Czechoslovak Mathematical Journal

Fumi-Yuki Maeda; Yoshihiro Mizuta; Takao Ohno; Tetsu Shimomura

Approximate identities and Young type inequalities in Musielak-Orlicz spaces

Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 4, 933–948

Czechoslovak Mathematical Journal, 63 (138) (2013), 933–948

APPROXIMATE IDENTITIES AND YOUNG TYPE INEQUALITIES IN MUSIELAK-ORLICZ SPACES

Fumi-Yuki Maeda, Yoshihiro Mizuta, Hiroshima, Takao Ohno, Oita,
Tetsu Shimomura, Hiroshima

(Received June 25, 2012)

Abstract. We discuss the convergence of approximate identities in Musielak-Orlicz spaces extending the results given by Cruz-Uribe and Fiorenza (2007) and the authors F.-Y. Maeda, Y. Mizuta and T. Ohno (2010). As in these papers, we treat the case where the approximate identity is of potential type and the case where the approximate identity is defined by a function of compact support. We also give a Young type inequality for convolution with respect to the norm in Musielak-Orlicz spaces.

Keywords: approximate identity; Musielak-Orlicz space; Young type inequality

MSC 2010: 46E30

1. Introduction

Let κ be an integrable function on $\mathbb{R}^N$. For each $t > 0$, define the function κ_t by $\kappa_t(x) = t^{-N}\kappa(x/t)$. Note that by a change of variables, $\|\kappa_t\|_{L^1(\mathbb{R}^N)} = \|\kappa\|_{L^1(\mathbb{R}^N)}$. We say that the family $\{\kappa_t\}_{t>0}$ is an *approximate identity* if $\int_{\mathbb{R}^N} \kappa(x)\,\mathrm{d}x = 1$. Define the radial majorant of κ to be the function

$$\hat{\kappa}(x) = \sup_{|y|\geqslant|x|} |\kappa(y)|.$$

If $\hat{\kappa}$ is integrable, we say that the family $\{\kappa_t\}_{t>0}$ is *of potential-type*.

It is well known (see, e.g., [9]) that if $\{\kappa_t\}_{t>0}$ is a potential-type approximate identity, then $\kappa_t * f \to f$ in $L^p(\mathbb{R}^N)$ as $t \to 0$ for every $f \in L^p(\mathbb{R}^N)$ $(p \geqslant 1)$.

Variable exponent Lebesgue spaces and Sobolev spaces were introduced to discuss nonlinear partial differential equations with non-standard growth conditions (see [3]). Cruz-Uribe and Fiorenza [1] gave sufficient conditions for the convergence of approximate identities in variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^N)$ when $p(\cdot)$ is a variable exponent satisfying the log-Hölder conditions on $\mathbb{R}^N$, locally and at ∞, as an extension of [2], [9], etc. In fact they proved the following:

Theorem A. *Let $\{\kappa_t\}_{t>0}$ be an approximate identity. Suppose that either*

(1) *$\{\kappa_t\}_{t>0}$ is of potential-type, or*

(2) *$\kappa \in L^{(p^-)'}(\mathbb{R}^N)$ and has compact support, where $p^- := \inf_{x\in\mathbb{R}^N} p(x)$ $(\geqslant 1)$ and $1/p^- + 1/(p^-)' = 1$.*

Then

$$\sup_{0<t\leqslant 1} \|\kappa_t * f\|_{L^{p(\cdot)}(\mathbb{R}^N)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^N)}$$

and

$$\lim_{t\to 0} \|\kappa_t * f - f\|_{L^{p(\cdot)}(\mathbb{R}^N)} = 0$$

for all $f \in L^{p(\cdot)}(\mathbb{R}^N)$.

Recently, Theorem A was extended to the two variable exponents spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbb{R}^N)$ in [4]. These spaces are special cases of the so-called Musielak-Orlicz spaces ([8]).

Our aim in this paper is to extend these results to the Musielak-Orlicz spaces $L^{\Phi}(\mathbb{R}^N)$ (see Section 2 for the definition of Φ). As a related topic, we also give a Young type inequality for convolution with respect to the norm in $L^{\Phi}(\mathbb{R}^N)$.

2. Preliminaries

We consider a function

$$\Phi(x,t) = t\varphi(x,t)\colon\ \mathbb{R}^N \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1)–(Φ4):

(Φ1) $\varphi(\cdot,t)$ is measurable on $\mathbb{R}^N$ for each $t \geqslant 0$ and $\varphi(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in \mathbb{R}^N$;

(Φ2) there exists a constant $A_1 \geqslant 1$ such that

$$A_1^{-1} \leqslant \varphi(x,1) \leqslant A_1 \quad \text{for all } x \in \mathbb{R}^N;$$

(Φ3) $\varphi(x,\cdot)$ is uniformly almost increasing, namely there exists a constant $A_2 \geqslant 1$ such that

$$\varphi(x,t) \leqslant A_2\varphi(x,s) \quad \text{for all } x \in \mathbb{R}^N \text{ whenever } 0 \leqslant t < s;$$

(Φ4) there exists a constant $A_3 \geqslant 1$ such that

$$\varphi(x, 2t) \leqslant A_3 \varphi(x, t) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t > 0.$$

Note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x \in \mathbb{R}^N} \varphi(x, t) \leqslant \sup_{x \in \mathbb{R}^N} \varphi(x, t) < \infty$$

for each $t > 0$.

If $\Phi(x, \cdot)$ is convex for each $x \in \mathbb{R}^N$, then (Φ3) holds with $A_2 = 1$; namely $\varphi(x, \cdot)$ is non-decreasing for each $x \in \mathbb{R}^N$.

Example 2.1. Let $p_1(\cdot)$, $p_2(\cdot)$, $q_1(\cdot)$ and $q_2(\cdot)$ be measurable functions on $\mathbb{R}^N$ such that

$$\text{(P1)} \qquad 1 \leqslant p_j^- := \inf_{x \in \mathbb{R}^N} p_j(x) \leqslant \sup_{x \in \mathbb{R}^N} p_j(x) =: p_j^+ < \infty, \quad j = 1, 2,$$

and

$$\text{(Q1)} \qquad -\infty < q_j^- := \inf_{x \in \mathbb{R}^N} q_j(x) \leqslant \sup_{x \in \mathbb{R}^N} q_j(x) =: q_j^+ < \infty, \quad j = 1, 2.$$

Then

$$\Phi(x, t) = (1 + t)^{p_1(x)} (1 + 1/t)^{-p_2(x)} (\log(\mathrm{e} + t))^{q_1(x)} (\log(\mathrm{e} + 1/t))^{-q_2(x)}$$

satisfies (Φ1), (Φ2) and (Φ4). It satisfies (Φ3) if $p_j^- > 1$, $j = 1, 2$, or $q_j^- \geqslant 0$, $j = 1, 2$. As a matter of fact, it satisfies (Φ3) if and only if $p_j(\cdot), q_j(\cdot)$ satisfy the following conditions:

(1) $q_j(x) \geqslant 0$ at points x where $p_j(x) = 1$, $j = 1, 2$;

(2) $\sup\limits_{x:\, p_j(x) > 1} \{\min(q_j(x), 0) \log(p_j(x) - 1)\} < \infty$, $j = 1, 2$.

Let $\bar{\varphi}(x, t) = \sup\limits_{0 \leqslant s \leqslant t} \varphi(x, s)$ and

$$\overline{\Phi}(x, t) = \int_0^t \bar{\varphi}(x, r)\, \mathrm{d}r$$

for $x \in \mathbb{R}^N$ and $t \geqslant 0$. Then $\overline{\Phi}(x, \cdot)$ is convex and

$$\frac{1}{2A_3} \Phi(x, t) \leqslant \overline{\Phi}(x, t) \leqslant A_2 \Phi(x, t) \tag{2.1}$$

for all $x \in \mathbb{R}^N$ and $t \geqslant 0$. In fact, the first inequality is seen as follows:

$$\overline{\Phi}(x,t) \geqslant \int_{t/2}^{t} \bar{\varphi}(x,r)\,\mathrm{d}r \geqslant \frac{t}{2}\varphi(x,t/2) \geqslant \frac{1}{2A_3}\Phi(x,t).$$

Corresponding to (Φ2) and (Φ4), we have by (2.1)

$$(2.2) \qquad (2A_1A_3)^{-1} \leqslant \overline{\Phi}(x,1) \leqslant A_1A_2 \quad \text{and} \quad \overline{\Phi}(x,2t) \leqslant 2A_3\overline{\Phi}(x,t)$$

for all $x \in \mathbb{R}^N$ and $t > 0$.

Given $\Phi(x,t)$ as above, the associated Musielak-Orlicz space

$$L^{\Phi}(\mathbb{R}^N) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^N);\ \int_{\mathbb{R}^N} \Phi(y,|f(y)|)\,\mathrm{d}y < \infty \right\}$$

is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{R}^N)} = \inf\left\{ \lambda > 0;\ \int_{\mathbb{R}^N} \overline{\Phi}(y,|f(y)|/\lambda)\,\mathrm{d}y \leqslant 1 \right\}$$

(cf. [8]).

By (2.2), we have the following lemma (see [7]).

Lemma 2.2.

$$(2.3) \qquad \|f\|_{L^{\Phi}(\mathbb{R}^N)} \leqslant 2\left(\int_{\mathbb{R}^N} \overline{\Phi}(x,|f(x)|)\,\mathrm{d}x\right)^{\sigma}$$

with $\sigma = \log 2/\log(2A_3)$, *if* $\|f\|_{L^{\Phi}(\mathbb{R}^N)} \leqslant 1$.

We shall also consider the following conditions:

(Φ5) or every $\gamma > 0$, there exists a constant $B_\gamma \geqslant 1$ such that

$$\varphi(x,t) \leqslant B_\gamma \varphi(y,t)$$

whenever $|x-y| \leqslant \gamma t^{-1/N}$ and $t \geqslant 1$;

(Φ6) there exist a function $g \in L^1(\mathbb{R}^N)$ and a constant $B_\infty \geqslant 1$ such that $0 \leqslant g(x) < 1$ for all $x \in \mathbb{R}^N$ and

$$B_\infty^{-1}\Phi(x,t) \leqslant \Phi(x',t) \leqslant B_\infty\Phi(x,t)$$

whenever $|x'| \geqslant |x|$ and $g(x) \leqslant t \leqslant 1$.

If $\Phi(x,t)$ satisfies (Φ5) (resp. (Φ6)), then so does $\overline{\Phi}(x,t)$ with $\overline{B}_\gamma = 2A_2A_3B_\gamma$ in place of B_γ (resp. $\overline{B}_\infty = 2A_2A_3B_\infty$ in place of B_∞).

Example 2.3. Let $\Phi(x,t)$ be as in Example 2.1. It satisfies ($\Phi 5$) if

(P2) $p_1(\cdot)$ is log-Hölder continuous, namely

$$|p_1(x)-p_1(y)| \leqslant \frac{C_p}{\log(1/|x-y|)} \quad \text{for } |x-y| \leqslant \frac{1}{2}$$

with a constant $C_p \geqslant 0$,

and

(Q2) $q_1(\cdot)$ is log-log-Hölder continuous, namely

$$|q_1(x)-q_1(y)| \leqslant \frac{C_q}{\log(\log(1/|x-y|))} \quad \text{for } |x-y| \leqslant \mathrm{e}^{-2}$$

with a constant $C_q \geqslant 0$.

$\Phi(x,t)$ satisfies ($\Phi 6$) with $g(x) = 1/(1+|x|)^{N+1}$ if

(P3) $p_2(\cdot)$ is log-Hölder continuous at ∞, namely

$$|p_2(x)-p_2(x')| \leqslant \frac{C_{p,\infty}}{\log(\mathrm{e}+|x|)} \quad \text{whenever } |x'| \geqslant |x|$$

with a constant $C_{p,\infty} \geqslant 0$,

and

(Q3) $q_2(\cdot)$ is log-log-Hölder continuous at ∞, namely

$$|q_2(x)-q_2(x')| \leqslant \frac{C_{q,\infty}}{\log(\mathrm{e}+\log(\mathrm{e}+|x|))} \quad \text{whenever } |x'| \geqslant |x|$$

with a constant $C_{q,\infty} \geqslant 0$.

In fact, if $1/(1+|x|)^{N+1} < t \leqslant 1$, then $(1+t)^{|p_1(x)-p_1(x')|} \leqslant 2^{p_1^+-1}$, $(1+1/t)^{|p_2(x)-p_2(x')|} \leqslant \mathrm{e}^{(N+1)C_{p,\infty}}$, $(\log(\mathrm{e}+t))^{|q_1(x)-q_1(x')|} \leqslant (\log(\mathrm{e}+1))^{q_1^+-q_1^-}$ and $(\log(\mathrm{e}+1/t))^{|q_2(x)-q_2(x')|} \leqslant C(N, C_{q,\infty})$ for $|x'| \geqslant |x|$.

3. The case of potential type

Throughout this paper, let C denote various positive constants independent of the variables in question.

First, we recall the following classical result (see, e.g., Stein [9]).

Lemma 3.1. *Let* $1 \leqslant p < \infty$ *and* $\{\kappa_t\}_{t>0}$ *be a potential-type approximate identity. Then,* $\kappa_t * f$ *converges to* f *in* $L^p(\mathbb{R}^N)$ *for every* $f \in L^p(\mathbb{R}^N)$.

We denote by $B(x,r)$ the open ball centered at $x \in \mathbb{R}^N$ and with radius $r > 0$. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

For a nonnegative $f \in L^1_{\rm loc}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$ and $r > 0$, let

$$I(f;x,r) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,\mathrm{d}y$$

and

$$J(f;x,r) = \frac{1}{|B(x,r)|}\int_{B(x,r)} \overline{\Phi}(y,f(y))\,\mathrm{d}y$$

in this section.

The following lemmas are due to [5], [6].

Lemma 3.2 ([5, Lemma 2.1], [6, Lemma 3.1])**.** *Suppose* $\Phi(x,t)$ *satisfies* (Φ5)*. Then there exists a constant* $C > 0$ *such that*

$$\overline{\Phi}(x, I(f;x,r)) \leqslant CJ(f;x,r)$$

for all $x \in \mathbb{R}^N$, $r > 0$ *and for all nonnegative* $f \in L^1_{\rm loc}(\mathbb{R}^N)$ *such that* $f(y) \geqslant 1$ *or* $f(y) = 0$ *for each* $y \in \mathbb{R}^N$ *and* $\|f\|_{L^\Phi(\mathbb{R}^N)} \leqslant 1$.

Lemma 3.3 ([5, Lemma 2.2], [6, Lemma 3.2])**.** *Suppose* $\Phi(x,t)$ *satisfies* (Φ6)*. Then there exists a constant* $C > 0$ *such that*

$$\overline{\Phi}(x, I(f;x,r)) \leqslant C\,\{J(f;x,r) + g(x)\}$$

for all $x \in \mathbb{R}^N$, $r > 0$ *and for all nonnegative* $f \in L^1_{\rm loc}(\mathbb{R}^N)$ *such that* $g(y) \leqslant f(y) \leqslant 1$ *or* $f(y) = 0$ *for each* $y \in \mathbb{R}^N$*, where* g *is the function appearing in* (Φ6)*.*

By using Lemmas 3.2 and 3.3, we show the following theorem.

Theorem 3.4. *Suppose* $\Phi(x,t)$ *satisfies* (Φ5) *and* (Φ6)*. If* $\{\kappa_t\}_{t>0}$ *is of potential-type, then*

$$\|\kappa_t * f\|_{L^\Phi(\mathbb{R}^N)} \leqslant C\|\hat{\kappa}\|_{L^1(\mathbb{R}^N)}\|f\|_{L^\Phi(\mathbb{R}^N)}$$

for all $t > 0$ *and* $f \in L^\Phi(\mathbb{R}^N)$.

Proof. Suppose $\|\hat{\kappa}\|_{L^1(\mathbb{R}^N)} = 1$ and let f be a nonnegative measurable function on $\mathbb{R}^N$ such that $\|f\|_{L^\Phi(\mathbb{R}^N)} \leqslant 1$. Write

$$f = f\chi_{\{y\in\mathbb{R}^N\colon\, f(y)\geqslant 1\}} + f\chi_{\{y\in\mathbb{R}^N\colon\, g(y)<f(y)<1\}} + f\chi_{\{y\in\mathbb{R}^N\colon\, f(y)\leqslant g(y)\}} = f_1 + f_2 + f_3,$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbb{R}^N$ and g is the function appearing in (Φ6).

Since $\hat{\kappa}_t$ is a radial function, we write $\hat{\kappa}_t(r)$ for $\hat{\kappa}_t(x)$ when $|x| = r$. First note that

$$|\kappa_t * f_j(x)| \leqslant \int_{\mathbb{R}^N} \hat{\kappa}_t(|x-y|) f_j(y)\,\mathrm{d}y = \int_0^\infty I(f_j; x, r)|B(x,r)|\,\mathrm{d}(-\hat{\kappa}_t(r)),$$

$j = 1, 2$, and

$$\int_{\mathbb{R}^N} |B(x,r)|\,\mathrm{d}(-\hat{\kappa}_t(r)) = \|\hat{\kappa}_t\|_{L^1(\mathbb{R}^N)} = 1,$$

so that Jensen's inequality yields

$$\overline{\Phi}(x, |\kappa_t * f_j(x)|) \leqslant \int_0^\infty \overline{\Phi}\left(x, I(f_j; x, r)\right) |B(x,r)|\,\mathrm{d}(-\hat{\kappa}_t(r)),$$

$j = 1, 2$.

Hence, by Lemma 3.2

$$\overline{\Phi}(x, |\kappa_t * f_1(x)|) \leqslant C \int_0^\infty J(f_1; x, r)|B(x,r)|\,\mathrm{d}(-\hat{\kappa}_t(r)) \leqslant C(\hat{\kappa}_t * h)(x),$$

where $h(y) = \overline{\Phi}(y, f(y))$. The usual Young inequality for convolution gives

$$\begin{aligned}\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_t * f_1(x)|)\,\mathrm{d}x &\leqslant C \int_{\mathbb{R}^N} (\hat{\kappa}_t * h)(x)\,\mathrm{d}x \\ &\leqslant C\|\hat{\kappa}_t\|_{L^1(\mathbb{R}^N)}\|h\|_{L^1(\mathbb{R}^N)} \leqslant C.\end{aligned}$$

Similarly, noting that $g \in L^1(\mathbb{R}^N)$ and applying Lemma 3.3, we derive the same result for f_2.

Finally, noting that $|\kappa_t * f_3(x)| \leqslant \|\kappa_t\|_{L^1(\mathbb{R}^N)} \leqslant 1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_t * f_3(x)|)\,\mathrm{d}x &\leqslant C \int_{\mathbb{R}^N} |\kappa_t * f_3(x)|\,\mathrm{d}x \\ &\leqslant C\|\kappa_t\|_{L^1(\mathbb{R}^N)}\|g\|_{L^1(\mathbb{R}^N)} \leqslant C.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_t * f(x)|)\,\mathrm{d}x \leqslant C,$$

which implies the required assertion. □

Theorem 3.5. *Suppose $\Phi(x,t)$ satisfies (Φ5) and (Φ6). Let $\{\kappa_t\}_{t>0}$ be a potential-type approximate identity. Then $\kappa_t * f$ converges to f in $L^{\Phi}(\mathbb{R}^N)$:*

$$\lim_{t\to 0} \|\kappa_t * f - f\|_{L^{\Phi}(\mathbb{R}^N)} = 0$$

for every $f \in L^{\Phi}(\mathbb{R}^N)$.

Proof. Given $\varepsilon > 0$, we find a bounded function h in $L^{\Phi}(\mathbb{R}^N)$ with compact support such that $\|f-h\|_{L^{\Phi}(\mathbb{R}^N)} < \varepsilon$. By Theorem 3.4 we have

$$\begin{aligned}\|\kappa_t * f - f\|_{L^{\Phi}(\mathbb{R}^N)} &\leqslant \|\kappa_t * (f-h)\|_{L^{\Phi}(\mathbb{R}^N)} + \|\kappa_t * h - h\|_{L^{\Phi}(\mathbb{R}^N)} + \|f-h\|_{L^{\Phi}(\mathbb{R}^N)} \\ &\leqslant (C\|\hat{\kappa}\|_{L^1(\mathbb{R}^N)} + 1)\varepsilon + \|\kappa_t * h - h\|_{L^{\Phi}(\mathbb{R}^N)}.\end{aligned}$$

Since $|\kappa_t * h| \leqslant \|h\|_{L^{\infty}(\mathbb{R}^N)}$, we have

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_t * h(x) - h(x)| \,\mathrm{d}x \leqslant C' \int_{\mathbb{R}^N} |\kappa_t * h(x) - h(x)| \,\mathrm{d}x \to 0$$

as $t \to 0$ by Lemma 3.1. (Here C' depends on $\|h\|_{L^{\infty}(\mathbb{R}^N)}$.) Hence $\|\kappa_t * h - h\|_{L^{\Phi}(\mathbb{R}^N)} \to 0$ as $t \to 0$ by Lemma 2.2, so that

$$\limsup_{t\to 0} \|\kappa_t * f - f\|_{L^{\Phi}(\mathbb{R}^N)} \leqslant (C\|\hat{\kappa}\|_{L^1(\mathbb{R}^N)} + 1)\varepsilon,$$

which completes the proof. □

4. The case of compact support

We know the following result due to Zo [10]; see also [1, Theorem 2.2].

Lemma 4.1. *Let $1 \leqslant p < \infty$, $1/p + 1/p' = 1$ and $\{\kappa_t\}_{t>0}$ be an approximate identity. Suppose that $\kappa \in L^{p'}(\mathbb{R}^N)$ and has compact support. Then for every $f \in L^p(\mathbb{R}^N)$, $\kappa_t * f$ converges to f pointwise almost everywhere as $t \to 0$.*

In this section, we take $p_0 \geqslant 1$ as follows. Let P be the set of all $p \geqslant 1$ such that $t \mapsto t^{-p}\Phi(x,t)$ is uniformly almost increasing, and set $\tilde{p} = \sup P$. Note that $1 \in P$ by (Φ3), so that $\tilde{p} > 1$ if $\tilde{p} \notin P$. Let $p_0 = \tilde{p}$ if $\tilde{p} \in P$ and $1 < p_0 < \tilde{p}$ otherwise.

Example 4.2. For $\Phi(x,t)$ in Example 2.3, $\tilde{p} = \min\{p_1^-, p_2^-\}$, so that $p_0 = 1$ if $p_1^- = 1$ or $p_2^- = 1$; and $1 < p_0 \leqslant \min\{p_1^-, p_2^-\}$ if $p_j^- > 1$, $j = 1, 2$ (cf. [4]).

Since $t^{-p_0}\Phi(x,t)$ is uniformly almost increasing in t, there exists a constant $A_2' \geqslant 1$ such that

$$t^{-p_0}\Phi(x,t) \leqslant A_2' s^{-p_0}\Phi(x,s) \quad \text{for all } x \in \mathbb{R}^N \text{ whenever } 0 \leqslant t < s.$$

Set

$$\Phi_0(x,t) = \Phi(x,t)^{1/p_0}.$$

Then $\Phi_0(x,t)$ also satisfies all the conditions (Φj), $j = 1, 2, \ldots, 6$. In fact, it trivially satisfies (Φj) for $j = 1, 2, 4, 5, 6$ with the same g for ($\Phi 6$). Since

$$\Phi_0(x,t) = t\varphi_0(x,t) \quad \text{with} \quad \varphi_0(x,t) = [t^{-p_0}\Phi(x,t)]^{1/p_0},$$

$\Phi_0(x,t)$ satisfies ($\Phi 3$) with A_2 replaced by $A_4 = (A_2')^{1/p_0}$.

Lemma 4.3. *Suppose $\Phi(x,t)$ satisfies ($\Phi 5$). Let κ have compact support contained in $B(0,R)$ and let $\|\kappa\|_{L^{(p_0)'}(\mathbb{R}^N)} \leqslant 1$. Then there exists a constant $C > 0$, which depends on R, such that*

$$\Phi_0(x, |\kappa_t * f(x)|) \leqslant C \int_{\mathbb{R}^N} |\kappa_t(x-y)| \Phi_0(y, f(y))\,\mathrm{d}y$$

for all $x \in \mathbb{R}^N$, $0 < t \leqslant 1$ and for all nonnegative $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $f(y) \geqslant 1$ or $f(y) = 0$ for each $y \in \mathbb{R}^N$ and $\|f\|_{L^\Phi(\mathbb{R}^N)} \leqslant 1$.

P r o o f. Given f as in the statement of the lemma, $x \in \mathbb{R}^N$ and $0 < t \leqslant 1$, set

$$F = |\kappa_t * f(x)| \quad \text{and} \quad G = \int_{\mathbb{R}^N} |\kappa_t(x-y)| \Phi_0(y, f(y))\,\mathrm{d}y.$$

Note that $\|f\|_{L^\Phi(\mathbb{R}^N)} \leqslant 1$ implies

$$G \leqslant \|\kappa_t\|_{L^{(p_0)'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} \Phi(y, f(y))\,\mathrm{d}y \right)^{1/p_0} \leqslant t^{-N/p_0}(2A_3)^{1/p_0} \leqslant (2A_3)^{1/p_0} t^{-N}$$

by Hölder's inequality and (2.1).

By ($\Phi 2$), $\Phi_0(y, f(y)) \geqslant (A_1 A_4)^{-1} f(y)$, since $f(y) \geqslant 1$ or $f(y) = 0$. Hence $F \leqslant A_1 A_4 G$. Thus, if $G \leqslant 1$, then

$$\Phi_0(x,F) \leqslant (A_1 A_4 G) A_4 (A_1 A_4)^{(1-p_0)/p_0} \varphi(x, A_1 A_4)^{1/p_0} \leqslant CG.$$

Next, let $G > 1$. Since $\Phi_0(x,t) \to \infty$ as $t \to \infty$, there exists $K \geqslant 1$ such that

$$\Phi_0(x,K) = \Phi_0(x,1)G.$$

Then $K \leqslant A_4 G$, since $\Phi_0(x,K) \geqslant A_4^{-1} K \Phi_0(x,1)$. With this K, we have

$$F \leqslant K \int_{\mathbb{R}^N} |\kappa_t(x-y)| \,\mathrm{d}y + A_4 \int_{\mathbb{R}^N} |\kappa_t(x-y)| f(y) \frac{\varphi_0(y,f(y))}{\varphi_0(y,K)} \,\mathrm{d}y.$$

Since

$$1 \leqslant K \leqslant A_4 G \leqslant A_4 (2A_3)^{1/p_0} t^{-N} \leqslant C(tR)^{-N},$$

there is $\beta > 0$, independent of f, x, t, such that

$$\varphi_0(x,K) \leqslant \beta \varphi_0(y,K) \quad \text{for all } y \in B(x,tR)$$

by $(\Phi 5)$. Thus, we have

$$\begin{aligned} F &\leqslant K \|\kappa_t\|_{L^1(\mathbb{R}^N)} + \frac{A_4 \beta}{\varphi_0(x,K)} \int_{\mathbb{R}^N} |\kappa_t(x-y)| f(y) \varphi_0(y,f(y)) \,\mathrm{d}y \\ &= K \|\kappa\|_{L^1(\mathbb{R}^N)} + A_4 \beta \frac{G}{\varphi_0(x,K)} \\ &= K \Big(\|\kappa\|_{L^1(\mathbb{R}^N)} + \frac{A_4 \beta}{\varphi_0(x,1)} \Big) \\ &\leqslant K \big(\|\kappa\|_{L^1(\mathbb{R}^N)} + A_1^{1/p_0} A_4 \beta \big) \leqslant CK. \end{aligned}$$

Therefore by $(\Phi 3)$, $(\Phi 4)$, the choice of K and $(\Phi 2)$,

$$\Phi_0(x,F) \leqslant C \Phi_0(x,K) \leqslant CG$$

with constants $C > 0$ independent of f, x, t, as required. $\square$

Lemma 4.4. *Suppose $\Phi(x,t)$ satisfies $(\Phi 6)$. Let $M \geqslant 1$ and assume that $\|\kappa\|_{L^1(\mathbb{R}^N)} \leqslant M$. Then there exists a constant $C > 0$, depending on M, such that*

$$\overline{\Phi}(x, |\kappa_t * f(x)|) \leqslant C \Big\{ \int_{\mathbb{R}^N} |\kappa_t(x-y)| \overline{\Phi}(y, f(y)) \,\mathrm{d}y + g(x) \Big\}$$

for all $x \in \mathbb{R}^N$, $t > 0$ and for all nonnegative $f \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $g(y) \leqslant f(y) \leqslant 1$ or $f(y) = 0$ for each $y \in \mathbb{R}^N$, where g is the function appearing in $(\Phi 6)$.

Proof. Let f be as in the statement of the lemma, $x \in \mathbb{R}^N$ and $t > 0$. By $(\Phi 4)$, there is a constant $c_M \geqslant 1$ such that $\overline{\Phi}(x, Mt) \leqslant c_M \overline{\Phi}(x,t)$ for all $x \in \mathbb{R}^N$ and $t > 0$. By Jensen's inequality, we have

$$\begin{aligned} \overline{\Phi}(x, |\kappa_t * f(x)|) &\leqslant c_M \overline{\Phi}\Big(x, \int_{\mathbb{R}^N} (|\kappa_t(x-y)|/M) f(y) \,\mathrm{d}y \Big) \\ &\leqslant (c_M/M) \int_{\mathbb{R}^N} |\kappa_t(x-y)| \overline{\Phi}(x, f(y)) \,\mathrm{d}y. \end{aligned}$$

If $|x| \geqslant |y|$, then $\overline{\Phi}(x, f(y)) \leqslant \overline{B}_\infty \overline{\Phi}(y, f(y))$ by (Φ6).

If $|x| < |y|$ and $g(x) < f(y)$, then $\overline{\Phi}(x, f(y)) \leqslant \overline{B}_\infty \overline{\Phi}(y, f(y))$ by (Φ6) again.

If $|x| < |y|$ and $g(x) \geqslant f(y)$, then

$$\overline{\Phi}(x, f(y)) \leqslant \overline{\Phi}(x, g(x)) \leqslant g(x)\overline{\Phi}(x, 1) \leqslant A_1 A_2 g(x)$$

by (2.2).

Hence,

$$\overline{\Phi}(x, f(y)) \leqslant C\{\overline{\Phi}(y, f(y)) + g(x)\}$$

in any case. Therefore, we obtain the required inequality. $\square$

Theorem 4.5. *Suppose $\Phi(x,t)$ satisfies (Φ5) and (Φ6). Suppose that $\kappa \in L^{(p_0)'}(\mathbb{R}^N)$ and has compact support in $B(0,R)$. Then*

$$\|\kappa_t * f\|_{L^\Phi(\mathbb{R}^N)} \leqslant C\|\kappa\|_{L^{(p_0)'}(\mathbb{R}^N)}\|f\|_{L^\Phi(\mathbb{R}^N)}$$

for all $0 < t \leqslant 1$ and $f \in L^\Phi(\mathbb{R}^N)$, where $C > 0$ depends on R.

Proof. Let f be a nonnegative measurable function on $\mathbb{R}^N$ such that $\|f\|_{L^\Phi(\mathbb{R}^N)} \leqslant 1$ and assume that $\|\kappa\|_{L^{(p_0)'}(\mathbb{R}^N)} = 1$. Note that $\|\kappa\|_{L^1(\mathbb{R}^N)} \leqslant |B(0,R)|^{1/p_0}$ by Hölder's inequality.

Write

$$f = f\chi_{\{y\in\mathbb{R}^N :\, f(y)\geqslant 1\}} + f\chi_{\{y\in\mathbb{R}^N :\, g(y)<f(y)<1\}} + f\chi_{\{y\in\mathbb{R}^N :\, f(y)\leqslant g(y)\}} = f_1 + f_2 + f_3,$$

where g is the function appearing in (Φ6). We have by (2.1) and Lemma 4.3,

$$\overline{\Phi}(x, |\kappa_t * f_1(x)|) \leqslant A_2\Phi_0(x, |\kappa_t * f_1(x)|)^{p_0} \leqslant C(|\kappa_t| * h(x))^{p_0},$$

where $h(y) = \Phi(y, f(y))^{1/p_0}$. Since $\|h\|^{p_0}_{L^{p_0}(\mathbb{R}^N)} \leqslant 2A_3$, the usual Young's inequality for convolution gives

$$\begin{aligned}\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_t * f_1(x)|)\,\mathrm{d}x &\leqslant C\int_{\mathbb{R}^N} (|\kappa_t| * h(x))^{p_0}\,\mathrm{d}x \\ &\leqslant C\left(\|\kappa_t\|_{L^1(\mathbb{R}^N)}\|h\|_{L^{p_0}(\mathbb{R}^N)}\right)^{p_0} \leqslant C.\end{aligned}$$

Similarly, applying Lemma 4.4 with $M = |B(0,R)|^{1/p_0}$ and noting that $g \in L^1(\mathbb{R}^N)$, we derive the same result for f_2.

Finally, since $|\kappa_t * f_3(x)| \leqslant \|\kappa_t\|_{L^1(\mathbb{R}^N)} \leqslant M$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_t * f_3(x)|)\,\mathrm{d}x &\leqslant C\int_{\mathbb{R}^N} |\kappa_t * f_3(x)|\,\mathrm{d}x \\ &\leqslant C\|\kappa_t\|_{L^1(\mathbb{R}^N)}\|g\|_{L^1(\mathbb{R}^N)} \leqslant C.\end{aligned}$$

Thus, we have shown that

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_t * f(x)|)\,\mathrm{d}x \leqslant C,$$

which implies the required result. $\square$

Theorem 4.6. *Suppose $\Phi(x,t)$ satisfies ($\Phi5$) and ($\Phi6$). Let $\{\kappa_t\}_{t>0}$ be an approximate identity such that $\kappa \in L^{(p_0)'}(\mathbb{R}^N)$ and has compact support. Then $\kappa_t * f$ converges to f in $L^\Phi(\mathbb{R}^N)$:*

$$\lim_{t\to 0} \|\kappa_t * f - f\|_{L^\Phi(\mathbb{R}^N)} = 0$$

for every $f \in L^\Phi(\mathbb{R}^N)$.

Proof. Let $f \in L^\Phi(\mathbb{R}^N)$. Given $\varepsilon > 0$, choose a bounded function h with compact support such that $\|f - h\|_{L^\Phi(\mathbb{R}^N)} < \varepsilon$. As in the proof of Theorem 3.5, using Theorem 4.5 this time, we have

$$\|\kappa_t * f - f\|_{L^\Phi(\mathbb{R}^N)} \leqslant (C\|\kappa\|_{L^{(p_0)'}(\mathbb{R}^N)} + 1)\varepsilon + \|\kappa_t * h - h\|_{L^\Phi(\mathbb{R}^N)}.$$

Obviously, $h \in L^{p_0}(\mathbb{R}^N)$. Hence by Lemma 4.1, $\kappa_t * h \to h$ almost everywhere in $\mathbb{R}^N$, and hence

$$\overline{\Phi}(x, |\kappa_t * h(x) - h(x)|) \to 0$$

almost everywhere in $\mathbb{R}^N$. Since $\{\kappa_t * h - h\}$ is uniformly bounded and there is a compact set S containing all the supports of $\kappa_t * h$, $\{\overline{\Phi}(x, |\kappa_t * h(x) - h(x)|)\}$ is uniformly bounded and S contains all the supports of $\overline{\Phi}(x, |\kappa_t * h(x) - h(x)|)$. Hence the Lebesgue convergence theorem implies

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_t * h(x) - h(x)|)\,\mathrm{d}x \to 0$$

as $t \to 0$. Then, by Lemma 2.2, we see that $\|\kappa_t * h - h\|_{L^\Phi(\mathbb{R}^N)} \to 0$ as $t \to 0$, so that

$$\limsup_{t\to 0} \|\kappa_t * f - f\|_{L^\Phi(\mathbb{R}^N)} \leqslant (C\|\kappa\|_{L^{(p_0)'}(\mathbb{R}^N)} + 1)\varepsilon,$$

which completes the proof. $\square$

5. Young type inequality

Lemma 5.1. *Suppose $\Phi(x,t)$ satisfies (Φ6). Let $\kappa \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with $\|\kappa\|_{L^1(\mathbb{R}^N)} \leqslant 1$. For $f \in L^1_{\rm loc}(\mathbb{R}^N)$, set*

$$I(f;x) = \int_{\mathbb{R}^N \setminus B(0,|x|/2)} |\kappa(x-y) f(y)| \, \mathrm{d}y$$

and

$$J(f;x) = \int_{\mathbb{R}^N} |\kappa(x-y)| \overline{\Phi}(y, |f(y)|) \, \mathrm{d}y.$$

Then there exists a constant $C > 0$ (depending on $\|\kappa\|_{L^\infty(\mathbb{R}^N)}$) such that

$$\overline{\Phi}(x, I(f;x)) \leqslant C \{J(f;x) + g(x/2)\}$$

for all $x \in \mathbb{R}^N$ and $f \in L^\Phi(\mathbb{R}^N)$ with $\|f\|_{L^\Phi(\mathbb{R}^N)} \leqslant 1$, where g is the function appearing in (Φ6).

Proof. Let $k > 0$. Since $t \mapsto \overline{\Phi}(x,t)/t$ is nondecreasing,

$$I(f;x) \leqslant k \int_{\mathbb{R}^N} |\kappa(x-y)| \, \mathrm{d}y + k \int_{\mathbb{R}^N \setminus B(0,|x|/2)} \frac{|\kappa(x-y)| \overline{\Phi}(y,|f(y)|)}{\overline{\Phi}(y,k)} \, \mathrm{d}y.$$

If $g(x/2) \leqslant k \leqslant 1$, then $\overline{\Phi}(x,k) \leqslant C \overline{\Phi}(y,k)$ for $|y| > |x|/2$ by (Φ6). Hence

$$I(f;x) \leqslant k\Big(1 + \frac{CJ(f;x)}{\overline{\Phi}(x,k)}\Big) \quad \text{whenever } g(x/2) \leqslant k \leqslant 1. \tag{5.1}$$

Since $J(f;x) \leqslant \|\kappa\|_{L^\infty(\mathbb{R}^N)}$, there exists $K_x \in [0,1]$ such that

$$\overline{\Phi}(x, K_x) = \frac{J(f;x)}{\|\kappa\|_{L^\infty(\mathbb{R}^N)}} \overline{\Phi}(x,1).$$

If $K_x \geqslant g(x/2)$, then taking $k = K_x$ in (5.1), we have

$$I(f;x) \leqslant K_x \Big(1 + \frac{C\|\kappa\|_{L^\infty(\mathbb{R}^N)}}{\overline{\Phi}(x,1)}\Big) \leqslant CK_x,$$

so that

$$\overline{\Phi}(x, I(f;x)) \leqslant C\overline{\Phi}(x, K_x) \leqslant CJ(f;x).$$

If $K_x < g(x/2)$, then

$$J(f;x) = \|\kappa\|_{L^\infty(\mathbb{R}^N)} \frac{\overline{\Phi}(x,K_x)}{\overline{\Phi}(x,1)} \leqslant C\overline{\Phi}(x, g(x/2)).$$

Hence, taking $k = g(x/2)$ in (5.1), we have $I(f;x) \leqslant Cg(x/2)$, so that

$$\overline{\Phi}(x, I(f;x)) \leqslant C\overline{\Phi}(x, g(x/2)) \leqslant Cg(x/2).$$

Hence, we have the assertion of the lemma. $\square$

Here, we recall the following result on the boundedness of the maximal operator M on $L^{\Phi}(\mathbb{R}^N)$ (see [6, Corollary 4.4]):

Lemma 5.2. *Suppose $\Phi(x,t)$ satisfies ($\Phi5$), ($\Phi6$) and the other condition*

($\Phi3^*$) *$t \mapsto t^{-\varepsilon_0}\varphi(x,t)$ is uniformly almost increasing on $(0,\infty)$ for some $\varepsilon_0 > 0$.*

Then the maximal operator M is bounded from $L^{\Phi}(\mathbb{R}^N)$ into itself, namely

$$\|Mf\|_{L^{\Phi}(\mathbb{R}^N)} \leqslant C\|f\|_{L^{\Phi}(\mathbb{R}^N)}$$

for all $f \in L^{\Phi}(\mathbb{R}^N)$.

Theorem 5.3. *Suppose $\Phi(x,t)$ satisfies ($\Phi5$), ($\Phi6$) and ($\Phi3^*$). Let $p_0 = 1+\varepsilon_0$ (>1) and $R>0$. Assume that $\kappa \in L^1(\mathbb{R}^N) \cap L^{(p_0)'}(B(0,R))$ and $|\kappa(x)| \leqslant c_\kappa |x|^{-N}$ for $|x| \geqslant R$. Then there is a constant $C>0$ such that*

$$\|\kappa * f\|_{L^{\Phi}(\mathbb{R}^N)} \leqslant C(\|\kappa\|_{L^1(\mathbb{R}^N)} + \|\kappa\|_{L^{(p_0)'}(B(0,R))})\|f\|_{L^{\Phi}(\mathbb{R}^N)}$$

for all $f \in L^{\Phi}(\mathbb{R}^N)$.

Proof. Let $f \in L^{\Phi}(\mathbb{R}^N)$ and $f \geqslant 0$. Assume that $\|f\|_{L^{\Phi}(\mathbb{R}^N)} \leqslant 1$ and

$$\|\kappa\|_{L^1(\mathbb{R}^N)} + \|\kappa\|_{L^{(p_0)'}(B(0,R))} \leqslant 1.$$

Let $\kappa_0 = \kappa\chi_{B(0,R)}$ and $\kappa_\infty = \kappa\chi_{\mathbb{R}^N\setminus B(0,R)}$.

By Theorem 4.5,

$$\|\kappa_0 * f\|_{L^{\Phi}(\mathbb{R}^N)} \leqslant C.$$

Hence it is enough to show that

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, |\kappa_\infty| * f(x))\,\mathrm{d}x \leqslant C. \tag{5.2}$$

Write

$$\begin{aligned} |\kappa_\infty| * f(x) &= \int_{B(0,|x|/2)} |\kappa_\infty(x-y)| f(y)\,\mathrm{d}y + \int_{\mathbb{R}^N\setminus B(0,|x|/2)} |\kappa_\infty(x-y)| f(y)\,\mathrm{d}y \\ &= I_1(x) + I_2(x). \end{aligned}$$

Since $|\kappa_\infty(x-y)| \leqslant c_\kappa|x-y|^{-N}$ and $|x-y| \geqslant |x|/2$ for $|y| \leqslant |x|/2$,

$$I_1(x) \leqslant 2^N c_\kappa |x|^{-N} \int_{B(0,|x|/2)} f(y)\,\mathrm{d}y \leqslant 2^N c_\kappa |x|^{-N} \int_{B(x,3|x|/2)} f(y)\,\mathrm{d}y \leqslant CMf(x).$$

Hence,

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, I_1(x))\,\mathrm{d}x \leqslant C$$

by Lemma 5.2.

On the other hand, by Lemma 5.1,

$$\overline{\Phi}(x, I_2(x)) \leqslant C\{|\kappa_\infty| * h(x) + g(x/2)\},$$

where $h(y) = \overline{\Phi}(y, f(y))$. Since

$$\||\kappa_\infty| * h\|_{L^1(\mathbb{R}^N)} \leqslant \||\kappa_\infty|\|_{L^1(\mathbb{R}^N)} \|h\|_{L^1(\mathbb{R}^N)} \leqslant 1$$

and $g \in L^1(\mathbb{R}^N)$, it follows that

$$\int_{\mathbb{R}^N} \overline{\Phi}(x, I_2(x))\,\mathrm{d}x \leqslant C.$$

Hence we obtain (5.2), and the proof is complete. □

References

[1] *D. Cruz-Uribe, A. Fiorenza*: Approximate identities in variable L^p spaces. Math. Nachr. *280* (2007), 256–270.

[2] *L. Diening*: Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces. Bull. Sci. Math. *129* (2005), 657–700.

[3] *L. Diening, P. Harjulehto, P. Hästö, M. Růžička*: Lebesgue and Sobolev Spaces with Variable Exponents. Lecture Notes in Mathematics 2017, Springer, Berlin, 2011.

[4] *F.-Y. Maeda, Y. Mizuta, T. Ohno*: Approximate identities and Young type inequalities in variable Lebesgue-Orlicz spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}$. Ann. Acad. Sci. Fenn., Math. *35* (2010), 405–420.

[5] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces. Research on Potential Theory 2010. Ohita Univerzity, 2010, pp. 17–30.

[6] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Boundedness of maximal operators and Sobolev's inequality on Musielak-Orlicz-Morrey spaces. Bull. Sci. Math. *137* (2013), 76–96.

[7] *F.-Y. Maeda, Y. Mizuta, T. Ohno, T. Shimomura*: Mean continuity for potentials of functions in Musielak-Orlicz spaces. To appear in RIMS Kôkyûroku Bessatsu.

[8] *J. Musielak*: Orlicz Spaces and Modular Spaces. Lecture Notes in Mathematics 1034, Springer, Berlin, 1983.

[9] *E. M. Stein*: Singular Integrals and Differentiability Properties of Functions. Princeton Mathematical Series 30, Princeton University Press, Princeton, 1970.

[10] *F. Zo*: A note on approximation of the identity. Stud. Math. *55* (1976), 111–122.

Authors' addresses: Fumi-Yuki Maeda, 4-24 Furue-higashi-machi, Nishi-ku, Hiroshima 733-0872, Japan, e-mail: fymaeda@h6.dion.ne.jp; Yoshihiro Mizuta, Department of Mechanical Systems Engineering, Hiroshima Institute of Technology 2-1-1 Miyake Saeki-ku Hiroshima 731-5193, Japan, e-mail: y.mizuta.5x@it-hiroshima.ac.jp; Takao Ohno, Faculty of Education and Welfare Science, Oita University Dannoharu Oita-city 870-1192, Japan, e-mail: t-ohno@oita-u.ac.jp; Tetsu Shimomura, Department of Mathematics, Graduate School of Education, Hiroshima University Higashi-Hiroshima 739-8524, Japan, e-mail: tshimo@hiroshima-u.ac.jp.