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CONGRUENCES INVOLVING THE FERMAT QUOTIENT

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Abstract. Let $p > 3$ be a prime, and let $q_p(2) = (2^{p-1} - 1)/p$ be the Fermat quotient of p to base 2. In this note we prove that

$$\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv q_p(2) - \frac{pq_p(2)^2}{2} + \frac{p^2 q_p(2)^3}{3} - \frac{7}{48} p^2 B_{p-3} \pmod{p^3},$$

which is a generalization of a congruence due to Z. H. Sun. Our proof is based on certain combinatorial identities and congruences for some alternating harmonic sums. Combining the above congruence with two congruences by Z. H. Sun, we show that

$$q_p(2)^3 \equiv -3 \sum_{k=1}^{p-1} \frac{2^k}{k^3} + \frac{7}{16} \sum_{k=1}^{(p-1)/2} \frac{1}{k^3} \pmod{p},$$

which is just a result established by K. Dilcher and L. Skula. As another application, we obtain a congruence for the sum $\sum\limits_{k=1}^{p-1} 1/(k^2 \cdot 2^k)$ modulo p^2 that also generalizes a related Sun's congruence modulo p.

Keywords: Fermat quotient; nth harmonic number of order m; Bernoulli number

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1. Introduction and main results

The Fermat Little Theorem states that if p is a prime and a is an integer not divisible by p, then $a^{p-1} \equiv 1 \pmod{p}$. This gives rise to the definition of the *Fermat quotient of p to base a*,

$$q_p(a) := \frac{a^{p-1} - 1}{p},$$

which is an integer according to the Fermat Little Theorem. This quotient has been extensively studied because of its links to numerous question in number theory. It is well known that divisibility of the Fermat quotient $q_p(a)$ by p has numerous applications which include the Fermat Last Theorem and squarefreeness testing (see [1], [4], [6], [12], [16], [22], [27] and [30]). In particular, solvability of the congruence $q_p(2) \equiv 0 \pmod{p}$ for a prime p with $p \equiv 1 \pmod{4}$ and the congruences $q_p(a) \equiv 0 \pmod{p}$ with $a \in \{2, 3, 5\}$ were studied by S. Jakubec in [18] and [19], respectively.

A classical congruence, due to F. G. Eisenstein [11] in 1850, asserts that for a prime $p \geqslant 3$,

$$q_p(2) \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \pmod{p},$$

which was extended in 1861 by J. J. Sylvester [41] and in 1901 by Glaisher [14, pp. 21–22] as

$$q_p(2) \equiv -\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod{p}.$$

The above congruence was generalized in 1905 by M. Lerch in the first paper of substance on Fermat quotients [23] (see also [1, pp. 32–35]). Lerch developed equivalent results entailing fewer terms (that is, related to the sums of the form $s(k, N) =: \sum_{k=[jp/N]+1}^{[(j+1)p/N]} 1/k$), and his result was recently generalized by L. Skula [31] and by J. B. Dobson [9]. Notice that the congruences $s(0, 4) \equiv -3q_p(2) \pmod{p}$, $s(0, 3) \equiv -(3/2)q_p(3) \pmod{p}$ and $s(0, 6) \equiv -2q_p(2) - \frac{3}{2}q_p(3) \pmod{p}$ were established by Glaisher [14, p. 23], Lerch [23, p. 476, equation 14] and E. Lehmer [22, p. 356], respectively. A complete list of Lerch's sums $s(k, N)$ (with $k < N/2$) which can be evaluated solely in terms of Fermat quotients is given in [9, p. 23, Table 1].

For an odd prime p not dividing xyz, A. Wieferich [43] showed that $x^p + y^p + z^p = 0$ implies $q_p(2) \equiv 0 \pmod{p}$. The only known such primes (the so called *Wieferich primes*) 1093 and 3511 have long been known, and it was reported in [5] that there exist no new Wieferich primes $p < 4 \times 10^{12}$. Quite recently, F. G. Dorais and D. Klyve [10] extended this bound up to 6.7×10^{15}.

The connection of Fermat quotients with the first case of the Fermat Last Theorem retains its historical interest despite the complete proof of this theorem by A. Wiles in 1995, and Skula's demonstration in 1992 [30] that the failure of the first case of the Fermat Last Theorem would imply the vanishing of many similar sums but with much smaller ranges (sums of Lerch's type which cannot be evaluated in terms of Fermat quotients). Some criteria concerning the first case of the Fermat Last Theorem on Lerch's type sums were established in Ribenboim's book [27], in 1995 by Dilcher and Skula [6] (cf. [9, Section 8]) and quite recently by J. B. Dobson [9].

Further, the Fermat quotient was extended and investigated for composite moduli in 1997 by T. Agoh, K. Dilcher and L. Skula [1] and in 1998 by L. Skula [29] (see also [2, Section 5]). Moreover, using the p-adic limit, L. Skula [29] transferred the notion of the Fermat quotient for composite moduli to those for *p-adic integers* and established related results.

Some combinatorial congruences for harmonic type sums modulo p^3 involving both the Fermat quotients $q_p(a)$ (with $a = 2$ or/and $a = 3$) and the Bernoulli number B_{p-3} can be found in [34, Theorem 5.2 (c) and Remark 5.3] and [35, Theorem 3.1 (i)–(iii) and Corollaries 3.1 and 3.2]. Also, certain similar combinatorial congruences modulo p^3 (or p^2) expressed in terms of Fermat quotients $q_p(a)$ (with $a = 2$ or/and $a = 3$) and some Euler numbers E_n (and/or the Bernoulli number B_{p-3}) can be found in [20] and [35, Theorems 3.2 (i)–(iii), 3.7 and Corollaries 3.3 and 3.9].

This paper is focussed on another type of sums arising from congruences modulo prime powers involving the Fermat quotient $q_p(2)$. In 1900 J. W. L. Glaisher [13] proved that for a prime $p \geqslant 3$ we have a curious congruence

$$q_p(2) \equiv -\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k} \pmod{p}. \tag{1.1}$$

Observe that comparing this congruence and Eisenstein's congruence given above, using the substitution trick $k \to p-k$ and the fact that by Fermat Little Theorem $2^p \equiv 2 \pmod{p}$, we immediately obtain

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k} \equiv \sum_{k=1}^{p-1}\frac{2^k}{k} \equiv -2\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k} \pmod{p}$$

which was also established in 1997 by W. Kohnen [21].

Recently L. Skula [17] conjectured that

$$q_p(2)^2 \equiv -\sum_{k=1}^{p-1}\frac{2^k}{k^2} \pmod{p}. \tag{1.2}$$

Applying a certain polynomial congruence, Granville [17] proved the congruence (1.2). In [25] we established a simple and elementary proof of the congruence (1.2).

In [17] Granville also remarked that, based on calculations, an obvious extension of (1.1) and (1.2) probably does not exist. However, using methods similar to those in [17], Dilcher and Skula ([7, Theorem 1, the congruence (5)]) established that

$$q_p(2)^3 \equiv -3\sum_{k=1}^{p-1}\frac{2^k}{k^3} + \frac{7}{16}\sum_{k=1}^{(p-1)/2}\frac{1}{k^3} \pmod{p}. \tag{1.3}$$

As noticed in [7], the congruences (1.1)–(1.3) give rise to the obvious question whether there exist similar formulas for higher powers of $q_p(2)$. The authors also remarked that their method of Section 2 in [7] does not appear to extend to higher powers. Recently, Agoh and Skula [3, Theorem 3.3] deduced an explicit formula for $q_p(2) \pmod{p^4}$ represented by a linear combination of Mirimanoff polynomial values (including Bernoulli numbers).

Further, note that by the Fermat Little Theorem and (1.1), we have

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k} &\equiv 2^{p-1}\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}=\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^{p-k}}{k}\\ &\equiv -\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^{p-k}}{p-k}=-\frac{1}{2}\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv q_p(2)\pmod{p}.\end{aligned}\tag{1.4}$$

Notice also that the above congruence may be extended by the following well known congruences (e.g., see [39, Proof of Corollary 1.2]):

$$q_p(2)\equiv\frac{1}{2}\sum_{k=1}^{p-1}\frac{(-1)^{k-1}}{k}\equiv-\frac{1}{2}\sum_{j=1}^{(p-1)/2}\frac{1}{j}\pmod{p}.$$

Similarly, using (1.2), we obtain

$$\sum_{k=1}^{p-1}\frac{1}{k^2\cdot 2^k}\equiv-\frac{1}{2}q_p(2)^2\pmod{p},\tag{1.5}$$

and using (1.3), we get

$$\sum_{k=1}^{p-1}\frac{1}{k^3\cdot 2^k}\equiv\frac{1}{6}q_p(2)^3+\frac{7}{48}B_{p-3}\pmod{p}.\tag{1.6}$$

In [35] Z. H. Sun presented the following extension of the previous congruences.

Theorem 1.1 ([35, Theorem 4.1])**.** *Let $p>3$ be a prime. Then*

(i) $\displaystyle\sum_{k=1}^{p-1}\frac{2^k}{k}\equiv-2q_p(2)-\frac{7p^2}{12}B_{p-3}\pmod{p^3}$,

(ii) $\displaystyle\sum_{k=1}^{p-1}\frac{2^k}{k^2}\equiv-q_p(2)^2+p\Big(\frac{2}{3}q_p(2)^3+\frac{7}{6}B_{p-3}\Big)\pmod{p^2}$,

(iii) $\displaystyle\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\equiv q_p(2)-\frac{p}{2}q_p(2)^2\pmod{p^2}$,

(iv) $\displaystyle\sum_{k=1}^{p-1}\frac{1}{k^2\cdot 2^k}\equiv -\frac{1}{2}q_p(2)^2 \pmod p$,

where B_{p-3} is the $(p-3)$rd Bernoulli number.

Recall that the *Bernoulli numbers B_k* are defined by the generating function

$$\sum_{k=0}^{\infty} B_k\frac{x^k}{k!}=\frac{x}{\mathrm{e}^x-1}.$$

It is easy to find the values $B_0=1$, $B_1=-1/2$, $B_2=1/6$, $B_4=-1/30$, and $B_n=0$ for odd $n\geqslant 3$. Furthermore, $(-1)^{n-1}B_{2n}>0$ for all $n\geqslant 1$ (see, e.g., [8]).

Note that the congruences (iv) and (1.5) are the same, while the congruences (i), (ii) and (iii) are generalizations of congruences (1.1), (1.2) and (1.4), respectively.

In this paper we generalize Sun's congruence (iii) modulo p^3 as follows.

Theorem 1.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}\equiv q_p(2)-\frac{p}{2}q_p(2)^2+\frac{p^2}{3}q_p(2)^3-\frac{7p^2}{48}B_{p-3} \pmod{p^3}. \tag{1.7}$$

As an application, we prove the congruence (1.3) due to Dilcher and Skula in [7].

Corollary 1.1 ([7, Theorem 1])**.** *Let $p>3$ be a prime. Then*

$$\begin{aligned} q_p(2)^3 &\equiv -3\sum_{k=1}^{p-1}\frac{2^k}{k^3}+\frac{7}{16}\sum_{k=1}^{(p-1)/2}\frac{1}{k^3} \pmod p\\ &\equiv -3\sum_{k=1}^{p-1}\frac{2^k}{k^3}-\frac{7}{8}B_{p-3} \pmod p. \end{aligned} \tag{1.8}$$

Remark 1.1. In [35, Remark 4.1] Z. H. Sun noticed that our congruence (1.7) may be derived from the congruence (1.8) and the congruence (4.5) in [35] related to the value of the Mirimanoff polynomial associated with p (for more information on Mirimanoff polynomials see [28]). Observe that congruential properties of Mirimanoff polynomials are in fact used in all the methods by Agoh and Skula [3], Dilcher and Skula [7], Granville [17] and Sun [35]. However, our proof of Theorem 1.2 is elementary and is based on certain combinatorial identities and related congruences. In this proof we additionally use certain congruences by H. Pan (Lemma 2.4) which have been derived in [26] via combinatorial methods. We also use some congruences (Lemma 2.5) which were proved by Z. H. Sun in [34] via a standard technique expressing sum of powers in terms of Bernoulli numbers.

We also point out that in a recent paper of the author [24, Theorem 2] the congruence (i) of Theorem 1.1 is proved in an elementary way and extended in terms of the harmonic sum.

The following result may be considered in some sense the "reversal congruence" of (1.7).

Corollary 1.2. *Let $p>3$ be a prime. Then*

$$q_p(2) \equiv \sum_{k=1}^{p-1} \frac{1}{k\cdot 2^k} - \frac{p}{2}\sum_{k=1}^{p-1}\frac{2^k}{k^2} + \frac{35p^2}{48}B_{p-3} \pmod{p^3}. \tag{1.9}$$

In particular,

$$q_p(2) \equiv \sum_{k=1}^{p-1} \frac{1}{k\cdot 2^k} - \frac{p}{2}\sum_{k=1}^{p-1}\frac{2^k}{k^2} \pmod{p^2}. \tag{1.10}$$

The following consequence is an improvement of Sun's congruence (iv) in Theorem 1.1.

Corollary 1.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{1}{k^2\cdot 2^k} \equiv -\frac{1}{2}q_p(2)^2 + \frac{p}{2}q_p(2)^3 + \frac{7p}{24}B_{p-3} \pmod{p^2}. \tag{1.11}$$

Proofs of Theorem 1.2 and its corollaries are given in Section 3 and are based on several combinatorial identities and congruences for some alternating harmonic sums presented in Section 2, and on some congruences due to H. Pan [26] and Z. H. Sun [34].

Remark 1.2. Note that the congruences (1.1)–(1.11) determine all the expressions for the sums $\sum_{k=1}^{p-1} 1/(k^r\cdot 2^k) \pmod{p^e}$ and $\sum_{k=1}^{p-1} 2^k/k^r \pmod{p^e}$ in terms of the Fermat quotient and the Bernoulli number B_{p-3}, where r and e are arbitrary positive integers such that $r+e\leqslant 4$. Thus a natural question arises: Is it possible to deduce analogous expressions for some values r and e such that $r+e\geqslant 5$? A recent result of Agoh and Skula ([3, Theorem 3.3]) concerning an explicit formula for

$q_p(2) \pmod{p^4}$ in terms of Mirimanoff polynomial values at 2, suggests that "the anti-derivative method" used in [3], [7], [17] and [35] cannot be applied to the case when $r + e \geqslant 5$. However, we believe that the method exposed in this paper can be applied for some pairs (r, e) with $r + e \geqslant 5$.

More recently, given a prime p and a positive integer $r < p - 1$, R. Tauraso [42, Theorem 2.3] established the congruence $\sum_{k=1}^{p-1} 2^k/k^r \pmod p$ in terms of an alternating r-tuple harmonic sum. For example, combining this result when $r = 2$ with the congruence (1.2) [42, Corollary 2.4], it follows that

$$\sum_{1 \leqslant i < j \leqslant p-1} \frac{(-1)^j}{ij} \equiv q_p(2)^2 \equiv -\sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod p.$$

Remark 1.3. Many curious congruences for the sums of the form $\sum_{k=1}^{(p-1)/2} a^k/k$ and $\sum_{k=1}^{(p-1)/2} 1/(k \cdot a^k)$ modulo odd prime with $a \in \{2, 3, 5\}$ were established by Z. W. Sun in [38, Theorem], [40, Theorem 3 (1.13)] and by Z. H. Sun in [37, Theorem 2.6], [36, congruences (1.1)–(1.5)].

2. Preliminary results

For a nonnegative integer n let

$$H_n := 1 + \frac{1}{2} + \ldots + \frac{1}{n}$$

be the nth *harmonic number* (we assume that $H_0 = 0$). The following identity is established in [33] by using finite differences.

Lemma 2.1 ([33, Identity 14, p. 3135])**.** *For a positive integer n we have*

$$\sum_{k=1}^{n} \binom{n}{k} H_k = 2^n H_n - 2^n \sum_{k=1}^{n} \frac{1}{k \cdot 2^k}. \tag{2.1}$$

We give here a simple induction proof of (2.1) which is based on the following identity.

Lemma 2.2 ([33, Identity 13, p. 3135]). *For a positive integer n we have*

$$\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{2^{n+1}-1}{n+1}. \tag{2.2}$$

P r o o f. Using the binomial formula and the identity $(n+1)^{-1}\binom{n+1}{k} = k^{-1}\binom{n}{k-1}$ with $1 \leqslant k \leqslant n+1$, we find that

$$\frac{2^{n+1}-1}{n+1} = \frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k} = \sum_{k=1}^{n+1}\frac{1}{k}\binom{n}{k-1} = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k},$$

as desired. $\square$

P r o o f of Lemma 2.1. We proceed by induction on $n \geqslant 1$. As (2.1) holds trivially for $n = 1$, we suppose that this is also true for some $n \geqslant 1$. Then using the induction hypothesis, the identities $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ and $H_k = H_{k-1} + 1/k$ with $1 \leqslant k \leqslant n+1$, we get

$$\begin{aligned}
\sum_{k=1}^{n+1}\binom{n+1}{k}H_k &= \sum_{k=1}^{n+1}\left(\binom{n}{k-1} + \binom{n}{k}\right)H_k \\
&= \sum_{k=1}^{n+1}\binom{n}{k-1}\Big(H_{k-1} + \frac{1}{k}\Big) + \sum_{k=1}^{n+1}\binom{n}{k}H_k \\
&= \sum_{k=1}^{n+1}\binom{n}{k-1}H_{k-1} + \sum_{k=1}^{n+1}\frac{1}{k}\binom{n}{k-1} + \sum_{k=1}^{n}\binom{n}{k}H_k \\
&= \sum_{k=1}^{n}\binom{n}{k}H_k + \sum_{k=1}^{n}\binom{n}{k}H_k + \sum_{k=1}^{n+1}\frac{1}{k}\binom{n}{k-1} \\
&= 2\sum_{k=1}^{n}\binom{n}{k}H_k + \sum_{k=1}^{n+1}\frac{1}{k}\binom{n}{k-1} \\
&= 2^{n+1}H_n - 2^{n+1}\sum_{k=1}^{n}\frac{1}{k\cdot 2^k} + \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}.
\end{aligned}$$

Hence, the induction proof will be completed if we prove that

$$2^{n+1}H_n - 2^{n+1}\sum_{k=1}^{n}\frac{1}{k\cdot 2^k} + \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k} = 2^{n+1}H_{n+1} - 2^{n+1}\sum_{k=1}^{n+1}\frac{1}{k\cdot 2^k}.$$

Substituting $H_{n+1} = H_n + 1/(n+1)$ into the above equality, it immediately reduces to

$$\sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k} = 2^{n+1}\Big(\frac{1}{n+1} - \frac{1}{(n+1)2^{n+1}}\Big) = \frac{2^{n+1}-1}{n+1}.$$

Since the above equality is in fact the identity (2.2) of Lemma 2.2, the induction proof is completed. □

Given positive integers n and m, the *harmonic numbers of order* m are the rational numbers $H_{n,m}$ defined as

$$H_{n,m}=\sum_{k=1}^{n}\frac{1}{k^m}.$$

Lemma 2.3. *Let n be an arbitrary positive integer. Then*

$$\sum_{k=1}^{2n}(-1)^k H_k=\frac{1}{2}H_n, \tag{2.3}$$

$$\sum_{k=1}^{2n}(-1)^k H_k^2=\frac{1}{4}H_{n,2}+2\sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2|j}}\frac{1}{ij}, \tag{2.4}$$

$$\sum_{k=1}^{2n}(-1)^k H_k\cdot H_{k,2}=\frac{1}{8}H_{n,3}+\sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2|j}}\frac{1}{i^2 j}+\sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2|j}}\frac{1}{ij^2}. \tag{2.5}$$

Proof. The identity (2.3) easily follows by induction on n, and hence its proof may be omitted.

We will prove the identity (2.4) also by induction on $n\geqslant 1$. For $n=1$ both sides of (2.4) are equal to $5/4$. If we suppose that (2.4) holds for some $n\geqslant 1$, then

$$\begin{aligned}
\sum_{k=1}^{2n+2}(-1)^k H_k^2 &= \sum_{k=1}^{2n}(-1)^k H_k^2+(H_{2n+2}^2-H_{2n+1}^2)\\
&=\frac{1}{4}H_{n,2}+2\sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2|j}}\frac{1}{ij}+\Big(H_{2n+1}+\frac{1}{2n+2}\Big)^2-H_{2n+1}^2\\
&=\frac{1}{4}H_{n,2}+2\sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2|j}}\frac{1}{ij}+\frac{2}{2n+2}\cdot H_{2n+1}+\frac{1}{(2n+2)^2}\\
&=\frac{1}{4}\Big(H_{n,2}+\frac{1}{(n+1)^2}\Big)+2\Big(\sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2|j}}\frac{1}{ij}+\frac{1}{2n+2}\sum_{l=1}^{2n+1}\frac{1}{l}\Big)\\
&=\frac{1}{4}H_{n+1,2}+2\sum_{\substack{1\leqslant i<j\leqslant 2n+2\\ 2|j}}\frac{1}{ij}.
\end{aligned}$$

This completes the induction proof of (2.4).

Similarly, we prove (2.5) by induction on n as

$$\begin{aligned}\sum_{k=1}^{2n+2}(-1)^k H_k\cdot H_{k,2} &= \sum_{k=1}^{2n}(-1)^k H_k\cdot H_{k,2} - H_{2n+1}\cdot H_{2n+1,2} + H_{2n+2}\cdot H_{2n+2,2}\\ &= \frac{1}{8}H_{n,3} + \sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2\mid j}}\frac{1}{i^2j} + \sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2\mid j}}\frac{1}{ij^2}\\ &\quad - H_{2n+1}\cdot H_{2n+1,2} + \Big(H_{2n+1}+\frac{1}{2n+2}\Big)\Big(H_{2n+1,2}+\frac{1}{(2n+2)^2}\Big)\\ &= \frac{1}{8}\Big(H_{n,3}+\frac{1}{(n+1)^3}\Big) + \sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2\mid j}}\frac{1}{i^2j} + \frac{1}{2n+2}\cdot H_{2n+1,2}\\ &\quad + \sum_{\substack{1\leqslant i<j\leqslant 2n\\ 2\mid j}}\frac{1}{ij^2} + \frac{1}{(2n+2)^2}\cdot H_{2n+1}\\ &= \frac{1}{8}H_{n+1,3} + \sum_{\substack{1\leqslant i<j\leqslant 2n+2\\ 2\mid j}}\frac{1}{i^2j} + \sum_{\substack{1\leqslant i<j\leqslant 2n+2\\ 2\mid j}}\frac{1}{ij^2}.\end{aligned}$$

This concludes the induction proof. □

Lemma 2.4. *Let $p>3$ be a prime. Then*

$$\sum_{\substack{1\leqslant i<j\leqslant p-1\\ 2\mid j}}\frac{1}{ij} \equiv \frac{1}{2}q_2(p)^2 - \frac{p}{2}q_2(p)^3 - \frac{7p}{16}B_{p-3} \pmod{p^2}, \tag{2.6}$$

$$\sum_{\substack{1\leqslant i<j<k\leqslant p-1\\ 2\mid k}}\frac{1}{ijk} \equiv -\frac{1}{6}q_2(p)^3 - \frac{7}{48}B_{p-3} \pmod{p}, \tag{2.7}$$

$$\sum_{\substack{1\leqslant i<j\leqslant p-1\\ 2\mid j}}\frac{1}{ij^2} \equiv \frac{5}{8}B_{p-3} \pmod{p}, \tag{2.8}$$

and

$$\sum_{\substack{1\leqslant i<j\leqslant p-1\\ 2\mid j}}\frac{1}{i^2j} \equiv -\frac{3}{8}B_{p-3} \pmod{p}. \tag{2.9}$$

P r o o f. The congruences (2.6) and (2.7) are in fact the congruences (2.9) and (2.10) in [26, Proof of Theorem 1.1], respectively. The congruences (2.8) and (2.9) are just the congruences (2.4) and (2.5) in [26, Lemma 2.2], respectively. □

Lemma 2.5. *If $p>3$ is a prime, then*

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k}\equiv -2q_2(p)+pq_2(p)^2-\frac{2p^2}{3}q_2(p)^3-\frac{7p^2}{12}B_{p-3}\pmod{p^3}, \tag{2.10}$$

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv \frac{7p}{3}B_{p-3}\pmod{p^2}, \tag{2.11}$$

and

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^3}\equiv -2B_{p-3}\pmod{p}. \tag{2.12}$$

Proof. The congruence (2.10) is in fact the congruence (c) in [34, Theorem 5.2]. Further, the congruences (2.11) and (2.12) are the congruences (a) with $k=2$ and (b) with $k=3$ in [34, Corollary 5.2], respectively. $\square$

Lemma 2.6. *Let $p>3$ be any prime. Then*

$$H_{p-1}\equiv -\frac{p^2}{3}B_{p-3}\pmod{p^3}, \tag{2.13}$$

$$\sum_{k=1}^{p-1}(-1)^kH_k\equiv -q_2(p)+\frac{p}{2}q_2(p)^2-\frac{p^2}{3}q_2(p)^3-\frac{7p^2}{24}B_{p-3}\pmod{p^3}, \tag{2.14}$$

$$\sum_{k=1}^{p-1}(-1)^kH_k^2\equiv q_2(p)^2-pq_2(p)^3-\frac{7p}{24}B_{p-3}\pmod{p^2}, \tag{2.15}$$

$$\sum_{k=1}^{p-1}(-1)^kH_k^3\equiv -q_2(p)^3-\frac{3}{8}B_{p-3}\pmod{p} \tag{2.16}$$

and

$$\sum_{k=1}^{p-1}(-1)^kH_k\cdot H_{k,2}\equiv 0\pmod{p}. \tag{2.17}$$

Proof. The congruence (2.13) is a classical result of Glaisher [13, p. 331]; see also [34, Theorem 5.1(a)]. The identity (2.3) in Lemma 2.3 with $n=(p-1)/2$ and the congruence (2.10) of Lemma 2.5 immediately yield (2.14). Similarly, the identity (2.4) in Lemma 2.3 with $n=(p-1)/2$, the congruences (2.11) of Lemma 2.5 and (2.6) of Lemma 2.4 immediately yield (2.15). Inserting the congruences (2.12), (2.8) and (2.9) into equality (2.5) of Lemma 2.3 with $n=(p-1)/2$, we obtain (2.17).

In order to prove (2.16), we will expand by the multinomial formula each term $H_k^3 = (1+1/2+\ldots+1/k)^3$ of the alternating sum $S := \sum_{k=1}^{p-1}(-1)^k H_k^3$. Accordingly, we will separately determine sums of all terms in S of the following forms: $\pm 1/i^3$ with $1 \leqslant i \leqslant p-1$, denoted by S_1, $\pm 3/(i^2 j)$ with $1 \leqslant i,j \leqslant p-1$ and $i \neq j$, denoted by S_2, and $\pm 6/(ijk)$ with $1 \leqslant i < j < k \leqslant p-1$, denoted by S_3. Since $p-1$ is even, for such an i the sum of all terms in S of the form $\pm 1/(2i-1)^3$ with $1 \leqslant i \leqslant (p-1)/2$ is equal to 0. Similarly, the sum of all terms in S of the form $\pm 1/(2i)^3$ with $1 \leqslant i \leqslant (p-1)/2$ is equal to $1/(2i)^3$. Therefore, applying (2.12) of Lemma 2.5, we have

$$S_1 = \frac{1}{8}\sum_{i=1}^{(p-1)/2}\frac{1}{i^3} \equiv -\frac{1}{4}B_{p-3} \pmod{p}. \tag{2.18}$$

Further, it is easy to see that the sum of all terms in S of the form $\pm 3/(i^2\cdot(2j-1))$ with $1 \leqslant i < 2j-1 \leqslant p-1$ is equal to 0. Similarly, the sum of all terms in S of the form $\pm 3/(i^2 \cdot 2j)$ with $1 \leqslant i < 2j \leqslant p-1$ is equal to $3/(i^2\cdot 2j)$. Hence, for a fixed $2i$ with $1 \leqslant i \leqslant (p-1)/2$, the subsum of the sum S_2 containing all the terms of the form $\pm 3/((2i)^2 \cdot j)$ with $j > 2i$ is

$$S_{1,2i} = \sum_{2i<2j\leqslant p-1}\frac{3}{(2i)^2\cdot 2j} = \frac{3}{4i^2}\sum_{j=i+1}^{(p-1)/2}\frac{1}{2j}. \tag{2.19}$$

In the same way, for a fixed $2i-1$ with $1 \leqslant i \leqslant (p-1)/2$, the subsum of the sum S_2 containing all the terms of the form $\pm 3/((2i-1)^2\cdot j)$ with $j > 2i-1$ is

$$S_{1,2i-1} = \sum_{2i-1<2j\leqslant p-1}\frac{3}{(2i-1)^2\cdot 2j} = \frac{3}{(2i-1)^2}\sum_{j=i}^{(p-1)/2}\frac{1}{2j}. \tag{2.20}$$

Similarly, for a fixed $2i$ with $1 \leqslant i \leqslant (p-1)/2$, the subsum of the sum S_2 containing all the terms of the form $\pm 3/((2i)^2\cdot j)$ with $j < 2i$ is

$$S'_{1,2i} = \sum_{j=1}^{2i-1}\frac{3}{(2i)^2\cdot j} = \frac{3}{4i^2}\sum_{j=1}^{2i-1}\frac{1}{j}. \tag{2.21}$$

Further, for a fixed $2i-1$ with $1 \leqslant i \leqslant (p-1)/2$, the subsum of the sum S_2 containing all the terms of the form $\pm 3/((2i-1)^2\cdot j)$ with $j < 2i-1$ is

$$S'_{1,2i-1} = 0. \tag{2.22}$$

From equalities (2.19) and (2.21) we see that for any fixed $2i$ with $1 \leqslant i \leqslant (p-1)/2$, the subsum of the sum S_2 containing all the terms of the form $\pm 3/((2i)^2 \cdot k)$ such that $1 \leqslant k \leqslant p-1$ and $k \neq 2i$ is

$$S_{2i} = S_{1,2i} + S'_{1,2i} = \frac{3}{4i^2} \sum_{j=i+1}^{(p-1)/2} \frac{1}{2j} + \frac{3}{4i^2} \sum_{j=1}^{2i-1} \frac{1}{j}. \tag{2.23}$$

Next, from equalities (2.20) and (2.22) we see that for any fixed $2i-1$ with $1 \leqslant i \leqslant (p-1)/2$, the subsum of the sum S_2 containing all the terms of the form $\pm 3/((2i-1)^2 \cdot k)$ such that $1 \leqslant k \leqslant p-1$ and $k \neq 2i-1$ is

$$S_{2i-1} = S_{1,2i-1} + S'_{1,2i-1} = \frac{3}{(2i-1)^2} \sum_{j=i}^{(p-1)/2} \frac{1}{2j}. \tag{2.24}$$

Note that (2.23) may be written as

$$\begin{aligned} S_{2i} &= \frac{3}{4i^2}\left(\left(\frac{1}{2i+2} + \frac{1}{2i+4} + \ldots + \frac{1}{p-1}\right) + \left(1 + \frac{1}{2} + \ldots + \frac{1}{2i-1}\right)\right) \\ &= \frac{3}{4i^2}\left(H_{p-1} - \frac{1}{2i} - \left(\frac{1}{2i+1} + \frac{1}{2i+3} + \ldots + \frac{1}{p-2}\right)\right). \end{aligned} \tag{2.25}$$

By *Wolstenholme's theorem* (see, e.g., [44] or [15]; for its generalizations see [32, Theorems 1 and 2]), if p is a prime greater than 3, then the numerator of the fraction $H_{p-1} = 1+1/2+1/3+\ldots+1/(p-1)$ is divisible by p^2. Substituting this into (2.25), we obtain

$$S_{2i} \equiv -\frac{3}{8i^3} - \frac{3}{4i^2}\left(\frac{1}{2i+1} + \frac{1}{2i+3} + \ldots + \frac{1}{p-2}\right) \pmod{p^2}. \tag{2.26}$$

Now (2.26) and (2.24) with $p-2i$ instead of $2i-1$, for each $1 \leqslant i \leqslant (p-1)/2$ give

$$\begin{aligned} S_{2i} + S_{p-2i} &\equiv -\frac{3}{8i^3} - \frac{3}{4i^2}\left(\frac{1}{2i+1} + \frac{1}{2i+3} + \ldots + \frac{1}{p-2}\right) \\ &\quad + \frac{3}{(p-2i)^2}\left(\frac{1}{p-2i+1} + \frac{1}{p-2i+3} + \ldots + \frac{1}{p-1}\right) \\ &\equiv -\frac{3}{8i^3} - \frac{3}{4i^2}\left(\frac{1}{2i+1} + \frac{1}{2i+3} + \ldots + \frac{1}{p-2}\right) \\ &\quad - \frac{3}{(2i)^2}\left(\frac{1}{2i-1} + \frac{1}{2i-3} + \ldots + \frac{1}{3} + 1\right) \pmod{p} \\ &= -\frac{3}{8i^3} - \frac{3}{4i^2}\left(1 + \frac{1}{3} + \ldots + \frac{1}{2i-1} + \frac{1}{2i+1} + \frac{1}{2i+3} + \ldots + \frac{1}{p-2}\right) \\ &= -\frac{3}{8i^3} - \frac{3}{4i^2}\left(H_{p-1} - \frac{1}{2} - \frac{1}{4} - \ldots - \frac{1}{p-1}\right) \pmod{p}. \end{aligned} \tag{2.27}$$

As $H_{p-1} \equiv 0 \pmod p$, (2.27) yields

$$S_{2i} + S_{p-2i} \equiv -\frac{3}{8i^3} + \frac{3}{8i^2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \pmod p. \tag{2.28}$$

From (2.28) we have
(2.29)

$$S_2 = \sum_{i=1}^{(p-1)/2} (S_{2i} + S_{p-2i}) \equiv -\frac{3}{8} \sum_{i=1}^{(p-1)/2} \frac{1}{i^3} + \frac{3}{8}\Bigl(\sum_{k=1}^{(p-1)/2} \frac{1}{k}\Bigr)\Bigl(\sum_{i=1}^{(p-1)/2} \frac{1}{i^2}\Bigr) \pmod p,$$

whence, by (2.11) and (2.12) of Lemma 2.5, we get

$$S_2 \equiv \frac{3}{4} B_{p-3} \pmod p. \tag{2.30}$$

It remains to determine the subsum S_3 modulo p. It is easy to see that the sum of all terms in S of the form $\pm 6/(ijk)$ such that $1 \leqslant i < j < k \leqslant p-1$ and k is odd, is equal to 0. Similarly, the sum of all terms in S of the form $\pm 6/(ijk)$ with $1 \leqslant i < j < k \leqslant p-1$ and $2 \mid k$ is equal to $6/(ijk)$. Consequently,

$$S_3 = 6 \sum_{\substack{1 \leqslant i < j < k \leqslant p-1 \\ 2 \mid k}} \frac{1}{ijk}, \tag{2.31}$$

whence, by (2.7) of Lemma 2.4, we obtain

$$S_3 \equiv -q_2(p)^3 - \frac{7}{8} B_{p-3} \pmod p. \tag{2.32}$$

Finally, the congruences (2.18), (2.30) and (2.32) immediately yield

$$\sum_{k=1}^{p-1} (-1)^k H_k^3 = S = S_1 + S_2 + S_3 \equiv -q_2(p)^3 - \frac{3}{8} B_{p-3} \pmod p.$$

This proves (2.16), and we are done. □

Lemma 2.7. *If $p > 3$ is a prime, then*

$$\binom{p-1}{k} \equiv (-1)^k - (-1)^k p H_k + (-1)^k \frac{p^2}{2} (H_k^2 - H_{k,2}) \pmod{p^3} \tag{2.33}$$

for each $k = 1, 2, \dots, p-1$.

P r o o f. For a fixed $1 \leqslant k \leqslant p-1$ we have

$$(-1)^k \binom{p-1}{k} = \prod_{i=1}^{k}\Big(1-\frac{p}{i}\Big) \equiv 1 - \sum_{i=1}^{k}\frac{p}{i} + \sum_{1\leqslant i<j\leqslant k}\frac{p^2}{ij} \pmod{p^3}$$

$$= 1 - pH_k + \frac{p^2}{2}\bigg(\bigg(\sum_{i=1}^{k}\frac{1}{i}\bigg)^2 - \sum_{i=1}^{k}\frac{1}{i^2}\bigg) = 1 - pH_k + \frac{p^2}{2}(H_k^2 - H_{k,2}) \pmod{p^3},$$

whence we obtain (2.33). □

3. Proof of Theorem 1.2 and corollaries

P r o o f of Theorem 1.2. By Lemma 2.1, for $n = p-1$ we have

$$\sum_{k=1}^{p-1}\binom{p-1}{k}H_k = 2^{p-1}H_{p-1} - 2^{p-1}\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k}. \tag{3.1}$$

In particular, the identity (3.1) yields

$$\sum_{k=1}^{p-1}\binom{p-1}{k}H_k \equiv 2^{p-1}H_{p-1} - 2^{p-1}\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k} \pmod{p^3}. \tag{3.2}$$

On the other hand, multiplying (2.33) of Lemma 2.7 by H_k, after summation over k we find

$$\begin{aligned}\sum_{k=1}^{p-1}\binom{p-1}{k}H_k \equiv{}& \sum_{k=1}^{p-1}(-1)^kH_k - p\sum_{k=1}^{p-1}(-1)^kH_k^2 + \frac{p^2}{2}\sum_{k=1}^{p-1}(-1)^kH_k^3\\ &-\frac{p^2}{2}\sum_{k=1}^{p-1}(-1)^kH_kH_{k,2} \pmod{p^3}.\end{aligned} \tag{3.3}$$

Substituting the congruences (2.13)–(2.17) of Lemma 2.6 into (3.3), we immediately obtain

$$\sum_{k=1}^{p-1}\binom{p-1}{k}H_k \equiv -q_2(p) - \frac{p}{2}q_2(p)^2 + \frac{p^2}{6}q_2(p)^3 - \frac{3p^2}{16}B_{p-3} \pmod{p^3}. \tag{3.4}$$

Comparing (3.4) and (3.2), and using the fact that by (2.13) and the Fermat Little Theorem, $2^{p-1}H_{p-1} \equiv -(p^2/3)B_{p-3} \pmod{p^3}$ holds, we obtain

$$2^{p-1}\sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k} \equiv q_p(2) + \frac{p}{2}q_p(2)^2 - \frac{p^2}{6}q_p(2)^3 - \frac{7p^2}{48}B_{p-3} \pmod{p^3}. \tag{3.5}$$

Substituting the identity $2^{p-1} = pq_p(2) + 1$ into (3.5) and reducing the modulus, we get

$$\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv -pq_p(2) \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} + q_p(2) + \frac{p}{2} q_p(2)^2 \pmod{p^2}. \tag{3.6}$$

Inserting the form (1.4) of Glaisher's congruence into (3.6), we obtain

$$\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv q_p(2) - \frac{p}{2} q_p(2)^2 \pmod{p^2}. \tag{3.7}$$

Again substituting $2^{p-1} = pq_p(2) + 1$ in (3.5), and inserting (3.7) into the result, we find that

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} &\equiv -pq_p(2) \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} + q_p(2) + \frac{p}{2} q_p(2)^2 - \frac{p^2}{6} q_p(2)^3 - \frac{7p^2}{48} B_{p-3} \\ &\equiv -pq_p(2)\Big(q_p(2) - \frac{p}{2} q_p(2)^2\Big) + q_p(2) + \frac{p}{2} q_p(2)^2 - \frac{p^2}{6} q_p(2)^3 - \frac{7p^2}{48} B_{p-3} \\ &= q_p(2) - \frac{p}{2} q_p(2)^2 + \frac{p^2}{3} q_p(2)^3 - \frac{7p^2}{48} B_{p-3} \pmod{p^3}.\end{aligned}$$

This is actually the congruence (1.7), and the proof is completed. $\square$

P r o o f of Corollary 1.1. Since $1/(p-k) \equiv -(p^2 + pk + k^2)/k^3 \pmod{p^3}$ for each $1 \leqslant k \leqslant p-1$, we have

$$\begin{aligned}-2^p \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} &= -2^p \sum_{k=1}^{p-1} \frac{1}{(p-k) \cdot 2^{p-k}} = -\sum_{k=1}^{p-1} \frac{2^k}{p-k} \\ &\equiv \sum_{k=1}^{p-1} \frac{(p^2 + pk + k^2) 2^k}{k^3} \pmod{p^3} \\ &= p^2 \sum_{k=1}^{p-1} \frac{2^k}{k^3} + p \sum_{k=1}^{p-1} \frac{2^k}{k^2} + \sum_{k=1}^{p-1} \frac{2^k}{k} \pmod{p^3},\end{aligned} \tag{3.8}$$

whence we have

$$p^2 \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -2^p \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} - p \sum_{k=1}^{p-1} \frac{2^k}{k^2} - \sum_{k=1}^{p-1} \frac{2^k}{k} \pmod{p^3}. \tag{3.9}$$

Multiplying by 2^p the congruence (1.7) of Theorem 1.2 and using $2^p \equiv 2 \pmod p$ and the identity $2^p = 2(pq_p(2)+1)$, we get
(3.10)

$$\begin{aligned}
-2^p \sum_{k=1}^{p-1} \frac{1}{k\cdot 2^k} &\equiv -2(pq_p(2)+1)\Big(q_p(2) - \frac{p}{2}q_p(2)^2\Big) - 2^p p^2\Big(\frac{1}{3}q_p(2)^3 - \frac{7}{48}B_{p-3}\Big)\\
&\equiv -2pq_p(2)^2 + p^2 q_p(2)^3 - 2q_p(2) + pq_p(2)^2 - \frac{2p^2}{3}q_p(2)^3 + \frac{7p^2}{24}B_{p-3}\\
&= -pq_p(2)^2 - 2q_p(2) + \frac{p^2}{3}q_p(2)^3 + \frac{7p^2}{24}B_{p-3} \pmod{p^3}.
\end{aligned}$$

Note that Sun's congruences (i) and (ii) in Theorem 1.1 give, respectively,

$$-\sum_{k=1}^{p-1}\frac{2^k}{k} \equiv 2q_p(2) + \frac{7p^2}{12}B_{p-3} \pmod{p^3}, \tag{3.11}$$

$$-p\sum_{k=1}^{p-1}\frac{2^k}{k^2} \equiv pq_p(2)^2 - \frac{2p^2}{3}q_p(2)^3 - \frac{7p^2}{6}B_{p-3} \pmod{p^3}. \tag{3.12}$$

Finally, inserting (3.10), (3.11) and (3.12) into (3.9), it simplifies to

$$p^2\sum_{k=1}^{p-1}\frac{2^k}{k^3} \equiv -\frac{p^2}{3}q_p(2)^3 - \frac{7p^2}{24}B_{p-3} \pmod{p^3},$$

which divided by $-p^2/3$ gives

$$q_p(2)^3 \equiv -3\sum_{k=1}^{p-1}\frac{2^k}{k^3} - \frac{7}{8}B_{p-3} \pmod p, \tag{3.13}$$

as desired. Finally, observe that the first congruence in (1.8) is immediate from (3.13) and the congruence (2.12) of Lemma 2.5. $\square$

Proof of Corollary 1.2. Multiplying by $p/2$ Sun's congruence (ii) in Theorem 1.1, we immediately obtain

$$-\frac{p}{2}q_p(2)^2 \equiv \frac{p}{2}\sum_{k=1}^{p-1}\frac{2^k}{k^2} - \frac{p^2}{3}q_p(2)^3 - \frac{7p^2}{12}B_{p-3} \pmod{p^3}. \tag{3.14}$$

Replacing the term $-(p/2)q_p(2)^2$ on the right hand side of (1.7) from Theorem 1.2 by the right hand side of (3.14), it becomes

$$q_p(2) \equiv \sum_{k=1}^{p-1}\frac{1}{k\cdot 2^k} - \frac{p}{2}\sum_{k=1}^{p-1}\frac{2^k}{k^2} + \frac{35p^2}{48}B_{p-3} \pmod{p^3},$$

which is just the congruence (1.9). $\square$

P r o o f of Corollary 1.3. We proceed in a way similar to that in the proof of Corollary 1.1. Since $1/(p-k)^2 \equiv (p+k)^2/k^4 \pmod{p^2}$ for each $1 \leqslant k \leqslant p-1$, we have

$$
\begin{aligned}
(3.15) \qquad 2^p \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} &= \sum_{k=1}^{p-1} \frac{2^{p-k}}{k^2} = \sum_{k=1}^{p-1} \frac{2^k}{(p-k)^2} \\
&\equiv \sum_{k=1}^{p-1} \frac{(p+k)^2 \cdot 2^k}{k^4} \equiv \sum_{k=1}^{p-1} \frac{(2pk+k^2) \cdot 2^k}{k^4} \pmod{p^2} \\
&= 2p \sum_{k=1}^{p-1} \frac{2^k}{k^3} + \sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p^2}.
\end{aligned}
$$

Taking $2^{p-1} = pq_p(2) + 1$ into (3.15), we obtain

$$
(3.16) \qquad \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} \equiv -pq_p(2) \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} + p \sum_{k=1}^{p-1} \frac{2^k}{k^3} + \frac{1}{2} \sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p^2}.
$$

By (1.8) of Corollary 1.1, we have

$$
(3.17) \qquad \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3} q_p(2)^3 - \frac{7}{24} B_{p-3} \pmod{p}.
$$

Finally, substituting the congruences (3.17), (ii) and (iv) of Theorem 1.1 into (3.16), we immediately obtain

$$
\sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} \equiv -\frac{1}{2} q_p(2)^2 + \frac{p}{2} q_p(2)^3 + \frac{7p}{24} B_{p-3} \pmod{p^2},
$$

as asserted. □

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