection $f\colon\ \mathbb{R} \to \mathbb{R}$ such that $f \subseteq A$.

A set $A^{-1} = \{(x, y)\colon\ (y, x) \in A\}$ also fulfils the assumptions of Theorem 4.4. So there exists an injection $g\colon\ \mathbb{R} \to \mathbb{R}$ such that $g \subseteq A^{-1}$.

By Theorem 4.5 we can construct a bijection $F_0\colon\ \mathbb{R} \to \mathbb{R}$ of the form $F_0 = (f \restriction A) \cup (g^{-1} \restriction (\mathbb{R} \setminus A))$. So, $F_0 \subseteq A$.

Let us notice that $A \setminus F_0$ is such that its intersection with every horizontal and vertical line has exactly $n-1$ elements. So, the proof can be completed by a simple induction. $\square$

We get the immediate corollary:

Corollary 4.7. *Let $n \geqslant 2$ be a natural number. Any n point set can be decomposed into n bijections.*

One can ask whether any two point set can be decomposed into two measurable (with Baire property) functions. We will prove that this is not the case. Moreover, there is a two point set which does not admit a measurable (with Baire property) uniformization.

We will use the following, probably well-known, lemma. We give a short proof of it for the reader's convenience.

Lemma 4.8. *There exists an unbounded F_σ set $C \subseteq \mathbb{R}_+$ of measure zero such that its intersection with any interval in $\mathbb{R}_+$ is of cardinality $\mathfrak{c}$. (In particular, C is meager.)*

P r o o f. Let $\mathbb{C}$ denote the standard ternary Cantor set. Let $\mathbb{Q}_+$ denote the set of positive rationals. Set

$$C = \mathbb{C} + \mathbb{Q}_+ = \{x + y\colon\ x \in \mathbb{C} \wedge y \in \mathbb{Q}_+\}.$$

This completes the proof. $\square$

Theorem 4.9. *For any Bernstein set $B \subseteq \mathbb{R}$ there exists a two point set $A \subseteq \mathbb{R}^2$ which is null and meager such that for any function $f \subseteq A$, $f^{-1}((0,1))$ is B.*

P r o o f. Let $B \subseteq \mathbb{R}$ be a Bernstein set and let $\{l_\xi\colon\ \xi < \mathfrak{c}\}$ be an enumeration of all straight lines in the plane $\mathbb{R}^2$. Let $C^* = \{r \cdot \mathrm{e}^{\mathrm{i}t}\colon\ t \in [0, 2\pi], r \in C\}$ where C is the set from Lemma 4.8. Notice that C^* is an F_σ-set. By Fubini's Theorem and Ulam's Theorem the set C^* is meager and of measure zero in the plane $\mathbb{R}^2$. Notice that $|l_\xi \cap C^*| = \mathfrak{c}$ for any $\xi < \mathfrak{c}$. We will define, by induction on $\xi < \mathfrak{c}$, a sequence $\{A_\xi\colon\ \xi < \mathfrak{c}\}$ of subsets of C^* such that for every $\xi < \mathfrak{c}$,

(1) $|A_\xi| < \omega$;

(2) $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta$ does not have three collinear points;

(3) $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta$ contains precisely two points of l_ξ;

(4) if l_ξ is a vertical line with x-coordinate $x_\xi \in B$ then $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta \cap l_\xi \subseteq \{x_\xi\} \times (0,1)$;

(5) if l_ξ is a horizontal line with y-coordinate $y_\xi \in (0,1)$ then $\bigcup\limits_{\zeta \leqslant \xi} A_\zeta \cap l_\xi \subseteq B \times \{y_\xi\}$;

(6) if neither (4) nor (5) then $\Big(\bigcup\limits_{\zeta \leqslant \xi} A_\zeta \cap l_\xi\Big) \cap (B \times (0,1)) = \emptyset$.

Assume that for some $\xi < \mathfrak{c}$ the sequence $\{A_\zeta\colon\ \zeta < \xi\}$ is already defined. Set $A_{<\xi} = \bigcup\limits_{\zeta < \xi} A_\zeta$. Let $\mathcal{L}$ be the family of all lines which meet $A_{<\xi}$ in exactly two points.

Then $|\mathcal{L}| \leqslant |A^2_{<\xi}| < \mathfrak{c}$. Note that $l_\xi \cap A_{<\xi}$ has at most two elements. Consider three cases.

Case 1 (l_ξ *is a vertical line with x-coordinate* $x_\xi \in B$). If $|l_\xi \cap A_{<\xi}| = 2$ then put $A_\xi = \emptyset$. If $|l_\xi \cap A_{<\xi}| < 2$, then $|l_\xi \cap l| \leqslant 1$ for any $l \in \mathcal{L}$. Choose two numbers $y^1_\xi, y^2_\xi \in (0,1)$ such that $(x_\xi, y^1_\xi), (x_\xi, y^2_\xi) \in (C^* \cap l_\xi) \setminus \Big(\bigcup_{l\in\mathcal{L}} l \cap l_\xi\Big)$. This is possible since $|C^* \cap l_\xi| = \mathfrak{c}$ and $\Big|\bigcup_{l\in\mathcal{L}} l \cap l_\xi\Big| < \mathfrak{c}$. Set $A_\xi = \{(x_\xi, y^1_\xi), (x_\xi, y^2_\xi)\}$ if $l_\xi \cap A_{<\xi} = \emptyset$ or $A_\xi = \{(x_\xi, y^1_\xi)\}$ if $|l_\xi \cap A_{<\xi}| = 1$.

Case 2 (l_ξ *is a horizontal line with y-coordinate* $y_\xi \in (0,1)$). Since $l_\xi \cap C^*$ is uncountable F_σ, it contains a perfect set and $|\pi_1[l_\xi \cap C^*] \cap B| = \mathfrak{c}$. If $|l_\xi \cap A_{<\xi}| = 2$ then put $A_\xi = \emptyset$. If $|l_\xi \cap A_{<\xi}| < 2$, then $|l_\xi \cap l| \leqslant 1$ for any $l \in \mathcal{L}$ and choose arbitrary two points $x^1_\xi, x^2_\xi \in B$ such that $(x^1_\xi, y_\xi), (x^2_\xi, y_\xi) \in (C^* \cap l_\xi) \setminus \Big(\bigcup_{l\in\mathcal{L}} l \cap l_\xi\Big)$. Set $A_\xi = \{(x^1_\xi, y_\xi), (x^2_\xi, y_\xi)\}$ if $l_\xi \cap A_{<\xi} = \emptyset$ or $A_\xi = \{(x_\xi, y^1_\xi)\}$ if $|l_\xi \cap A_{<\xi}| = 1$.

Case 3 (*otherwise*). If $|l_\xi \cap A_{<\xi}| = 2$ then set $A_\xi = \emptyset$. If $|l_\xi \cap A_{<\xi}| < 2$ then $|l_\xi \cap l| \leqslant 1$ for any $l \in \mathcal{L}$ and choose arbitrary $(x^1_\xi, y^1_\xi), (x^2_\xi, y^2_\xi) \in (C^* \cap l_\xi) \setminus \Big(\bigcup_{l\in\mathcal{L}} l \cap l_\xi\Big)$ with $x^1_\xi, x^2_\xi \notin B$ and $y^1_\xi, y^2_\xi \notin (0,1)$. It is possible since $|\pi_1[l_\xi \cap C^*] \cap (\mathbb{R} \setminus B)| = \mathfrak{c}$. Set $A_\xi = \{(x^1_\xi, y^1_\xi), (x^2_\xi, y^2_\xi)\}$ if $l_\xi \cap A_{<\xi} = \emptyset$ or $A_\xi = \{(x^1_\xi, y^1_\xi)\}$ if $|l_\xi \cap A_{<\xi}| = 1$.

Finally, set $A = \bigcup_{\xi<\mathfrak{c}} A_\xi$. Since $A \subseteq C^*$, it is meager and null. By (4)–(6) if $f \subseteq A$ then $f^{-1}((0,1)) = B$.

□

5. Luzin and Sierpiński sets

We start this section with the definitions of special subsets of the real plane $\mathbb{R}^2$.

Definition 5.1. We say that a subset $A \subseteq \mathbb{R}^2$ is a *Luzin set* if the intersection of the set A with every meager set is countable.

Moreover, a set $A \subseteq \mathbb{R}^2$ is a *strongly Luzin set* if A is a Luzin set and the intersection of A with every Borel nonmeager set has cardinality $\mathfrak{c}$.

Definition 5.2. We say that a subset $A \subseteq \mathbb{R}^2$ is a *Sierpiński set* if the intersection of the set A with every null set is countable.

Moreover, a set $A \subseteq \mathbb{R}^2$ is a *strongly Sierpinński set* if A is a Sierpiński set and the intersection of A with every Borel set of positive Lebesgue measure has cardinality $\mathfrak{c}$.

The following remark holds.

Remark 5.3. Assume $A \subseteq \mathbb{R}^2$ is a two point set. Then
(1) A is not Bernstein,

(2) A is not Luzin,
(3) A is not Sierpiński.

Proof. (1) Each line l is a perfect set such that $|A \cap l| = 2$, so A cannot be a Bernstein set.

(2) and (3) Let N be a perfect null subset of $\mathbb{R}$. Then N is a nowhere dense set and then $N \times \mathbb{R}$ is null and meager set with

$$|(N \times \mathbb{R}) \cap A| = 2|N| = \mathfrak{c}.$$

So, A cannot be a Luzin set and a Sierpiński set. □

Let us give the following definition.

Definition 5.4. A set $A \subseteq \mathbb{R}^2$ is a *partial two point set* if A intersects every line in at most two points.

Theorem 5.5 ([CH]).
(1) *There exists a partial two point set A that is a strong Luzin set.*
(2) *There exists a partial two point set B that is a strong Sierpiński set.*

Proof. Let us focus on the Luzin set. The case of the Sierpiński set is similar.

Fix a base $\{B_\alpha\colon \alpha < \omega_1\}$ of the ideal of meager sets and let $\{D_\alpha\colon \alpha < \omega_1\}$ be the enumeration of Borel nonmeager sets such that each set appears ω_1 many times. We will construct a sequence $\{x_\alpha\colon \alpha < \omega_1\}$ having the following properties:
(1) $A_\alpha = \{x_\xi\colon \xi \leqslant \alpha\}$ does not contain three collinear points,
(2) $x_\alpha \in D_\alpha \setminus \bigcup_{\xi<\alpha} B_\xi$.
We will show that at any α step we can pick x_α such that (1) and (2) are fulfilled. Since A_ξ is countable so is $\bigcup_{\xi<\alpha} A_\xi$. Therefore the set

$$\mathcal{L}_{<\alpha} = \Big\{l\colon\ l \text{ is a line and } \Big|l \cup \bigcup_{\xi<\alpha} A_\xi\Big| = 2\Big\}$$

is countable. Hence, both $\mathcal{L}_{<\alpha}$ and $\bigcup_{\xi<\alpha} B_\xi$ are meager. Consequently, one can pick a point x_α from D_α that meets neither $\mathcal{L}_{<\alpha}$ nor $\bigcup_{\xi<\alpha} B_\xi$. So, the inductive construction is done.

Finally, set $A = \{x_\alpha\colon \alpha < \omega_1\}$. It is a required partial two point set that is strong Luzin. □

Let us remark that Luzin sets and Sierpiński sets are s_0. Moreover, A is strongly null and B is strongly meager. For the definitions of strongly meager and strongly null we refer the reader to [2].

Theorem 5.5 can be strengthen. If we assume that $\operatorname{add}(\mathcal{M}) = \operatorname{cof}(\mathcal{M}) = \kappa$ then we can construct a partial two point set A such that $|A| = \kappa$ and for every Borel set B, $|B \cap A| < \kappa$ if and only if $B \in \mathcal{M}$.

An analogous observation is true in the case of null sets $\mathcal{N}$.

6. κ-COVERING

At the beginning of this section we will recall the notion of a κ-covering and a κ-I-covering (see [7]).

Definition 6.1. Let κ be a cardinal number. A set $A \subseteq \mathbb{R}^2$ is called a κ-*covering* if

$$(\forall X \in [\mathbb{R}^2]^\kappa)(\exists y \in \mathbb{R}^2)\ y + X \subseteq A$$

where $y + X$ stands for $\{y + x\colon\ x \in X\}$.

Let $\operatorname{Iso}(\mathbb{R}^2)$ be the group of all isometries of the real plane $\mathbb{R}^2$.

Definition 6.2. Let κ be a cardinal number. A set $A \subseteq \mathbb{R}^2$ is called a κ-I-*covering* if

$$(\forall X \in [\mathbb{R}^2]^\kappa)(\exists g \in \operatorname{Iso}(\mathbb{R}^2))\ g[X] \subseteq A.$$

Obviously, if A is a κ-covering then A is a κ-I-covering and if $\lambda < \kappa$, then A is a κ-covering (κ-I-covering) implies that A is a λ-covering (λ-I-covering).

Let us start with the following result.

Theorem 6.3. *There exists an $\aleph_0$ point set which is not a* 2-I-*covering.*

P r o o f. Let us enumerate the set of all lines $\text{Lines} = \{l_\xi\colon\ \xi < \mathfrak{c}\}$ in $\mathbb{R}^2$. We construct a transfinite sequence $(A_\xi\colon\ \xi < \mathfrak{c})$ of countable subsets of $\mathbb{R}^2$ such that for every $\xi < \mathfrak{c}$:

(1) $l \cap A_\xi = \emptyset$ for every $l \in \mathcal{L}_{<\xi}$,
(2) if $l_\xi \notin \mathcal{L}_{<\xi}$ then $|l_\xi \cap A_\xi| = \aleph_0$,
(3) $d(a,b) \neq 1$ for every $a, b \in \bigcup\limits_{\zeta<\xi} A_\zeta$

where $\mathcal{L}_{<\xi} = \Big\{l \in \text{Lines}\colon\ \Big|l \cap \bigcup\limits_{\zeta<\xi} A_\zeta\Big| = \aleph_0\Big\}$ and $d\colon\ \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_+$ denotes the standard metric on $\mathbb{R}^2$.

Let us notice that $\mathcal{L}_{<\xi} \subseteq \Big\{l \in \text{Lines}\colon\ \Big|l \cap \bigcup\limits_{\zeta<\xi} A_\zeta\Big| \geqslant 2\Big\}$. So, $|\mathcal{L}_{<\xi}| < \mathfrak{c}$ and the inductive construction can be done.

Now, setting $A = \bigcup_{\xi<\mathfrak{c}} A_\xi$, we obtain the requested set. Indeed, (1) and (2) imply that A is an $\aleph_0$ point set and (3) guarantees that A is not a 2-I-covering. $\square$

Theorem 6.4. *There exists an $\aleph_0$ point set which is an $\aleph_0$-covering.*

P r o o f. Let us enumerate the set of all lines $\text{Lines} = \{l_\xi \colon \xi < \mathfrak{c}\}$ and the family of all countable subsets of the real plane $[\mathbb{R}^2]^\omega = \{X_\xi \colon \xi < \mathfrak{c}\}$. We construct a transfinite sequence $((A_\xi, y_\xi) \in [\mathbb{R}^2]^\omega \times \mathbb{R}^2 \colon \xi < \mathfrak{c})$ with the following properties:
(1) $l \cap A_\xi = \emptyset$ for every $l \in \mathcal{L}_{<\xi}$,
(2) if $l_\xi \notin \mathcal{L}_{<\xi}$ then $|l_\xi \cap A_\xi| = \aleph_0$,
(3) $y_\xi + X_\xi \subseteq A_\xi$
where $\mathcal{L}_{<\xi} = \left\{ l \in \text{Lines} \colon \left| l \cap \bigcup_{\zeta<\xi} A_\zeta \right| = \aleph_0 \right\}$.

Let us notice that

$$\left\{ y \colon y + X_\xi \cap \bigcup \mathcal{L}_{<\xi} \neq \emptyset \right\} = \{y \colon \exists x \in X_\xi \; \exists l \in \mathcal{L}_{<\xi} y + x \in l\} = \bigcup_{l \in \mathcal{L}_{<\xi}} \bigcup_{x \in X_\xi} l - x.$$

This set, as a union of $< \mathfrak{c}$ many lines, does not cover the whole $\mathbb{R}^2$. Set y_ξ in such a way that $y_\xi \notin \bigcup_{l \in \mathcal{L}_{<\xi}} \bigcup_{x \in X_\xi} l - x$. The rest of the inductive construction is similar to that in Theorem 6.7.

The resulting set $A = \bigcup_{\xi<\mathfrak{c}} A_\xi$ is an $\aleph_0$ point set by (1) and (2). So, y_ξ's constructed in (3) witness that A is an $\aleph_0$-covering. $\square$

Theorem 6.5. *If there is a family $\mathcal{F} \subseteq [\mathfrak{c}]^{\omega_1}$ of size $\mathfrak{c}$ such that for every $X \in [\mathfrak{c}]^{\omega_1}$ there exists $Y \in \mathcal{F}$ with $X \subseteq Y$, then there exists an $\aleph_1$ point set in the plane which is an $\aleph_1$-covering.*

P r o o f. Let us consider V, a model of ZFC such that $V \models \mathfrak{c} = 2^{\aleph_1} = \aleph_2$. Such a model can be obtained by adding ω_2 Cohen reals to the constructible universe L. The rest of the proof goes in way similar to the proof of Theorem 6.4. $\square$

Moreover, we can state the following theorem provided by referee.

Theorem 6.6. *Suppose the continuum $\mathfrak{c}$ is singular of cofinality ω_1, e.g. $\mathfrak{c} = \aleph_{\omega_1}$, then there is no $\aleph_1$ point set in the plane which is an $\aleph_1$-*I*-covering.*

P r o o f. Suppose $X \subseteq \mathbb{R} \times \mathbb{R}$ were such set. Let $Y_\alpha \in [\mathbb{R} \times \{0\}]^{\omega_1}$ for $\alpha < \mathfrak{c}$ list all subsets of the x-axis isometric to $l \cap X$ for some line l. Let κ_α for $\alpha < \omega_1$ be strictly increasing with $\sup \mathfrak{c}$. For each $\alpha < \omega_1$ choose

$$p_\alpha \in \mathbb{R} \times \{0\} \setminus \bigcup_{\beta<\kappa_\alpha} Y_\beta.$$

Then X fails to contain an isometric copy of $\{p_\alpha\colon \alpha < \omega_1\}$, contradicting that it is an $\aleph_1$-I-covering. $\square$

We can obtain the following result.

Theorem 6.7. *Fix an integer $n \geqslant 2$.*
▷ *There exists an n point set which is not a 2-I-covering.*
▷ *There exists an n point set which is a n-covering.*

Proof. The proof of this theorem is similar to the proofs of Theorem 6.3 and Theorem 6.4. $\square$

Let us recall that A is a 2-covering iff $A - A = \mathbb{R}^2$. This gives the following result.

Corollary 6.8. *There exists a two point set A such that $A - A = \mathbb{R}^2$.*

7. Combinatorial properties

Let us recall that a family $\mathcal{A}$ of infinite subsets of ω is an *almost disjoint family* (ad) if any two distinct members of $\mathcal{A}$ have finite intersection. $\mathcal{A}$ is a *maximal almost disjoint family* (mad) if it is an ad family which is maximal with respect to inclusion.

Analogously, we say that $\mathcal{B} \subseteq \omega^\omega$ is a *family of eventually different functions* if every two distinct members $x, y \in \mathcal{B}$ coincide only on a finite subset of ω.

Let κ be a cardinal number. We say that the family $\{A_\xi \in [\omega]^\omega\colon \xi < \kappa\}$ is a *tower* if
▷ $(\forall\, \xi, \eta < \kappa) \xi < \eta \Rightarrow A_\eta \subseteq^* A_\xi$ and
▷ there is no $B \in [\omega]^\omega$ $(\forall \xi < \kappa)$ $B \subseteq^* A_\xi$. Here, $A \subseteq^* B$ means that $|A \setminus B| < \omega$.

Theorem 7.1 ([CH])**.** *Let $h\colon \mathbb{R} \to \omega^\omega$ be a bijection. There exists a partial two point set $A \subseteq \mathbb{R}^2$ such that the family $h[\pi_1[A] \cup \pi_2[A]]$ forms a maximal family of eventually different functions.* (π_i *denotes the projection on the i-th coordinate.*)

Proof. Let $\omega^\omega = \{f_\alpha\colon \alpha < \omega_1\}$. By transfinite induction we will construct a set $A = \{a_\xi\colon \xi < \omega_1\} \subseteq \mathbb{R}^2$ such that for every $\alpha < \omega_1$
(1) $A_\alpha = \{a_\xi\colon \xi < \alpha\}$ is a partial two point set,
(2) $F_\alpha = h[\pi_1[A_\alpha] \cup \pi_2[A_\alpha]]$ is a family of eventually different functions,
(3) $(\exists \xi \leqslant \alpha)(\exists i \in \{0, 1\})$ $|f_\alpha \cap h(\pi_i(a_\xi))| = \aleph_0$.
Assume now that we have already constructed the set A_α.

Case 1. (f_α *is eventually different from every function of the form $h(\pi_i(a_\xi))$ for $\xi < \alpha$ and $i \in \{0, 1\}$*) Set $x_\alpha = h^{-1}(f_\alpha)$. We can find $y_\alpha \in \mathbb{R}$ such that
▷ (x_α, y_α) does not belong to any line from $\mathcal{L}(A_\alpha)$,

▷ $h(y_\alpha)$ is eventually different from every function from $F_\alpha \cup \{f_\alpha\}$, where $\mathcal{L}(A_\alpha)$ denotes the family of all lines intersecting A_α in exactly two points. A point y_α can be found since A_α is countable.

Case 2. ($|f_\alpha \cap h(\pi_i(a_\xi))| = \aleph_0$ *for some* $\xi < \alpha$ *and* $i \in \{0,1\}$) Then we can find $x_\alpha,\ y_\alpha \in \mathbb{R}$ such that

▷ (x_α, y_α) does not belong to any line from $\mathcal{L}(A_\alpha)$,

▷ $F_\alpha \cup \{h(x_\alpha), h(y_\alpha)\}$ is a family of eventually different functions. Again, the construction is possible since A_α is countable.

Set $a_\alpha = (x_\alpha, y_\alpha)$. The inductive step is proved.

Let us notice that the resulting set $A = \bigcup_{\alpha<\omega_1} A_\alpha$ is a partial two point set by (1). $h[\pi_1[A] \cup \pi_2[A]]$ is a family of eventually different functions by (2). The maximality of this family follows from (3). □

Remark 7.2. The same result is true if we replace a maximal family of eventually different functions by a mad family. (In this case we consider a bijection $h\colon \mathbb{R} \to [\omega]^\omega$.)

In the proof of the next theorem we adopt the method from Kunen's theorem about the existence of an indestructible mad family (see [8]).

Theorem 7.3. *Let us fix a standard Borel bijection* $h\colon \mathbb{R} \to [\omega]^\omega$*. It is consistent with* ZFC$+\neg$CH *that there exists a partial two point set* A *such that* $h[\pi_1[A] \cup \pi_2[A]]$ *forms a mad family of size* ω_1*.*

Proof. Let us consider a model V' obtained from $V \models \mathrm{CH}$ by adding $\kappa > \omega_1$ Cohen reals (i.e. using forcing $\mathrm{Fn}(\kappa, 2)$). It suffices to construct a partial two point set A in V which remains maximal in the generic extension V'.

Let us notice that, since every subset of ω has a name in $\mathrm{Fn}(I, 2)$ for some countable $I \subseteq \kappa$, it is enough to consider names in $\mathrm{Fn}(\omega, 2)$.

In V, let us enumerate all possible pairs $(p_\xi, \tau_\xi)\colon \omega \leqslant \xi < \omega_1$ (by CH), where $p_\xi \in \mathrm{Fn}(\omega, 2)$ and τ_ξ is a nice name for an infinite subset of ω. Take any countable sequence $(F_n^i\colon n \in \omega \wedge i \in \{0,1\})$ of pairwise disjoint countable subsets of ω.

Now we define a transfinite sequence $(F_\xi^i\colon \omega \leqslant \xi < \omega_1 \wedge i \in \{0,1\})$ satisfying the following conditions for every $\xi < \omega_1$:

(1) $(F_\zeta^i\colon \zeta < \xi \wedge i \in \{0,1\})$ is an almost disjoint family,

(2) if $(\forall \eta < \xi)(\forall i \in 2) p_\xi \Vdash |\tau_\xi \cap F_\eta^i| < \omega$ then $p_\xi \Vdash |\tau_\xi \cap F_\xi^0| = \omega$ or $p_\xi \Vdash |\tau_\xi \cap F_\xi^1| = \omega$,

(3) $\{a_\zeta = (h^{-1}[\{F_\zeta^0\}], h^{-1}[\{F_\zeta^1\}])\colon \zeta < \xi\}$ forms a partial two point set.

To see that this recursion is possible let us assume that the construction at the step $\xi < \omega_1$ is done. Now let us enumerate $\{F_\eta^i\colon \eta < \xi \wedge i \in 2\} = \{B_n\colon n \in \omega\}$ by ω. If the assumption in condition (2) is not fulfilled then choose any F_ξ^1 almost disjoint with every F^i_η for $\eta < \xi$ and $i \in 2$ what is possible since $|\xi| = \omega$. Now, let us assume that the assumption of (2) is fulfilled. We show that

$$(\star\star) \qquad (\forall n \in \omega)(\forall q \leqslant p_\xi)(\exists m > n)(\exists r < q)\ r \Vdash m \in \tau_\xi \setminus (B_0 \cup \ldots B_n).$$

Let us fix any $n \in \omega$ and $q < p_\xi$. By assumption $p_\xi \Vdash |\tau_\xi \cap (B_0 \cup \ldots B_n)| < \omega$. So

$$p_\xi \Vdash (\exists m > n)\ m \in \tau \setminus (B_0 \cup \ldots \cup B_n).$$

q is stronger than p_ξ, so it forces the same sentence. Now, we can find a stronger condition $r < q$ and a positive integer $m > n$ such that

$$r \Vdash m \in \tau \setminus (B_0 \cup \ldots B_n).$$

This completes the proof of $(\star\star)$. $\square$

Now let us enumerate the set $\omega \times \{q \in \mathrm{Fn}(\omega, 2)\colon\ q \leqslant p_\xi\} = \{(n_j, q_j)\colon\ j < \omega\}$. Then for every $j < \omega$ there exist $m_j \in \omega$ and $r_j < q_j$ such that $n_j < m_j$ and

$$r_j \Vdash m_j \in \tau_\xi \setminus (B_0 \cup \ldots B_{n_j}).$$

Let $F^1_\xi = \{m_j\colon\ j < \omega\}$. Then $F^i_\eta \cap F^1_\xi$ is finite, so $y_\xi = h^{-1}[\{F^1_\xi\}]$ is a real different from the other coordinates appearing in the previous step of the construction.

Now we will construct the first coordinate of the new point. To do this, set $A_{<\xi} = \{(h^{-1}(F^0_\eta), h^{-1}(F^1_\eta))\colon\ \eta < \xi\} \subset \mathbb{R}^2$. Denote by $\mathcal{L}_{<\xi}$ the set of all lines $l \subseteq \mathbb{R}^2$ in the real plane such that $|l \cap A_{<\xi}| = 2$. Let us observe that the set

$$Y = \{z \in \mathbb{R}\colon\ (\exists l \in \mathcal{L}_{<\xi})(z, y_\xi) \in l\}$$

is countable. Let us enumerate $Y = \{z_n\colon\ n < \omega\}$. Now, consider the sequence $C_n = h(z_n),\ n \in \omega$.

To define the set F^0_ξ we will use a diagonal argument. Let us arrange elements of each set $C_n = \{c^n_i\colon\ i \in \omega\}$ in an increasing sequence and let us define the increasing sequence $(d_n)_{n\in\omega}$ of nonnegative integers by

$$d_n = \max\{c^n_i\colon\ i \leqslant n\}.$$

Now, let us choose an increasing sequence $(m_n)_{n\in\omega}$ such that for every $n \in \omega$ we have

- ▷ $d_n < m_n$ and
- ▷ $m_n \in \omega \setminus F^1_\xi \cup B_0 \cup \ldots \cup B_n$.

Set $F^0_\xi = \{m_n\colon\ n\in\omega\}$. It is easy to see that

(1) $F^0_\xi \neq C_n$ for every $n\in\omega$,

(2) $|F^0_\xi \cap B_n| < \omega$ for every $n\in\omega$,

(3) $|F^0_\xi \cap F^1_\xi| < \omega$.

The first property ensures that the set $A_{<\xi} \cup \{(h^{-1}(F^0_\xi), h^{-1}(F^1_\xi))\}$ does not contain three collinear points. The second and third properties imply that the set $\{F^i_\eta\colon \eta \leqslant \xi \wedge i\in 2\}$ forms an almost disjoint family.

Our construction of the sequences $(F^0_\xi\colon\ \xi<\omega)$ and $(F^1_\xi\colon\ \xi<\omega_1)$ is completed. It remains to prove that

$$\Vdash_{\mathrm{Fn}(\omega,2)}\ \{F^0_\xi\colon\ \xi<\omega_1\}\cup\{F^1_\xi\colon\ \xi<\omega_1\} \text{ is a mad family.}$$

If not then there exists a condition $p\in \mathrm{Fn}(\omega,2)$ and a nice name $\tau\in V^{\mathrm{Fn}(\omega,2)}$ for an element of $P(\omega)$ such that

$$p\Vdash (\forall\xi<\omega_1)(\forall(i\in 2))\ |\tau\cap F^i_\xi|<\omega.$$

There exists $\xi<\omega_1$ such that $(p,\tau)=(p_\xi,\tau_\xi)$. So, the assumption in the condition (2) is fulfilled. We know that τ witnesses that there exist $q<p$ and $n\in\omega$ such that

$$q\Vdash \tau\cap F^i_\xi\subset n.$$

On the other hand, there exist $r<q$ and $m>n$ such that $r\Vdash m\in\tau\cap F^0_\xi$ or there exist $r'<q$ and $m'>n$ such that $r'\Vdash m'\in\tau\cap F^1_\xi$, a contradiction. $\square$

Theorem 7.4. *Let us fix a standard Borel bijection $h\colon\ \mathbb{R}\to[\omega]^\omega$. It is consistent with* ZFC$+\neg$CH *that there exists a partial two point set A such that $h[\pi_1[A]\cup\pi_2[A]]$ forms a tower of size ω_1.*

We will omit the proof because it is very similar to the proof of Theorem 7.3.

Theorem 7.5. *It is consistent with* ZFC $+\neg$CH *that there exists a partial two point set $C\subseteq\mathbb{R}^2$ of size ω_2 such that C is a Luzin set and*

$$(\exists A\in\mathcal{N})(\forall D\in[C]^{\omega_1})\ A+D=\mathbb{R}^2.$$

P r o o f. Let us start with $V\vDash$ CH. Consider the generic extension $V[c_\alpha\colon\ \alpha<\omega_2]$ obtained by adding ω_2 independent Cohen reals. We can assume that $c_\alpha\in\mathbb{R}^2$ for every $\alpha<\omega_2$. Set $C=\{c_\alpha\colon\ \alpha<\omega_2\}$.

C is a partial two point set. Indeed, take any line l which intersects two different points of C: c_α, c_β. Take any $\gamma \in \omega_2 \setminus \{\alpha, \beta\}$. Then c_γ is a Cohen real over $V[c_\alpha, c_\beta]$ and l is a meager set coded in $V[c_\alpha, c_\beta]$. So, $c_\gamma \notin l$.

C is a Luzin set. Take any Borel meager set M from $V[c_\alpha\colon \alpha < \omega_2]$. Then M is coded in $V[c_\alpha\colon \alpha \in I]$ for some countable I. So, $M \cap \{c_\alpha\colon \alpha \in \omega_2 \setminus I\} = \emptyset$.

Now, let us fix the Marczewski decomposition: $\mathbb{R}^2 = A \cup B$, where $A \in \mathcal{N}$, $B \in \mathcal{M}$ and $A \cap B = \emptyset$. Let us recall that A, B are coded in V. Take any $D \subseteq C$ of size ω_1. Take any $x \in \mathbb{R}^2$ (in $V[c_\alpha\colon \alpha < \omega_2]$). Then x is in $V[c_\alpha\colon \alpha \in J]$ for some countable J. So, $x - B$ is a meager set coded in $V[c_\alpha\colon \alpha \in J]$. Take $c \in D \setminus \{c_\alpha\colon \alpha \in J\}$. Then $c \notin x - B$. So, $x \in A + c$. This shows that $\mathbb{R}^2 \subseteq A + D$. □

In a similar way one can show the following result.

Theorem 7.6. *It is consistent with* ZFC + ¬CH *that there exists a partial two point set* $R \subseteq \mathbb{R}^2$ *of size* ω_2 *such that* R *is a Sierpiński set and*

$$(\exists B \in \mathcal{M})(\forall D \in [R]^{\omega_1})\ B + D = \mathbb{R}^2.$$

Acknowledgment. Authors would like to thank the referee for Theorem 6.6, for finding mistakes in an earlier version of this work and for giving valuable suggestions.

References

[1] *J. L. Bell, A. B. Slomson*: Models and Ultraproducts. An Introduction. North-Holland Publishing Company, Amsterdam, 1969.

[2] *T. J. Carlson*: Strong measure zero and strongly meager sets. Proc. Am. Math. Soc. *118* (1993), 577–586.

[3] *J. Cichoń, M. Morayne, R. Rałowski, C. Ryll-Nardzewski, S. Żeberski*: On nonmeasurable unions. Topology Appl. *154* (2007), 884–893.

[4] *J. J. Dijkstra, K. Kunen, J. van Mill*: Hausdorff measures and two point set extensions. Fundam. Math. *157* (1998), 43–60.

[5] *J. J. Dijkstra, J. van Mill*: Two point set extensions—a counterexample. Proc. Am. Math. Soc. *125* (1997), 2501–2502.

[6] *T. Jech*: Set Theory. The third millennium edition, revised and expanded. Springer Monographs in Mathematics, Springer, Berlin, 2003.

[7] *J. Kraszewski, R. Rałowski, P. Szczepaniak, S. Żeberski*: Bernstein sets and κ-coverings. Math. Log. Q. *56* (2010), 216–224.

[8] *K. Kunen*: Set Theory. An Introduction to Independence Proofs. Studies in Logic and the Foundations of Mathematics vol. 102, North-Holland Publishing Company, Amsterdam, 1980.

[9] *D. D. Larman*: A problem of incidence. J. Lond. Math. Soc. *43* (1968), 407–409.

[10] *R. D. Mauldin*: On sets which meet each line in exactly two points. Bull. Lond. Math. Soc. *30* (1998), 397–403.

[11] *S. Mazurkiewicz*: O pewnej mnogości płaskiej, która ma z każdą prostą dwa i tylko dwa punkty wspólne. Comptes Rendus des Séances de la Société des Sciences et Lettres de

Varsovie *7* (1914), 382–384 (In Polish.); Sur un ensemble plan qui a avec chaque droite deux et seulement deux points communs. Stefan Mazurkiewicz, Traveaux de Topologie et ses Applications (K. Borsuk et al., eds.). Wydawnictwo naukowe PWN, Warsaw, 1969, 46–47. (In French transl.)

[12] *A. W. Miller*: Infinite combinatorics and definability. Ann. Pure Appl. Logic *41* (1989), 179–203.

[13] *A. W. Miller, S. G. Popvassiliev*: Vitali sets and Hamel base that are Marczewski measurable. Fundam. Math. *166* (2000), 269–279.

[14] *R. Ralowski*: Remarks on nonmeasurable unions of big point families. Math. Log. Q. *55* (2009), 659–665.

[15] *R. Rałowski, S. Żeberski*: Completely nonmeasurable unions. Cent. Eur. J. Math. *8* (2010), 683–687.

[16] *J. H. Schmerl*: Some 2-point sets. Fundam. Math. *208* (2010), 87–91.

[17] *E. Szpilrajn*: Sur une classe de fonctions de M. Sierpiński et la classe correspondante d'ensembles. Fundam. Math. *24* (1934), 17–34. (In French.)

[18] *S. Żeberski*: On completely nonmeasurable unions. Math. Log. Q. *53* (2007), 38–42.

Authors' addresses: Marek Bienias, Szymon Głąb, Institute of Mathematics, Technical University of Łódź, Wólczańska 215, 93-005 Łódź, Poland, e-mail: marek.bienias88@gmail.com, szymon.glab@p.lodz.pl; Robert Rałowski, Szymon Żeberski, Institute of Mathematics and Computer Science, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland, e-mail: robert.ralowski@pwr.wroc.pl, szymon.zeberski@pwr.wroc.pl.