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MONOTONE ITERATIVE METHOD FOR ABSTRACT IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN BANACH SPACES

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Abstract. In this paper we use a monotone iterative technique in the presence of the lower and upper solutions to discuss the existence of mild solutions for a class of semilinear impulsive integro-differential evolution equations of Volterra type with nonlocal conditions in a Banach space E

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), Gu(t)), \quad t \in J, \ t \neq t_k, \\ \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = g(u) + x_0, \end{cases}$$

where $A\colon D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a strongly continuous semigroup $T(t)$ $(t \geqslant 0)$ on E, $f \in C(J \times E \times E, E)$, $J = [0, a]$, $0 < t_1 < t_2 < \ldots < t_m < a$, $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$, and g constitutes a nonlocal condition. Under suitable monotonicity conditions and noncompactness measure conditions, we obtain the existence of the extremal mild solutions between the lower and upper solutions assuming that $-A$ generates a compact semigroup, a strongly continuous semigroup or an equicontinuous semigroup. The results improve and extend some relevant results in ordinary differential equations and partial differential equations. Some concrete applications to partial differential equations are considered.

Keywords: evolution equation; impulsive integro-differential equation; nonlocal condition; lower and upper solutions; monotone iterative technique; mild solution

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1. Introduction

The theory of impulsive differential equations describes processes which experience a sudden change in their states at certain moments. Processes with such a character arise naturally and often, especially in phenomena studied in physics, chemistry, biology, population and dynamics, engineering and economics. The theory of impulsive differential equations has emerged as an important area of research in the previous decades, see [5], [16], [23], [27], [30], and the references therein. Particularly, the theory of impulsive evolution equations has become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population, dynamics, biotechnology and economics. There has been a significant development in impulsive evolution equations in Banach spaces. For more details on this theory and its applications, we refer to the references [1], [2], [10], [11], [19], [20], [26], [32], [35], [38].

In 1990, Byszewski and Lakshmikantham [9] were the first to investigate the nonlocal problems. They studied and obtained the existence and uniqueness of mild solutions for nonlocal differential equations without impulsive conditions. Since it has been demonstrated that the nonlocal problems have better effects in applications than the traditional Cauchy problems, differential equations with nonlocal conditions have been studied by many authors and some basic results on nonlocal problems have been obtained, see [6], [7], [8], [13], [17], [18], [21], [25], [31], [33], [34], [36], [40], [41], [42], and the references therein for more comments and citations. In 2009, Liang et al. [32] combined the impulsive conditions and the nonlocal conditions, and investigated the nonlocal problem of impulsive evolution equations in Banach spaces. Later on, Fan [19], Fan and Li [20], Ji et al. [26] studied the impulsive evolution equations with nonlocal conditions. In previous works, nonlocal problems have been studied by many authors using different tools, such as Banach contraction mapping principal, Schauder's fixed-point theorem, Sadovskii's fixed-point theorem and Mönch fixed-point theorem. However, to the best of our knowledge, no results yet exist for the nonlocal problems by using the method of the lower and upper solutions coupled with the monotone iterative technique.

In this paper we use a monotone iterative technique in the presence of the lower and upper solutions to discuss the existence of the extremal mild solutions to the nonlocal problem of first order semilinear impulsive integro-differential evolution equations of Volterra type in an ordered Banach space E

$$\begin{cases} u'(t) + Au(t) = f(t, u(t), Gu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = g(u) + x_0, \end{cases} \tag{1}$$

where $A\colon D(A) \subset E \to E$ is a closed linear operator and $-A$ generates a strongly continuous semigroup (C_0-semigroup, in short) $T(t)$ $(t \geqslant 0)$ on E; $f \in C(J \times E \times E, E)$, $J = [0, a]$, $a > 0$ is a constant, $0 < t_1 < t_2 < \ldots < t_m < a$; $I_k \in C(E, E)$ is an impulsive function, $k = 1, 2, \ldots, m$; $x_0 \in E$, g is a nonlocal function; and

$$Gu(t) = \int_0^t K(t, s)u(s)\,\mathrm{d}s \tag{2}$$

is a Volterra integral operator with integral kernel $K \in C(\nabla, \mathbb{R}^+)$, $\nabla = \{(t, s)\colon\ 0 \leqslant s \leqslant t \leqslant a\}$; $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right and the left limits of $u(t)$ at $t = t_k$, respectively.

It is well known that the monotone iterative technique in the presence of the lower and upper solutions is an important method for seeking solutions of differential equations in abstract spaces. Early on, Du and Lakshmikantham [15], Sun and Zhao [39] investigated the existence of extremal solutions to the initial value problem of ordinary differential equations without impulse by using the method of the lower and upper solutions coupled with the monotone iterative technique. Later, Guo and Liu [23] developed the monotone iterative method for impulsive integro-differential equations, and built a monotone iterative method for the initial value problem (IVP, in short) of impulsive ordinary integro-differential equations in an ordered Banach space E:

$$\begin{cases} u'(t) = f(t, u(t), Gu(t)), \quad t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) = x_0. \end{cases} \tag{3}$$

They proved that if IVP (3) has a lower solution v_0 and an upper solution w_0 with $v_0 \leqslant w_0$, and the nonlinear term f and the impulsive function I_k satisfy the monotonicity conditions

$$\begin{gathered} f(t, x_2, y_2) - f(t, x_1, y_1) \geqslant -M(x_2 - x_1) - M^*(y_2 - y_1), \quad I_k(x_2) \geqslant I_k(x_1), \\ v_0(t) \leqslant x_1 \leqslant x_2 \leqslant w_0(t),\ Gv_0(t) \leqslant y_1 \leqslant y_2 \leqslant Gw_0(t) \quad \forall t \in J, \end{gathered} \tag{4}$$

with positive constants M and M^*, and the noncompactness measure conditions

$$\alpha(f(t, U, V)) \leqslant L_1\alpha(U) + L_2\alpha(V), \tag{5}$$

$$\alpha(I_k(D)) \leqslant M_k\alpha(D), \quad k = 1, 2, \ldots, m, \tag{6}$$

where $U, V, D \subset E$ are arbitrary bounded sets, L_1, L_2 and M_k are positive constants satisfying

$$2a(M + L_1 + aK_0L_2) + \sum_{k=1}^{m} M_k < 1, \tag{7}$$

where $K_0 = \max_{(t,s)\in\nabla} K(t,s)$, $\alpha(\cdot)$ denotes the Kuratowski measure of noncompactness of E, then IVP (3) has the minimal and the maximal solutions between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively. Recently, Li and Liu [30] extended the results in [23] by removing the noncompactness measure condition (6) for the impulsive function I_k and the restriction condition (7).

The purpose of this paper is to improve and extend the above-mentioned results. By combining the theory of semigroups of linear operators and the method of the lower and upper solutions coupled with the monotone iterative technique, we construct two monotone iterative sequences, and prove that the sequences monotonically converge to the minimal and maximal mild solutions of problem (1), respectively, under the suitable conditions on A, f, I_k, and g.

The outline of this paper is as follows. In Section 2, some notation and preliminaries are introduced, which are used throughout the paper. The existence of the extremal mild solutions of problem (1) is given in Section 3. Finally, two examples are given to illustrate our abstract results in Section 4.

2. Preliminaries

Let E be an ordered Banach space with the norm $\|\cdot\|$ and partial order "$\leqslant$", whose positive cone $P = \{x \in E;\ x \geqslant \theta\}$ is normal with normal constant N. Let $PC(J,E) = \{u\colon J \to E;\ u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$, then $PC(J,E)$ is a Banach space with the norm $\|u\|_{PC} = \sup_{t\in J} \|u(t)\|$. Evidently, $PC(J,E)$ is also an ordered Banach space with the partial order "$\leqslant$" induced by the positive cone $K_{PC} = \{u \in PC(J,E);\ u(t) \geqslant \theta,\ t \in J\}$. The cone K_{PC} is also normal with the same normal constant N. For $v,\ w \in PC(J,E)$ with $v \leqslant w$, we use $[v,w]$ to denote the order interval $\{u \in PC(J,E);\ v \leqslant u \leqslant w\}$ on $PC(J,E)$, and $[v(t), w(t)]$ to denote the order interval $\{u \in E;\ v(t) \leqslant u(t) \leqslant w(t),\ t \in J\}$ on E. We use E_1 to denote the Banach space $D(A)$ with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. Let $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $J'' = J \setminus \{0, t_1, t_2, \ldots, t_m\}$. An abstract function $u \in PC(J,E) \cap C^1(J'',E) \cap C(J',E_1)$ is called a solution of the problem (1) if $u(t)$ satisfies all the equalities in (1).

Let $C(J,E)$ denote the Banach space of all continuous E-valued functions on the interval J with the norm $\|u\|_C = \max\limits_{t\in J}\|u(t)\|$. Then $C(J,E)$ is an ordered Banach space induced by the convex cone $P_C = \{u \in C(J,E);\ u(t) \geqslant \theta,\ t \in J\}$, and P_C is also a normal cone. Let $\alpha(\cdot)$ denote the Kuratowski measure of noncompactness of the bounded set. For the details of the definition and properties of the measure of noncompactness, see [3], [12]. For any $B \subset C(J,E)$ and $t \in J$, set $B(t) = \{u(t);\ u \in B\} \subset E$. If $B \subset C(J,E)$ is bounded, then $B(t)$ is bounded on E and $\alpha(B(t)) \leqslant \alpha(B)$.

We first give lemmas which are used further in this paper.

Lemma 1 ([3]). *Let E be a Banach space, let $B \subset C(J,E)$ be bounded and equicontinuous. Then $\alpha(B(t))$ is continuous on J, and*

$$\alpha(B) = \max_{t\in J}\alpha(B(t)) = \alpha(B(J)).$$

Lemma 2 ([24]). *Let E be a Banach space, let $B = \{u_n\} \subset PC(J,E)$ be a bounded and countable set. Then $\alpha(B(t))$ is Lebesgue integrable on J, and*

$$\alpha\biggl(\biggl\{\int_J u_n(t)\,\mathrm{d}t;\ n \in \mathbb{N}\biggr\}\biggr) \leqslant 2\int_J \alpha(B(t))\,\mathrm{d}t.$$

Lemma 3 ([28]). *Let E be a Banach space, let $D \subset E$ be bounded. Then there exists a countable set $D_0 \subset D$ such that $\alpha(D) \leqslant 2\alpha(D_0)$.*

P r o o f. We give the proof of this lemma here for the convenience of readers. Without loss of generality, assume that $\alpha(D) > 0$. If $r_n = (1 - 1/2^n)\alpha(D)$, then $0 < r_n < \alpha(D)$. Choose $x_1^{(n)} \in D$, then $D \setminus B(x_1^{(n)}, r_n/2) \neq \emptyset$. Otherwise, if $D \subset B(x_1^{(n)}, r_n/2)$, by the definition of the noncompactness measure, $\alpha(D) \leqslant r_n$, which is a contradiction. This shows that $D \setminus B(x_1^{(n)}, r_n/2) \neq \emptyset$. Choose $x_2^{(n)} \in D \setminus B(x_1^{(n)}, r_n/2)$, similarly, $D \setminus (B(x_1^{(n)}, r_n/2) \cup B(x_2^{(n)}, r_n/2)) \neq \emptyset$. Therefore, we can choose $x_3^{(n)} \in D \setminus (B(x_1^{(n)}, r_n/2) \cup B(x_2^{(n)}, r_n/2))$. Continuing such a process, we obtain a sequence $\{x_k^{(n)};\ k = 1,2,\ldots\}$ such that $x_{k+1}^{(n)} \in D \setminus \bigcup\limits_{i=1}^{k} B(x_i^{(n)}, r_n/2)$, $k = 1,2,\ldots$. Letting $D_n = \{x_k^{(n)};\ k = 1,2,\ldots\}$, together with the definition of noncompactness measure, we know that $\alpha(D_n) \geqslant r_n/2$. Let $D_0 = \bigcup\limits_{n=1}^{\infty} D_n$. Then D_0 is a countable set. Since $\alpha(D_0) \geqslant \alpha(D_n) \geqslant r_n/2 \to (\alpha(D))/2$ $(n \to \infty)$, we have $\alpha(D) \leqslant 2\alpha(D_0)$. The proof is completed. $\square$

Lemma 4 ([22]). *Let P be a normal cone of the Banach space E and let v_0, $w_0 \in E$ with $v_0 \leqslant w_0$. Suppose that $Q\colon [v_0, w_0] \to E$ is a nondecreasing strict set contraction operator such that $v_0 \leqslant Qv_0$ and $Qw_0 \leqslant w_0$. Then Q has a minimal fixed point $\underline{u}$ and a maximal fixed point $\bar{u}$ in $[v_0, w_0]$; moreover, $v_n \to \underline{u}$ and $w_n \to \bar{u}$, where $v_n = Qv_{n-1}$ and $w_n = Qw_{n-1}$ $(n = 1, 2, \ldots)$ which satisfy $v_0 \leqslant v_1 \leqslant \ldots \leqslant v_n \leqslant \ldots \leqslant \underline{u} \leqslant \bar{u} \leqslant \ldots \leqslant w_n \leqslant \ldots \leqslant w_1 \leqslant w_0$.*

Let $A\colon D(A) \subset E \to E$ be a closed linear operator and let $-A$ generate a C_0-semigroup $T(t)$ $(t \geqslant 0)$ on E. Then there exist constants $C > 0$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leqslant C\mathrm{e}^{\delta t}, \quad t \geqslant 0.$$

Definition 1. A function $u \in PC(J, E)$ is said to be a mild solution of the problem (1) if it satisfies

$$u(t) = T(t)(g(u) + x_0) + \int_0^t T(t-s)f(s, u(s), Gu(s))\,\mathrm{d}s + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k)), \quad t \in J. \tag{8}$$

Definition 2. If a function $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ satisfies

$$\begin{cases} v_0'(t) + Av_0(t) \leqslant f(t, v_0(t), Gv_0(t)), & t \in J', \\ \Delta v_0|_{t=t_k} \leqslant I_k(v_0(t_k)), & k = 1, 2, \ldots, m, \\ v_0(0) \leqslant g(v_0) + x_0, \end{cases} \tag{9}$$

we call it a lower solution of the problem (1); if all the inequalities in (9) are reversed, we call it an upper solution of the problem (1).

Definition 3. A C_0-semigroup $T(t)$ $(t \geqslant 0)$ on E is said to be positive, if the order inequality $T(t)x \geqslant \theta$ holds for each $x \geqslant \theta$, $x \in E$, and $t \geqslant 0$.

It is easy to see that for any $M \geqslant 0$, $-(A + MI)$ also generates a C_0-semigroup $S(t) = \mathrm{e}^{-Mt}T(t)$ $(t \geqslant 0)$ on E. And $S(t)$ $(t \geqslant 0)$ is a positive C_0-semigroup if $T(t)$ $(t \geqslant 0)$ is a positive C_0-semigroup. For the details of the properties of the positive C_0-semigroup, see [4], [29].

3. The main results

Theorem 1. *Let E be an ordered Banach space whose positive cone P is normal, let $A\colon D(A)\subset E\to E$ be a closed linear operator, let the positive C_0-semigroup $T(t)$ $(t\geqslant 0)$ generated by $-A$ be compact on E, $f\in C(J\times E\times E,E)$, $I_k\in C(E,E)$, $k=1,2,\dots,m$, and let $g\colon PC(J,E)\to E$ be a compact operator. Assume that the problem* (1) *has a lower solution $v_0\in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ and an upper solution $w_0\in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ with $v_0\leqslant w_0$. Suppose also that the following conditions are satisfied:*

(H1) *There exists a constant $M>0$ such that*

$$f(t,u_2,v_2)-f(t,u_1,v_1)\geqslant -M(u_2-u_1)$$

for all $t\in J$, and $v_0(t)\leqslant u_1\leqslant u_2\leqslant w_0(t)$, $Gv_0(t)\leqslant v_1\leqslant v_2\leqslant Gw_0(t)$.

(H2) *The impulsive function $I_k(\cdot)$ satisfies*

$$I_k(u_1)\leqslant I_k(u_2),\quad k=1,2,\dots,m,$$

for any $t\in J$, and $v_0(t)\leqslant u_1\leqslant u_2\leqslant w_0(t)$.

(H3) *The nonlocal function $g(u)$ is increasing on the order interval $[v_0,w_0]$.*

Then the problem (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between v_0 and w_0.*

P r o o f. Letting $\overline{M}=\sup\limits_{t\in J}\|S(t)\|$, we define a mapping $Q\colon [v_0,w_0]\to PC(J,E)$ by

$$\begin{aligned}(10)\qquad Qu(t)&=S(t)(g(u)+x_0)+\int_0^t S(t-s)[f(s,u(s),Gu(s))+Mu(s)]\,\mathrm{d}s\\&\quad+\sum_{0<t_k<t}S(t-t_k)I_k(u(t_k)),\quad t\in J.\end{aligned}$$

Obviously, $Q\colon [v_0,w_0]\to PC(J,E)$ is continuous. By Definition 1, the mild solution of the problem (1) is equivalent to the fixed point of the operator Q. Since $S(t)$ $(t\geqslant 0)$ is a positive C_0-semigroup, together with the assumptions (H1), (H2), and (H3), Q is increasing in $[v_0,w_0]$.

We first show that $v_0\leqslant Qv_0$, $Qw_0\leqslant w_0$. Letting $h(t)=v_0'(t)+Av_0(t)+Mv_0(t)$, we have by (9) that $h\in PC(J,E)$ and $h(t)\leqslant f(t,v_0(t),Gv_0(t))+Mv_0(t)$, $t\in J'$.

By Definitions 1 and 2 we have

$$\begin{aligned}
v_0(t) &= S(t)v_0(0) + \int_0^t S(t-s)h(s)\,\mathrm{d}s + \sum_{0<t_k<t} S(t-t_k)\Delta v_0|_{t=t_k} \\
&\leqslant S(t)(g(u)+x_0) + \int_0^t S(t-s)[f(s,v_0(s),Gv_0(s)) + Mv_0(s)]\,\mathrm{d}s \\
&\quad + \sum_{0<t_k<t} S(t-t_k)I_k(v_0(t_k)) = Qv_0(t), \quad t\in J,
\end{aligned}$$

namely, $v_0 \leqslant Qv_0$. Similarly, it can be shown that $Qw_0 \leqslant w_0$. Therefore, $Q\colon [v_0,w_0]\to[v_0,w_0]$ is a continuous increasing operator.

Next, we show that $Q\colon [v_0,w_0]\to[v_0,w_0]$ is completely continuous. Let

$$\begin{aligned}
Wu(t) &= \int_0^t S(t-s)[f(s,u(s),Gu(s)) + Mu(s)]\,\mathrm{d}s, \\
Vu(t) &= \sum_{0<t_k<t} S(t-t_k)I_k(u(t_k)), \quad u\in[v_0,w_0].
\end{aligned} \tag{11}$$

On one hand, we prove that for any $0<t\leqslant a$, $X(t) \triangleq \{Wu(t)\colon u\in[v_0,w_0]\}$ is precompact on E. For $0<\varepsilon<t$ and $u\in[v_0,w_0]$,

$$\begin{aligned}
W_\varepsilon u(t) &= \int_0^{t-\varepsilon} S(t-s)[f(s,u(s),Gu(s)) + Mu(s)]\,\mathrm{d}s \\
&= S(\varepsilon)\int_0^{t-\varepsilon} S(t-s-\varepsilon)[f(s,u(s),Gu(s)) + Mu(s)]\,\mathrm{d}s.
\end{aligned} \tag{12}$$

For any $u\in[v_0,w_0]$, by the assumption (H1) we have

$$\begin{aligned}
f(t,v_0(t),Gv_0(t)) + Mv_0(t) &\leqslant f(t,u(t),Gu(t)) + Mu(t) \\
&\leqslant f(t,w_0(t),Gw_0(t)) + Mw_0(t).
\end{aligned}$$

By the normality of the cone P, there exists $\overline{M}_1>0$ such that

$$\|f(t,u(t),Gu(t)) + Mu(t)\| \leqslant \overline{M}_1, \quad u\in[v_0,w_0].$$

By the compactness of $S(\varepsilon)$, $X_\varepsilon(t) \triangleq \{W_\varepsilon u(t)\colon u\in[v_0,w_0]\}$ is precompact on E. Since

$$\begin{aligned}
\|Wu(t) - W_\varepsilon u(t)\| &\leqslant \int_{t-\varepsilon}^t \|S(t-s)\|\cdot\|f(s,u(s),Gu(s)) + Mu(s)\|\,\mathrm{d}s \\
&\leqslant \overline{M}\,\overline{M}_1\varepsilon,
\end{aligned} \tag{13}$$

the set $X(t)$ is totally bounded on E. Furthermore, $X(t)$ is precompact on E.

On the other hand, for any $0 \leqslant t_1 \leqslant t_2 \leqslant a$ we have

$$
\begin{aligned}
(14)\qquad &\|Wu(t_2) - Wu(t_1)\| \\
&= \Big\| \int_0^{t_1} (S(t_2 - s) - S(t_1 - s))[f(s, u(s), Gu(s)) + Mu(s)]\,\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} S(t_2 - s)[f(s, u(s), Gu(s)) + Mu(s)]\,\mathrm{d}s \Big\| \\
&\leqslant \overline{M}_1 \int_0^{t_1} \|S(t_2 - s) - S(t_1 - s)\|\,\mathrm{d}s + \overline{M}\,\overline{M}_1(t_2 - t_1) \\
&\leqslant \overline{M}_1 \int_0^{a} \|S(t_2 - t_1 + s) - S(s)\|\,\mathrm{d}s + \overline{M}\,\overline{M}_1(t_2 - t_1).
\end{aligned}
$$

Since $S(t)$ is continuous in the uniform operator topology for $t > 0$, it is easy to see that $\|Wu(t_2) - Wu(t_1)\|$ tends to zero independently of $u \in [v_0, w_0]$ as $t_2 - t_1 \to 0$, which means that $\{Wu,\ u \in [v_0, w_0]\}$ is equicontinuous.

Let $Y(t) = \{Vu(t) \colon u \in [v_0, w_0]\}$. Since $S(t)$ is compact for $t > 0$, $Y(t)$ is precompact on E. For any $u \in [v_0, w_0]$, by the assumption (H2) we have

$$I_k(v_0(t_k)) \leqslant I_k(u(t_k)) \leqslant I_k(w_0(t_k)), \quad k = 1, 2, \ldots, m.$$

By the normality of the cone P, there exists $\overline{M}_2 > 0$ such that

$$\|I_k(u(t_k))\| \leqslant \overline{M}_2, \quad u \in [v_0, w_0],\ k = 1, 2, \ldots, m.$$

For any $0 \leqslant t' < t'' \leqslant a$, we have

$$
\begin{aligned}
(15)\quad &\|Vu(t'') - Vu(t')\| \\
&= \Big\| \sum_{0<t_k<t} (S(t'' - t_k) - S(t' - t_k)) I_k(u(t_k)) \Big\| \\
&\leqslant \Big\| \sum_{0<t_k<t'} (S(t'' - t_k) - S(t' - t_k)) I_k(u(t_k)) \Big\| \\
&\quad + \Big\| \sum_{t' \leqslant t_k < t''} S(t'' - t_k) I_k(u(t_k)) \Big\| \\
&\leqslant \overline{M}_2 \sum_{0<t_k<t'} \|S(t'' - t_k) - S(t' - t_k)\| + \overline{M}_2 \sum_{t' \leqslant t_k < t''} \|S(t'' - t_k)\| \\
&\leqslant \overline{M}_2\,\overline{M} \sum_{0<t_k<t'} \Big\| S\Big(t'' - t' + \frac{t' - t_k}{2}\Big) - S\Big(\frac{t' - t_k}{2}\Big) \Big\| + \overline{M}_2\,\overline{M}(t'' - t').
\end{aligned}
$$

Since $S(t)$ is continuous in the uniform operator topology for $t > 0$, it is easy to see that $\|Vu(t'') - Vu(t')\|$ tends to zero independently of $u \in [v_0, w_0]$ as $t'' - t' \to 0$, which means that $\{Vu,\ u \in [v_0, w_0]\}$ is equicontinuous.

For $0 \leqslant t \leqslant a$, $\{Qu(t)\colon u \in [v_0, w_0]\} = \{S(t)(g(u)+x_0) + Wu(t) + Vu(t)\colon u \in [v_0, w_0]\}$. Obviously, $Qu(0) = g(u) + x_0$ is precompact on E owing to the compactness of g. Hence, $Q([v_0, w_0])$ is precompact by the Arzela-Ascoli Theorem. Thus, $Q\colon [v_0, w_0] \to [v_0, w_0]$ is completely continuous. Hence, the theory of monotone increasing operators implies that Q has a minimal fixed point $\underline{u}$ and a maximal fixed point $\bar{u}$ in $[v_0, w_0]$, and therefore, they are the minimal and the maximal mild solutions of the problem (1) in $[v_0, w_0]$, respectively. □

Theorem 2. *Let E be an ordered Banach space, whose positive cone P is normal, let $A\colon D(A) \subset E \to E$ be a closed linear operator and let $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geqslant 0)$ on E, $f \in C(J \times E \times E, E)$, $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$, and let $g\colon PC(J, E) \to E$ map a monotonic set into a precompact set. Assume that the problem* (1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \leqslant w_0$. If conditions* (H1), (H2), (H3) *and the condition*

(H4) *There exists a constant $L > 0$ such that*

$$\alpha(\{f(t, u_n, v_n)\}) \leqslant L(\alpha(\{u_n\}) + \alpha(\{v_n\}))$$

for all $t \in J$, and increasing or decreasing monotonic sequences $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$

hold, then the problem (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Proof. From Theorem 1 we know that $Q\colon [v_0, w_0] \to [v_0, w_0]$ is a continuous increasing operator. Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in $[v_0, w_0]$ by the iterative scheme

$$v_n = Qv_{n-1}, \quad w_n = Qw_{n-1}, \quad n = 1, 2, \ldots \tag{16}$$

Then from the monotonicity of Q it follows that

$$v_0 \leqslant v_1 \leqslant v_2 \leqslant \ldots \leqslant v_n \leqslant \ldots \leqslant w_n \leqslant \ldots \leqslant w_2 \leqslant w_1 \leqslant w_0. \tag{17}$$

Next, we prove that $\{v_n\}$ and $\{w_n\}$ are convergent on J. For convenience, let $B = \{v_n;\ n \in \mathbb{N}\}$ and $B_0 = \{v_{n-1};\ n \in \mathbb{N}\}$. Then $B = Q(B_0)$. Let $J_1 = [0, t_1]$, $J_k = (t_{k-1}, t_k]$, $k = 2, 3, \ldots, m+1$, $J_{m+1} = a$. From $B_0 = B \cup \{v_0\}$ it follows that $\alpha(B_0(t)) = \alpha(B(t))$ for $t \in J$. Let $\varphi(t) := \alpha(B(t))$, $t \in J$, go from J_1 to J_{m+1}. We show interval by interval that $\varphi(t) \equiv 0$ on J.

For $t \in J$, there exists a J_k such that $t \in J_k$. By (2) and Lemma 2, we have that

$$\begin{aligned}
\alpha(G(B_0)(t)) &= \alpha\Big(\Big\{\int_0^t K(t,s)v_{n-1}(s)\,\mathrm{d}s;\ n \in \mathbb{N}\Big\}\Big)\\
&\leqslant \sum_{j=1}^{k-1} \alpha\Big(\Big\{\int_{t_{j-1}}^{t_j} K(t,s)v_{n-1}(s)\,\mathrm{d}s;\ n \in \mathbb{N}\Big\}\Big)\\
&\quad + \alpha\Big(\Big\{\int_{t_{k-1}}^{t} K(t,s)v_{n-1}(s)\,\mathrm{d}s;\ n \in \mathbb{N}\Big\}\Big)\\
&\leqslant 2K_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \alpha(B_0(s))\,\mathrm{d}s + 2K_0 \int_{t_{k-1}}^{t} \alpha(B_0(s))\,\mathrm{d}s\\
&= 2K_0 \sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \varphi(s)\,\mathrm{d}s + 2K_0 \int_{t_{k-1}}^{t} \varphi(s)\,\mathrm{d}s\\
&= 2K_0 \int_0^t \varphi(s)\,\mathrm{d}s,
\end{aligned}$$

and therefore,

$$\int_0^t \alpha(G(B_0)(s))\,\mathrm{d}s \leqslant 2aK_0 \int_0^t \varphi(s)\,\mathrm{d}s. \tag{18}$$

For $t \in J_1$, from (10) and (18), using Lemma 2 and the assumption (H4), we have

$$\begin{aligned}
&\varphi(t) = \alpha(B(t)) = \alpha(Q(B_0)(t))\\
&= \alpha\Big(\Big\{S(t)(g(v_{n-1}) + x_0) + \int_0^t S(t-s)[f(s, v_{n-1}(s), Gv_{n-1}(s)) + Mv_{n-1}(s)]\,\mathrm{d}s\Big\}\Big)\\
&\leqslant 2\overline{M} \int_0^t \alpha(\{f(s, v_{n-1}(s), Gv_{n-1}(s)) + Mv_{n-1}(s)\})\,\mathrm{d}s\\
&\leqslant 2\overline{M} \int_0^t [L(\alpha(B_0(s)) + \alpha(G(B_0)(s))) + M\alpha(B_0(s))]\,\mathrm{d}s\\
&\leqslant 2\overline{M}(L + M + 2aLK_0) \int_0^t \varphi(s)\,\mathrm{d}s.
\end{aligned}$$

Hence by Gronwall's inequality, $\varphi(t) \equiv 0$ on J_1. In particular, $\alpha(B(t_1)) = \alpha(B_0(t_1)) = \varphi(t_1) = 0$, which implies that $B(t_1)$ and $B_0(t_1)$ are precompact on E. Thus $I_1(B_0(t_1))$ is precompact on E, and $\alpha(I_1(B_0(t_1))) = 0$.

Now, for $t \in J_2$, by (10) and the above argument for $t \in J_1$, we have

$$\begin{aligned}
\varphi(t) &= \alpha(B(t)) = \alpha(Q(B_0)(t)) \\
&= \alpha\Bigg(\bigg\{S(t)(g(v_{n-1}) + x_0) + \int_0^t S(t-s)[f(s, v_{n-1}(s), Gv_{n-1}(s)) + Mv_{n-1}(s)]\,\mathrm{d}s \\
&\qquad\qquad + S(t-t_1)I_1(v_{n-1}(t_1))\bigg\}\Bigg) \\
&\leqslant 2\overline{M}(L + M + 2aLK_0)\int_0^t \varphi(s)\,\mathrm{d}s \\
&= 2\overline{M}(L + M + 2aLK_0)\int_{t_1}^t \varphi(s)\,\mathrm{d}s.
\end{aligned}$$

Again by Gronwall's inequality, $\varphi(t) \equiv 0$ on J_2, from which we obtain that $\alpha(B_0(t_2)) = 0$ and $\alpha(I_2(B_0(t_2))) = 0$.

Continuing such a process interval by interval up to J_{m+1}, we can prove that $\varphi(t) \equiv 0$ on every $J_k, k = 1, 2, \ldots, m+1$. Hence, for any $t \in J$, $\{v_n(t)\}$ is precompact, and $\{v_n(t)\}$ has a convergent subsequence. Combining this with the monotonicity (17), we can easily prove that $\{v_n(t)\}$ itself is convergent, i.e., $\lim\limits_{n\to\infty} v_n(t) = \underline{u}(t)$, $t \in J$. Similarly, $\lim\limits_{n\to\infty} w_n(t) = \bar{u}(t)$, $t \in J$.

Evidently, $\{v_n(t)\} \subset PC(J, E)$, so $\underline{u}(t)$ is bounded and integrable on J. Since for any $t \in J$,

$$\begin{aligned}
v_n(t) &= Qv_{n-1}(t) \\
&= S(t)(g(v_{n-1}) + x_0) + \int_0^t S(t-s)[f(s, v_{n-1}(s), Gv_{n-1}(s)) + Mv_{n-1}(s)]\,\mathrm{d}s \\
&\quad + \sum_{0<t_k<t} S(t-t_k)I_k(v_{n-1}(t_k)),
\end{aligned}$$

letting $n \to \infty$, by the Lebesgue dominated convergence theorem we know that $\underline{u}(t) \in PC(J, E)$ and $\underline{u} = Q\underline{u}$. Similarly, $\bar{u}(t) \in PC(J, E)$ and $\bar{u} = Q\bar{u}$. Combining this with monotonicity (17), we can see that $v_0 \leqslant \underline{u} \leqslant \bar{u} \leqslant w_0$. By the monotonicity of Q, it is easy to see that $\underline{u}$ and $\bar{u}$ are the minimal and the maximal fixed points of Q in $[v_0, w_0]$. Therefore, $\underline{u}$ and $\bar{u}$ are the minimal and the maximal mild solutions of the problem (1) in $[v_0, w_0]$, respectively. □

Remark 1. If $G \equiv 0$, $A \equiv 0$, $I_k \equiv 0$ and $g \equiv 0$, then Theorem 2 in this paper is the main result in [15]; if $G \equiv 0$, $A \equiv 0$ and $g \equiv 0$, then Theorem 2 in this paper is the extension of the main result in [16]; if $A \equiv 0$, $g \equiv 0$, then Theorem 2 in this paper is the main result in [30].

Corollary 1. *Let E be an ordered Banach space whose positive cone P is normal, let $A\colon D(A) \subset E \to E$ be a closed linear operator and let $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geqslant 0)$ on E, $f \in C(J \times E \times E, E)$, $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$, and let $g\colon PC(J, E) \to E$ be a compact operator. If the problem* (1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \leqslant w_0$, and conditions* (H1), (H2), (H3) *and* (H4) *are satisfied, then the problem* (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Corollary 2. *Let E be an ordered Banach space whose positive cone P is regular, let $A\colon D(A) \subset E \to E$ be a closed linear operator and let $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geqslant 0)$ on E, $f \in C(J \times E \times E, E)$, $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$, and let $g\colon PC(J, E) \to E$ be a continuous function. If the problem* (1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \leqslant w_0$, and conditions* (H1), (H2) *and* (H3) *are satisfied, then the problem* (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Theorem 3. *Let E be an ordered Banach space whose positive cone P is normal, let $A\colon D(A) \subset E \to E$ be a closed linear operator and let $-A$ generate a positive C_0-semigroup $T(t)$ $(t \geqslant 0)$ on E, $f \in C(J \times E \times E, E)$, $I_k \in C(E, E)$, $k = 1, 2, \ldots, m$, and let $g\colon PC(J, E) \to E$ be a continuous function. Assume that the problem* (1) *has a lower solution $v_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ and an upper solution $w_0 \in PC(J, E) \cap C^1(J'', E) \cap C(J', E_1)$ with $v_0 \leqslant w_0$. If conditions* (H1), (H2), (H3), (H4) *and the condition*

(H5) *The sequences $v_n(0)$ and $w_n(0)$ are convergent, where v_n and w_n are defined by (16)*

hold, then the problem (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

P r o o f. The proof of Theorem 3 is similar to that of Theorem 2, we omit the details here. □

In Theorem 3, if E is weakly sequentially complete, the conditions (H4) and (H5) hold automatically. In fact, by Theorem 2.2 in [14], any monotonic and order-bounded sequence is precompact. By the monotonicity (17), we can easily see that $v_n(t)$ and $w_n(t)$ are convergent on J. In particular, $v_n(0)$ and $w_n(0)$ are convergent.

So, condition (H5) holds. Let $\{u_n\}$ and $\{v_n\}$ be increasing or decreasing sequences obeying condition (H4), then by condition (H1), $\{f(t,u_n,v_n)+Mu_n\}$ is a monotonic and order-bounded sequence, so, $\alpha(\{f(t,u_n,v_n)+Mu_n\})=0$. Hence, condition (H4) holds. From Theorem 3, we obtain

Corollary 3. *Let E be an ordered and weakly sequentially complete Banach space whose positive cone P is normal, let $A\colon D(A)\subset E\to E$ be a closed linear operator and let $-A$ generate a positive C_0-semigroup $T(t)$ $(t\geqslant 0)$ on E, $f\in C(J\times E\times E,E)$, $I_k\in C(E,E)$, $k=1,2,\ldots,m$, and let $g\colon PC(J,E)\to E$ be a continuous function. If the problem* (1) *has a lower solution $v_0\in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ and an upper solution $w_0\in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ with $v_0\leqslant w_0$, and conditions* (H1), (H2), *and* (H3) *are satisfied, then the problem* (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between v_0 and w_0, which can be obtained by a monotone iterative procedure starting from v_0 and w_0, respectively.*

Remark 2. In the application of partial differential equations, we often choose the Banach space L^p $(1\leqslant p<\infty)$ as the working space, which is a weakly sequentially complete space. Therefore, Corollary 3 is very valuable in applications.

If we replace the assumption (H4) by the assumption

(H6) There exist positive constants $\overline{C}$ and $\overline{L}$ such that

$$f(t,u_2,v_2)-f(t,u_1,v_1)\leqslant \overline{C}(u_2-u_1)+\overline{L}(v_2-v_1)$$

for any $t\in J$ and $v_0(t)\leqslant u_1\leqslant u_2\leqslant w_0(t)$, $Gv_0(t)\leqslant v_1\leqslant v_2\leqslant Gw_0(t)$,

we have the following existence result.

Theorem 4. *Let E be an ordered Banach space whose positive cone P is normal, let $A\colon D(A)\subset E\to E$ be a closed linear operator and let $-A$ generate a positive C_0-semigroup $T(t)$ $(t\geqslant 0)$ on E, $f\in C(J\times E\times E,E)$, $I_k\in C(E,E)$, $k=1,2,\ldots,m$, and let $g\colon PC(J,E)\to E$ be a continuous function. Assume that the problem* (1) *has a lower solution $v_0\in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ and an upper solution $w_0\in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ with $v_0\leqslant w_0$.*

Then we have:

(i) *If g maps a monotonic set into a precompact set, and conditions* (H1), (H2), (H3) *and* (H6) *are satisfied, then the problem* (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between v_0 and w_0.*

(ii) *If g is a compact operator, and conditions* (H1), (H2), (H3), *and* (H6) *are satisfied, then the problem* (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar{u}$ between v_0 and w_0.*

(iii) *If conditions* (H1), (H2), (H3), (H5), *and* (H6) *are satisfied, then the problem* (1) *has a minimal mild solution* $\underline{u}$ *and a maximal mild solution* $\bar{u}$ *between* v_0 *and* w_0.

P r o o f. For $t \in J$, let $\{u_n\} \subset [v_0(t), w_0(t)]$ and $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$ be two increasing sequences. For $m, n \in \mathbb{N}$ with $m > n$, by conditions (H1) and (H6),

$$\begin{aligned}\theta &\leqslant f(t, u_m, v_m) - f(t, u_n, v_n) + M(u_m - u_n)\\ &\leqslant (M + \overline{C})(u_m - u_n) + \overline{L}(v_n - v_m).\end{aligned}$$

From this and the normality of the cone P, it follows that

$$\begin{aligned}&\|f(t, u_m, v_m) - f(t, u_n, v_n)\|\\ &\qquad\leqslant N\|(M + \overline{C})(u_m - u_n) + \overline{L}(v_n - v_m)\| + M\|u_m - u_n\|\\ &\qquad\leqslant (N(M + \overline{C}) + M)\|u_m - u_n\| + N\overline{L}\|v_n - v_m\|.\end{aligned}$$

By this and the definition of the measure of noncompactness, we obtain that

$$\begin{aligned}\alpha(\{f(t, u_n, v_n)\}) &\leqslant (N(M + \overline{C}) + M)\alpha(\{u_n\}) + N\overline{L}\alpha(\{v_n\})\\ &\leqslant L(\alpha(\{u_n\}) + \alpha(\{v_n\})),\end{aligned}$$

where $L = \max\{(N(M + \overline{C}) + M), N\overline{L}\}$. If $\{u_n\}$ and $\{v_n\}$ are two decreasing sequences, the above inequality is also valid. Hence, the condition (H4) holds.

Therefore, our conclusions (i), (ii) and (iii) follow from Theorem 2, Corollary 1, and Theorem 3, respectively. □

R e m a r k 3. The condition (H6) is easy to be verified in applications. Therefore, using Theorem 4 in the application is very convenient.

If the nonlinear term f, the impulsive function I_k $(k = 1, 2, \ldots, m)$ and the nonlocal function g satisfy the noncompactness measure condition

(H7) There exist nonnegative constants L, M_k $(k = 1, 2, \ldots, m)$ and R with

$$2\overline{M}\left[R + 2a(L + M + 2aLK_0) + \sum_{k=1}^{m} M_k\right] < 1$$

such that

$$\begin{aligned}\alpha(\{f(t, u_n, v_n)\}) &\leqslant L(\alpha(\{u_n\}) + \alpha(\{v_n\})),\\ \alpha(\{I_k(w_n(t_k))\}) &\leqslant M_k\alpha(\{w_n(t_k)\}), \quad k = 1, 2, \ldots, m,\\ \alpha(\{g(x_n)\}) &\leqslant R\alpha(\{x_n\})\end{aligned}$$

for all $t \in J$, equicontinuous and countable sets $\{u_n\}, \{w_n\} \subset [v_0(t), w_0(t)]$, $\{v_n\} \subset [Gv_0(t), Gw_0(t)]$ and $\{x_n\} \subset [v_0, w_0]$,

we have the following existence result.

Theorem 5. *Let E be an ordered Banach space whose positive cone P is normal, let $A\colon D(A)\subset E\to E$ be a closed linear operator and let $-A$ generate a positive and equicontinuous C_0-semigroup $T(t)$ $(t\geqslant 0)$ on E, $f\in C(J\times E\times E,E)$, $I_k\in C(E,E)$, $k=1,2,\dots,m$, and let $g\colon PC(J,E)\to E$ be a continuous function. If the problem* (1) *has a lower solution $v_0\in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ and an upper solution $w_0\in PC(J,E)\cap C^1(J'',E)\cap C(J',E_1)$ with $v_0\leqslant w_0$, and conditions* (H1), (H2), (H3), *and* (H7) *hold, then the problem* (1) *has a minimal mild solution $\underline{u}$ and a maximal mild solution $\bar u$ in $[v_0,w_0]$; moreover,*

$$v_n(t)\to\underline{u}(t),\quad w_n(t)\to\bar u(t),\quad (n\to\infty)\ \text{uniformly for } t\in J,$$

where $v_n(t)=Qv_{n-1}(t)$, $w_n(t)=Qw_{n-1}(t)$ satisfy

$$\begin{aligned}v_0(t)\leqslant v_1(t)\leqslant\dots\leqslant v_n(t)\leqslant\dots\leqslant\underline{u}(t)\leqslant\bar u(t)\leqslant\dots\\ \leqslant w_n(t)\leqslant\dots\leqslant w_1(t)\leqslant w_0(t)\quad\forall t\in J.\end{aligned}$$

Proof. From the proof of Theorems 1 and 2, we know that $Q\colon[v_0,w_0]\to[v_0,w_0]$ is continuous, and for any $D\subset[v_0,w_0]$, $Q(D)$ is bounded and equicontinuous. So, by Lemma 3, there exists a countable set $D_0=\{u_n\}\subset D$ such that

$$\alpha(Q(D))\leqslant 2\alpha(Q(D_0)). \tag{19}$$

By assumption (H7) and Lemma 2, we have

$$\begin{aligned}\alpha(Q(D_0)(t))&=\alpha\biggl(\Bigl\{S(t)(g(u_n)+x_0)+\int_0^t S(t-s)[f(s,u_n(s),Gu_n(s))+Mu_n(s)]\,\mathrm{d}s\\&\qquad+\sum_{0<t_k<t}S(t-t_k)I_k(u_n(t_k))\Bigr\}\biggr)\\&\leqslant\|S(t)\|\alpha(\{g(u_n)+x_0\})\\&\qquad+2\int_0^t\|S(t-s)\|\alpha(f(s,D_0(s),G(D_0)(s))+MD_0(s))\,\mathrm{d}s\\&\qquad+\sum_{0<t_k<t}\|S(t-t_k)\|\alpha(I_k(D_0(t_k)))\\&\leqslant\overline{M}R\alpha(D_0)+2\overline{M}\int_0^t[L(\alpha(D_0(s))+\alpha(G(D_0)(s)))+M\alpha(D_0(s))]\,\mathrm{d}s\\&\qquad+\overline{M}\sum_{0<t_k<t}M_k\alpha(D_0(t_k))\\&\leqslant\Bigl[\overline{M}R+2\overline{M}a(L+M)+2\overline{M}aL\cdot 2aK_0+\overline{M}\sum_{0<t_k<t}M_k\Bigr]\alpha(D).\end{aligned}$$

Since $Q(D_0)$ is equicontinuous, by Lemma 1 we have $\alpha(Q(D_0)) = \max\limits_{t\in J}\alpha(Q(D_0)(t))$. Combining this with (19), we have

$$\alpha(Q(D)) \leqslant \gamma\alpha(D), \tag{20}$$

where $\gamma = 2\overline{M}\Big[R + 2a(L + M + 2aLK_0) + \sum\limits_{k=1}^{m} M_k\Big] < 1$.

Therefore, $Q\colon [v_0, w_0] \to [v_0, w_0]$ is a strict set contraction operator. Hence, our conclusion follows from Lemma 4. □

Remark 4. An analytic semigroup and a differentiable semigroup are equicontinuous semigroups [37]. In the application of partial differential equations, such as parabolic and strongly damped wave equations, the corresponding solution semigroups are analytic semigroups. So, Theorem 5 in this paper has broad applicability.

Remark 5. If $A \equiv 0$ and $g \equiv 0$, then Theorem 5 in this paper is a generalization of Theorem 1 in [23].

4. Applications

In this section, we give two examples to illustrate our abstract results obtained in Section 3.

First, consider the impulsive parabolic partial differential equation with nonlocal conditions

$$\begin{cases} \dfrac{\partial}{\partial t}u(x,t) - \dfrac{\partial^2}{\partial x^2}u(x,t) = f(x,t,u(x,t),Gu(x,t)), \quad x \in [0,\pi],\ t \in J,\ t \neq t_k, \\ \Delta u|_{t=t_k} = I_k(u(x,t_k)), \quad x \in [0,\pi],\ k = 1,2,\ldots,m, \\ u(0,t) = u(\pi,t) = 0, \quad t \in J, \\ u(x,0) = \displaystyle\int_0^a h(s)\log(1 + |u(x,s)|)\,\mathrm{d}s + \varphi(x), \quad x \in [0,\pi], \end{cases} \tag{21}$$

where $Gu(x,t) = \int_0^t K(t,s)u(x,s)\,\mathrm{d}s$, $J = [0,a]$, $0 < t_1 < t_2 < \ldots < t_m < a$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $J'' = J \setminus \{0, t_1, t_2, \ldots, t_m\}$, $f\colon [0,\pi] \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $I_k\colon \mathbb{R} \to \mathbb{R}$ is also continuous, $k = 1,2,\ldots,m$, $h(\cdot) \in L^1(J,\mathbb{R}^+)$, $\varphi \in L^2[0,\pi]$.

Let $E = L^2[0,\pi]$ with the norm $\|\cdot\|_2$, $P = \{u \in L^2[0,\pi];\ u(x) \geqslant 0 \text{ a.e. } x \in [0,\pi]\}$, and let us consider the operator $A\colon D(A) \subset E \to E$ defined by

$$D(A) = \{u \in L^2[0,\pi];\ u', u'' \in L^2[0,\pi],\ u(0) = u(\pi) = 0\}, \quad Au = -u''.$$

Then E is a Banach space, P is a regular cone of E, and $-A$ generates a positive, compact and analytic C_0-semigroup $T(t)$ ($t \geqslant 0$) on E (see [20], [37]). From [32] we know that g is a compact operator. Let $f(t, u(t), Gu(t)) = f(\cdot, t, u(\cdot, t), Gu(\cdot, t))$, $I_k(u(t_k)) = I_k(u(\cdot, t_k))$, $g(u) = \int_0^a h(s) \log(1 + |u(\cdot, s)|) \, ds$, $x_0 = \varphi(\cdot)$, then the problem (21) can be transformed into the form of problem (1).

Theorem 6. *If the conditions*

(F1) *let $f(x, t, 0, 0) \geqslant 0$, $I_k(0) \geqslant 0$, $\varphi(x) \geqslant 0$, $x \in \Omega$, and there exist a function $w = w(x, t) \in PC([0, \pi] \times J) \cap C^1([0, \pi] \times J'')$ such that*

$$\begin{cases} \dfrac{\partial w}{\partial t} - \dfrac{\partial^2 w}{\partial x^2} \geqslant f(x, t, w, Gw), & (x, t) \in [0, \pi] \times J, \ t \neq t_k, \\ \Delta w|_{t=t_k} \geqslant I_k(w(x, t_k)), & x \in [0, \pi], \ k = 1, 2, \ldots, m, \\ w(0, t) = w(\pi, t) = 0, & t \in J, \\ w(x, 0) \geqslant \displaystyle\int_0^a h(s) \log(1 + |w(x, s)|) \, ds + \varphi(x), & x \in [0, \pi], \end{cases}$$

(F2) *there exists a constant $M > 0$ such that*

$$f(x, t, u_2, v_2) - f(x, t, u_1, v_1) \geqslant -M(u_2 - u_1)$$

for any $t \in J$, and $0 \leqslant u_1 \leqslant u_2 \leqslant w(x, t)$, $0 \leqslant v_1 \leqslant v_2 \leqslant Gw(x, t)$,

(F3) *for any $u_1, u_2 \in [0, w(x, t)]$ with $u_1 \leqslant u_2$ we have*

$$I_k(u_1(x, t_k)) \leqslant I_k(u_2(x, t_k)), \quad x \in [0, \pi], \ k = 1, 2, \ldots, m,$$

hold, then the problem (21) *has a minimal mild solution and a maximal mild solution between* 0 *and $w(x, t)$, which can be obtained by a monotone iterative procedure starting from* 0 *and $w(x, t)$, respectively.*

Proof. Assumption (F1) implies that $v_0 \equiv 0$ and $w_0 = w(x, t)$ are the lower and upper solutions of the problem (21), respectively. From assumptions (F2), (F3), and the definition of the operator g it is easy to verify that conditions (H1), (H2), and (H3) are satisfied. So, our conclusion follows from Theorem 1. □

To complete this section, we consider the nonlocal problem of impulsive parabolic partial integro-differential equation

$$(22) \qquad \begin{cases} \dfrac{\partial}{\partial t} u(x, t) + A(x, D) u(x, t) = f(x, t, u(x, t), Gu(x, t)), & x \in \Omega, \ t \in J', \\ \Delta u|_{t=t_k} = I_k(u(x, t_k)), & x \in \Omega, \ k = 1, 2, \ldots, m, \\ u|_{\partial\Omega} = 0, \\ u(x, 0) = g(u) + \varphi(x), & x \in \Omega, \end{cases}$$

where $J = [0, a]$, $0 < t_1 < t_2 < \ldots < t_m < a$, $N \geqslant 1$ is an integer, $\Omega \subset \mathbb{R}^N$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$,

$$A(x, D) = -\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big)$$

is a uniformly elliptic differential operator of divergence form in $\overline{\Omega}$ with coefficients $a_{ij} \in C^{1+\mu}(\overline{\Omega})$ $(i, j = 1, 2, \ldots, N)$ for some $\mu \in (0, 1)$. That is, $[a_{ij}(x)]_{N\times N}$ is a positive definite symmetric matrix for every $x \in \overline{\Omega}$ and there exists a constant $\nu > 0$ such that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}(x)\eta_i\eta_j \geqslant \nu|\eta|^2 \quad \forall\eta = (\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^N,\ x \in \overline{\Omega},$$

$f\colon \overline{\Omega} \times J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $I_k\colon \mathbb{R} \to \mathbb{R}$ is also continuous, $k = 1, 2, \ldots, m$, g is a continuous mapping.

Let $E = L^p(\Omega)$ with $p \geqslant 2$, $P = \{u \in L^p(\Omega);\ u(x) \geqslant 0 \text{ a.e. } x \in \Omega\}$, and define operator $A_P\colon D(A_P) \subset E \to E$ as follows:

$$D(A_P) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad A_P u = A(x, D)u.$$

It is well known that E is a Banach space, P is a regular cone of E, and $-A_P$ generates a positive and analytic C_0-semigroup $T_P(t)$ $(t \geqslant 0)$ on E. Let $f(t, u(t), Gu(t)) = f(\cdot, t, u(\cdot, t), Gu(\cdot, t))$, $I_k(u(t_k)) = I_k(u(\cdot, t_k))$, $x_0 = \varphi(\cdot)$. Then the problem (22) can be rewritten into the abstract form (1).

Theorem 7. *If the condition* (F2) *and the conditions*

(F4) *there exist* $C \geqslant 0$, $h \in PC(\Omega \times J) \cap C^1(\overline{\Omega} \times J')$, $h(x, t) \geqslant 0$, $y_k \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $y_k(x) \geqslant 0$, $k = 1, 2, \ldots, m$, $\varphi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $\varphi(x) \geqslant 0$ *and* $g \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $g(u) \geqslant 0$, *such that*

$$f(x, t, u, Gu) \leqslant Cu + h(x, t),\ I_k(u) \leqslant y_k, \quad u \geqslant 0,$$
$$f(x, t, u, Gu) \geqslant Cu - h(x, t),\ I_k(u) \geqslant -y_k, \quad u \leqslant 0,$$

(F5) *for any* u_1, u_2 *in any bounded and ordered interval with* $u_1 \leqslant u_2$ *we have*

$$I_k(u_1(x, t_k)) \leqslant I_k(u_2(x, t_k)), \quad x \in \Omega,\ k = 1, 2, \ldots, m,\ g(u_1) \leqslant g(u_2),$$

hold, then the problem (22) *has a minimal mild solution and a maximal mild solution.*

P r o o f. First, we consider the nonlocal problem of linear impulsive parabolic partial differential equation

$$
(23)\qquad \begin{cases} \dfrac{\partial}{\partial t} u(x,t) + A(x,D)u(x,t) - Cu(x,t) = h(x,t), & x \in \Omega,\ t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad x \in \Omega,\ k = 1,2,\dots,m, \\ u|_{\partial\Omega} = 0, \\ u(x,0) = g(u) + \varphi(x), \quad x \in \Omega. \end{cases}
$$

From the above argument, problem (23) can be transformed into the abstract form

$$
(24)\qquad \begin{cases} u'(t) + (A_P - CI)u(t) = h(t), \quad t \in J', \\ \Delta u|_{t=t_k} = y_k, \quad k = 1,2,\dots,m, \\ u(0) = g(u) + x_0, \end{cases}
$$

where $h(t) = h(\cdot,t)$. Since $-(A_P - CI)$ generates a positive C_0-semigroup $S_P(t) = \mathrm{e}^{Ct} T_P(t)$ $(t \geqslant 0)$ on E, from [35] we know that for the linear problem (24) there exists a unique positive classical solution $u^* \in PC(J,E) \cap C^1(J'',E) \cap C(J',E_1)$. Let $v_0 = -u^*$, $w_0 = u^*$; from the assumption (F4) we know that v_0 and w_0 are the lower and the upper solutions of problem (1), respectively. From assumptions (F2) and (F5) it is easy to verify that conditions (H1), (H2), and (H3) are satisfied. Therefore, by Corollary 2, the problem (22) has a minimal mild solution and a maximal mild solution. □

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