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DYNAMIC ANALYSIS OF AN IMPULSIVE DIFFERENTIAL EQUATION WITH TIME-VARYING DELAYS

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Abstract. An impulsive differential equation with time varying delay is proposed in this paper. By using some analysis techniques with combination of coincidence degree theory, sufficient conditions for the permanence, the existence and global attractivity of positive periodic solution are established. The results of this paper improve and generalize some previously known results.

Keywords: periodic solution; permanence; attractivity; impulse; delay

MSC 2010: 34D23, 34D20, 93D20

1. Introduction

In natural world, for studying the control of a single population of cells, Nazarenko [9] proposed the following model:

$$x'(t) + px(t) - \frac{qx(t)}{r + x^n(t-\tau)} = 0. \tag{1}$$

The author established conditions for oscillation of all positive solutions about the unique positive fixed point and proved that every nonoscillatory solution tend to the fixed point.

Considering the effects of the periodically varying environment [11], [5], and the abrupt change of state (i.e., the effects of impulse, see [7], [1]), Saker and Alzabut

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[10] proposed the following impulsive delay population model:

$$(2)\qquad \begin{cases} x'(t) + p(t)x(t) - \dfrac{q(t)x(t)}{r + x^n(t - m\omega)} = \lambda(t), \quad t \neq t_k, \\ x(t_k^+) = \dfrac{1}{1 + b_k} x(t_k). \end{cases}$$

By using Brouwer's fixed point theorem and the comparison method, they studied the qualitative behavior of the model including the existence of periodic solutions, global attractivity and oscillation.

Delays often affect the dynamics of ecological systems, see [6]. The authors of [9], [10] investigated the effect of time delay on the dynamical behavior, but they all assumed that the delay is constant. However, in real world, delay is not always constant. It is often time-varying [2], [12]. Then how a time-varying delay affects the dynamical behavior of the system?

Motivated by the above discussion, in this paper we study the impulsive differential equation with periodic parameters and time-varying delay

$$(3)\qquad \begin{cases} x'(t) + p(t)x(t) - \dfrac{q(t)x(t)}{r + x^n(t - \sigma(t))} = 0, \quad t \neq t_k, \\ x(t_k^+) = \dfrac{1}{1 + b_k} x(t_k). \end{cases}$$

Eq. (3) is accompanied with the initial condition

$$y(t) = \varphi(t), \quad -r \leqslant t \leqslant 0, \ \varphi \in L([-r, 0], [0, \infty)), \ \varphi(0) > 0,$$

where $x(t_k^+) = \lim\limits_{t \to t_k^+} x(t)$, $L([-r, 0], [0, \infty))$ denotes the set of Lebesgue measurable functions on $[-r, 0]$ and $-r = \inf\limits_{t \geqslant 0} \{t - \sigma(t)\}$.

By employing Mawhin's continuation theorem and a comparison theorem, we aim at studying the permanence, existence and global attractivity of positive periodic solutions of system (3). It is of biological significance.

Obviously, by the transformation $x(t) = 1/(y(t))$, model (3) leads to the impulsive delay differential equation

$$(4)\qquad \begin{cases} y'(t) = y(t)\big(p(t) - \dfrac{Q(t)y^n(t - \sigma(t))}{R + y^n(t - \sigma(t))}\big), \quad t \neq t_k, \\ y(t_k^+) = (1 + b_k)y(t_k), \end{cases}$$

where $Q(t) = q(t)/r$, $R = 1/r$.

Hence, in order to study the dynamics of (3), it suffices to study the qualitative behavior of system (4).

For system (4), we assume that:

(H$_1$) $0 < t_1 < t_2 < \ldots$ are fixed impulsive points such that $\lim\limits_{k\to\infty} t_k = \infty$;

(H$_2$) $p(t), Q(t) \in ([0,\infty),(0,\infty))$ are locally summable functions, $\sigma \in ([0,\infty),(0,\infty))$ is a Lebesgue measurable function;

(H$_3$) $R > 0$ is a constant, $\{b_k\}$ is a real sequence such that $b_k > -1$, $k = 1, 2, \ldots$;

(H$_4$) $p(t)$, $Q(t)$, $\sigma(t)$, $\prod\limits_{0<t_k<t}(1+b_k)$ are positive periodic functions with period ω, and if the number of factors is zero, then the product is equal to unit.

In this paper, for convenience, we use the following notation:

$$f^L = \min_{0\leqslant t\leqslant\omega} f(t), \quad f^M = \max_{0\leqslant t\leqslant\omega} f(t), \quad \overline{f} = \frac{1}{\omega}\int_0^\omega f(t)\,\mathrm{d}t,$$

where f is a positive ω-periodic function.

This paper is organized as follows. In Section 2, preliminaries are introduced. In Section 3, the permanence and existence of a periodic solution of system (4) are studied. In Section 4, the uniqueness and global attractivity of a positive periodic solution of (4) are investigated. Finally, in Section 5, a brief discussion and an example are given to conclude this paper.

2. Preliminaries

In this section, we introduce some definitions and lemmas.

Definition 1. A function $y(t) \in ([-r,\infty),(0,\infty))$ is said to be a solution of (4) on $[-r,\infty)$, if

(i) $y(t)$ is absolutely continuous on each interval $[0,t_1]$ and $(t_k,t_{k+1}]$, $k = 1, 2, \ldots$,

(ii) for any t_k, $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k)$, $k = 1, 2, \ldots$,

(iii) $y(t)$ satisfies (4).

Definition 2. Suppose that $y_1(t)$ and $y_2(t)$ are two positive solutions of (4) on $[-r,\infty)$. The solution $y_2(t)$ is said to be asymptotically attractive to $y_1(t)$ provided that

$$\lim_{t\to\infty}(y_1(t) - y_2(t)) = 0.$$

Further, $y_2(t)$ is called globally attractive if $y_2(t)$ is asymptotically attractive to all positive solutions of (4).

Definition 3. A solution $y_1(t)$ of (4) is said to oscillate about $y_2(t)$ if $y_1(t)-y_2(t)$ has arbitrarily large zeros. Otherwise, $y_1(t)$ is called nonoscillatory.

Consider the nonimpulsive delay differential equation

$$z'(t) = \Big(p(t) - \frac{Q(t)z^n(t-\sigma(t))}{R(t) + z^n(t-\sigma(t))}\Big)z(t) \tag{5}$$

with the initial condition

$$z(t) = \varphi(t), \quad -r \leqslant t \leqslant 0,\ \varphi \in L([-r,0),(0,\infty)),\ \varphi(0) > 0, \tag{6}$$

where $R(t) = R/\prod\limits_{0<t_k<t-\sigma(t)} (1+b_k)^n$. By a solution of (5) and (6), we mean an absolutely continuous function $z(t)$ defined on $[-r,\infty)$ satisfying (5) almost everywhere for $t \geqslant 0$ and $z(t) = \varphi(t)$ on $[-r,0]$.

The following lemma will be used in the proof of our results; its proof is similar to that of Theorem 1 in [13] and hence is omitted.

Lemma 1. *Assume that* (H_1)–(H_4) *hold. Then*

(1) *if* $y(t)$ *is a solution of* (4) *on* $[-r,\infty)$, *then* $z(t) = \prod\limits_{0<t_k<t} (1+b_k)y(t)$ *is a solution of* (5) *on* $[-r,\infty)$;

(2) *if* $z(t)$ *is a solution of* (5) *on* $[-r,\infty)$, *then* $y(t) = \prod\limits_{0<t_k<t} (1+b_k)^{-1}z(t)$ *is a solution of* (4) *on* $[-r,\infty)$.

It is clear that the transformation $z(t) = \prod\limits_{0<t_k<t} (1+b_k)y(t)$ preserves the asymptotic properties of Eq. (4). Thus, in the proof of the asymptotic properties of (4), it suffices to consider the asymptotic properties of (5).

Lemma 2. *Assume that* (H_1)–(H_4) *hold. Then the solutions of* (4) *are defined on* $[-r,\infty)$ *and are positive on* $[0,\infty)$.

P r o o f. Clearly, by Lemma 1, we only need to prove that the solutions of (5) and (6) are defined on $[-r,\infty)$ and are positive on $[0,\infty)$. From (5) and (6), it is easy to obtain

$$z(t) = \varphi(0)\exp\bigg(\int_0^t \Big(p(s) - \frac{Q(s)z^n(s-\sigma(s))}{R(s) + z^n(s-\sigma(s))}\Big)\,\mathrm{d}s\bigg).$$

The assertion of the lemma follows immediately for all $t \in [0,\infty)$. The proof is complete. $\square$

The following lemma will be used repeatedly in the proof of our results. Its proof is straightforward and is omitted.

Lemma 3. *Assume that p, q, r are positive constants and let*

$$f(x) = p - \frac{qx^n}{r + x^n}, \quad x \geqslant 0.$$

Then

(i) *there exists a unique positive constant x_0 such that $f(x_0) = 0$ and*

$$f(x) > 0 \quad \text{for } 0 \leqslant x < x_0; \qquad f(x) < 0 \quad \text{for } x_0 < x < \infty.$$

(ii) *$f(x)$ attains its maximum at $x = 0$. Further, $f(x)$ is decreasing for $x \geqslant 0$.*

Lemma 4 ([3])**.** *Let X and Z be two Banach spaces, $L\colon \operatorname{Dom} L \subset X \to Z$ a Fredholm operator with index zero. Let $\Omega \subset X$ be an open bounded set, and let $N\colon \overline{\Omega} \to Z$ be L-compact on $\overline{\Omega}$. Assume that:*

(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*
(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Lemma 5 ([4])**.** *Let a, σ satisfy*

(i) *$a, \sigma \in ([0,\infty), [0,\infty))$, a is a locally summable function, σ is a bounded Lebesgue measurable function, $\sigma^* = \sup_{t \geqslant 0} \sigma(t)$;*
(ii)

$$\limsup_{t\to\infty} \int_t^{t+\sigma^*} a(s)\,\mathrm{d}s < \frac{3}{2} \quad \text{and} \quad \liminf_{t\to\infty} \int_t^{t+\sigma^*} a(s)\,\mathrm{d}s > 0.$$

Then all nontrivial solutions of $y'(t) + a(t)y(t - \sigma(t)) = 0$ satisfy $\lim_{t\to\infty} y(t) = 0$.

3. Permanence and existence of periodic solution

In this section, the permanence and existence of periodic solution of (4) are investigated.

First, we study the permanence of model (4). By Lemma 1, we only need to establish the permanence of (5). For $p(t)$ and $Q(t)$, we define two functions as follows:

$$f_1(z) = p^L - \frac{Q^M z^n}{R^L + z^n}, \quad f_2(z) = p^M - \frac{Q^L z^n}{R^M + z^n}. \tag{7}$$

Due to Lemma 3, there exist z_1 and z_2 such that

$$f_1(z_1) = 0, \quad f_2(z_2) = 0 \quad \text{with } 0 < z_1 \leqslant z_2,$$

where z_1, z_2 are the unique zero points respectively for $f_1(z)$ and $f_2(z)$. In the rest of this paper, we always assume that the roots of $f_1(z) = 0$ and $f_2(z) = 0$ are z_1 and z_2, respectively.

Theorem 1. *Assume that* (H_1)–(H_4) *hold. Then system* (4) *is permanent, that is, if* $y(t)$ *is a solution of* (4), *then there exists* $T > r > 0$ *such that*

$$\prod_{0<t_k<t} (1+b_k) z_1 \mathrm{e}^{-\mu_1} \leqslant y(t) \leqslant \prod_{0<t_k<t} (1+b_k) z_2 \mathrm{e}^{\mu_2}, \quad t \geqslant T, \tag{8}$$

where

$$\mu_2 = \sup_{t\to\infty} \int_{t-\sigma(t)}^{t} p(s)\,\mathrm{d}s, \quad \mu_1 = \sup_{t\to\infty} \int_{t-\sigma(t)}^{t} \Big(\frac{Q(s) z_2^n \mathrm{e}^{n\mu_2}}{R(s) + z_2^n \mathrm{e}^{n\mu_2}} - p(s) \Big)\,\mathrm{d}s.$$

P r o o f. First, we study the permanence of (5). Then by using Lemma 1 we prove that (8) holds for $t \geqslant T$, which leads to the permanence of system (4). By Lemma 2, the solutions of (5) are positive on $[0, \infty)$. There are four cases.

Case 1: the solution $z(t)$ of (5) oscillates about z_2 and satisfies $\sup\limits_{t\geqslant 0} z(t) > z_2$.

Then there exist two sequences of $\{t_n\}$ and $\{\xi_n\}$ such that

$$r < t_1 < t_2 < \ldots < t_n < t_{n+1} < \ldots, \quad \lim_{n\to\infty} t_n = \infty \quad \text{and} \quad z(t_n) = z_2,$$

$z(\xi_n)$ is the maximum of $z(t)$ on (t_n, t_{n+1}) with $z(\xi_n) > z_2$, $n = 1, 2, \ldots$ Thus, for any $\varepsilon > 0$ small enough, there exist $\delta > 0$ and ξ_n^* such that $\xi_n^* \in (\xi_n - \delta, \xi_n]$ with $z'(\xi_n^*) \geqslant 0$, $z(\xi_n^*) > z_2$ and

$$z(\xi_n) - z(\xi_n^*) < \varepsilon, \quad n = 1, 2, \ldots \tag{9}$$

From (5) we have

$$0 \leqslant z'(\xi_n^*) \leqslant z(\xi_n^*) \Big(p^M - \frac{Q^L z^n(\xi_n^* - \sigma(\xi_n^*))}{R^M + z^n(\xi_n^* - \sigma(\xi_n^*))} \Big).$$

Hence,

$$p^M - \frac{Q^L z^n(\xi_n^* - \sigma(\xi_n^*))}{R^M + z^n(\xi_n^* - \sigma(\xi_n^*))} > 0.$$

By Lemma 3 we get $z(\xi_n^* - \sigma(\xi_n^*)) < z_2$. Let ξ_n^0 be a zero of $z(t) - z_2$ in $(\xi_n^* - \sigma(\xi_n^*)) \cap [t_n, \xi_n^*)$, i.e., $z(\xi_n^0) = z_2$.

According to Lemma 3 and (5), for any $t > r$ we have

$$z'(t) = z(t)\Big(p(t) - \frac{Q(t)z^n(t-\sigma(t))}{R(t) + z^n(t-\sigma(t))}\Big) \leqslant z(t)p(t). \tag{10}$$

Integrating (10) from ξ_n^0 to ξ_n^* leads to

$$0 < \ln \frac{z(\xi_n^*)}{z(\xi_n^0)} \leqslant \int_{\xi_n^0}^{\xi_n^*} p(s)\,\mathrm{d}s \leqslant \int_{\xi_n^* - \sigma(\xi_n^*)}^{\xi_n^*} p(s)\,\mathrm{d}s \leqslant \mu_2, \quad n = 1, 2, \ldots \tag{11}$$

It follows from (9) and (11) that $z(\xi_n) \leqslant z(\xi_n^0)\mathrm{e}^{\mu_2}$, that is

$$z(t) \leqslant z_2 \mathrm{e}^{\mu_2}. \tag{12}$$

Case 2: the solution $z(t)$ of (5) is nonoscillatory about z_2.

Then we claim that for any $\varepsilon > 0$ there exists $T_1 > r$ such that

$$z(t) < z_2 + \varepsilon \quad \text{for all } t \geqslant T. \tag{13}$$

Otherwise, since $z(t) > z_2$, by (6) and Lemma 3 we have

$$z'(t) \leqslant z(t)\Big(p^M - \frac{Q^L z^n(t-\sigma(t))}{R^M + z^n(t-\sigma(t))}\Big) \leqslant z(t)\Big(p^M - \frac{Q^L(z_2+\varepsilon)^n}{R^M + (z_2+\varepsilon)^n}\Big) < 0,$$

which contradicts $z(t) > 0$ for all $t \geqslant 0$. This implies that (13) holds. Then there exists $T_2 > T_1$ such that for all $t \geqslant T_2$, $z(t) \leqslant z_2 \mathrm{e}^{\mu_2}$.

Case 3: the solution $z(t)$ of (5) oscillates about z_1 and $\inf\limits_{t\geqslant 0} z(t) < z_1$.

Then there exist two sequences $\{s_n\}$ and $\{\eta_n\}$ such that

$$0 < s_1 < s_2 < \ldots < s_n < s_{n+1} < \ldots, \quad \lim_{n\to\infty} s_n = \infty, \quad z(s_n) = z_1,$$

and $z(\eta_n)$ is the minimum of $z(t)$ on (s_n, s_{n+1}) with $z(\eta_n) < z_1$, $n = 1, 2, \ldots$ Similarly, for any $\varepsilon > 0$ small enough, there exist $\delta > 0$ and $\eta_n^* \in (\eta_n - \delta, \eta_n]$ with $z'(\eta_n^*) \leqslant 0$, $z(\eta_n^*) < z_1$ and

$$z(\eta_n^*) - z(\eta_n) < \varepsilon, \quad n = 1, 2, \ldots \tag{14}$$

From (5) we have

$$0 \geqslant z'(\eta_n^*) \geqslant z(\eta_n^*)\Big(p^L - \frac{Q^M z^n(\eta_n^* - \sigma(\eta_n^*))}{R^L + z^n(\eta_n^* - \sigma(\eta_n^*))}\Big).$$

In view of Lemma 3, then $z(\eta_n^* - \sigma(\eta_n^*)) > z_1$ and there exists $\eta_n^0 \in (\eta_n^* - \sigma(\eta_n^*), \eta_n^*)$ such that $z(\eta_n^0) = z_1$, $n = 1, 2, \dots$ Integrating (5) from η_n^0 to η_n^* for any $\eta_n^0 > T_2$, we can derive that

$$0 > \ln \frac{z(\eta_n^*)}{z(\eta_n^0)} \geqslant \int_{\eta_n^0}^{\eta_n^*} \Big(p(s) - \frac{Q(s) z_2^n \mathrm{e}^{n\mu_2}}{R(s) + z_2^n \mathrm{e}^{n\mu_2}}\Big)\, \mathrm{d}s.$$

Therefore,

$$\begin{aligned} z(\eta_n^*) &\geqslant z_1 \exp \int_{\eta_n^0}^{\eta_n^*} \Big(p(s) - \frac{Q(s) z_2^n \mathrm{e}^{n\mu_2}}{R(s) + z_2^n \mathrm{e}^{n\mu_2}}\Big)\, \mathrm{d}s \\ &\geqslant z_1 \exp \int_{\eta_n^* - \sigma(\eta_n^*)}^{\eta_n^*} \Big(p(s) - \frac{Q(s) z_2^n \mathrm{e}^{n\mu_2}}{R(s) + z_2^n \mathrm{e}^{n\mu_2}}\Big)\, \mathrm{d}s \\ &\geqslant z_1 \mathrm{e}^{-\mu_1}. \end{aligned}$$

Thus for any $t > T_2$, $z(t) \geqslant z(\eta_n) \geqslant z(\eta_n^*) - \varepsilon$ holds, that is

$$z(t) \geqslant z_1 \mathrm{e}^{-\mu_1}. \tag{15}$$

Case 4: the solution $z(t)$ of (5) is nonoscillatory about z_1 and $z(t) < z_1$.

We prove that for any $\varepsilon > 0$ there exists $T_3 > T_2$ such that $z(t) > z_1 - \varepsilon$.

If not, since $z(t) < z_1$, in view of Lemma 3, for any $t \geqslant \widetilde{T}$ there exists $\widetilde{T} > T_3$ such that

$$z'(t) \geqslant z(t)\Big(p^L - \frac{Q^M z^n(t - \sigma(t))}{R^L + z^n(t - \sigma(t))}\Big) \geqslant z(t)\Big(p^L - \frac{Q^M (z_1 - \varepsilon)^n}{R^L + (z_1 - \varepsilon)^n}\Big) > 0,$$

which leads to contradiction with $z(t) \leqslant z_2 \mathrm{e}^{\mu_2}$. By using Lemma 3 again, we have

$$p(t) - \frac{Q(t) z_2^n \mathrm{e}^{n\mu_2}}{R(t) + z_2^n \mathrm{e}^{n\mu_2}} < 0.$$

Hence,

$$\mu_1 = \sup_{t \to \infty} \int_{t - \sigma(t)}^{t} \Big(\frac{Q(s) z_2^n \mathrm{e}^{n\mu_2}}{R(s) + z_2^n \mathrm{e}^{n\mu_2}} - p(s)\Big)\, \mathrm{d}s > 0.$$

This implies that there exists $T_3 > T_2$ such that $z(t) \geqslant z_1 \mathrm{e}^{-\mu_1}$ holds for all $T \geqslant T_3$.

According to (12) and (15), we conclude that

$$z_1 \mathrm{e}^{-\mu_1} \leqslant z(t) \leqslant z_2 \mathrm{e}^{\mu_2} \quad \text{for all sufficiently large } t. \tag{16}$$

By Lemma 1, then (8) holds. The proof of Theorem 1 is complete. □

Theorem 2. *Suppose that* (H$_1$)–(H$_4$) *hold. Then system* (4) *has at least one* ω*-periodic solution* $\overline{y}(t)$.

Proof. By Lemma 1, we only need to prove that (5) has at least one ω-periodic solution $\overline{z}(t)$. Using the transformation $z(t) = \mathrm{e}^{x(t)}$, (5) leads to

$$x'(t) = p(t) - \frac{Q(t)\mathrm{e}^{nx(t-\sigma(t))}}{R(t) + \mathrm{e}^{nx(t-\sigma(t))}}, \quad t \geqslant 0. \tag{17}$$

Take $X = Z = \{x \in C([0,\infty), R)\colon\ x(t+\omega) = x(t)\}$ with the norm $\|x\| = \max\limits_{t\in[0,\omega]} |x(t)|$. Then both X and Z are Banach spaces. Define

$$L\colon\ \operatorname{Dom} L \cap X \to Z,\ Lx = x', \quad N\colon\ X \to Z,\ Nx = p(t) - \frac{Q(t)\mathrm{e}^{nx(t-\sigma(t))}}{R(t) + \mathrm{e}^{nx(t-\sigma(t))}},$$

where $\operatorname{Dom} L = \{x \in C'[0,\omega]\colon\ x(t+\omega) = x(t)\}$. Define operators P, Q as follows:

$$Px = \frac{1}{\omega}\int_0^\omega x(t)\,\mathrm{d}t, \quad Qz = \frac{1}{\omega}\int_0^\omega z(t)\,\mathrm{d}t.$$

Then $\operatorname{Ker} L = \{x \in X\colon\ x = h \in R\}$, $\operatorname{Im} L = \bigl\{z \in Z\colon\ \int_0^\omega z(t)\,\mathrm{d}t = 0\bigr\}$, and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1$. It is easy to obtain that the operators P, Q are continuous and satisfy $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. Hence, $\operatorname{Im} L$ is closed in Z and L is a Fredholm mapping of index zero.

Denote $L_p = L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}$. Then the generalized inverse $K_p = L_P^{-1}$ is given by

$$K_P(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t z(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Therefore,

$$\begin{aligned}
QNx &= \frac{1}{\omega}\int_0^\omega \Bigl(p(t) - \frac{Q(t)\mathrm{e}^{nx(t-\sigma(t))}}{R(t) + \mathrm{e}^{nx(t-\sigma(t))}}\Bigr)\,\mathrm{d}t,\\
K_P(I-Q)Nx &= \int_0^t \Bigl(p(s) - \frac{Q(s)\mathrm{e}^{nx(s-\sigma(s))}}{R(s) + \mathrm{e}^{nx(s-\sigma(s))}}\Bigr)\,\mathrm{d}s\\
&\quad - \frac{1}{\omega}\int_0^\omega\int_0^t \Bigl(p(s) - \frac{Q(s)\mathrm{e}^{nx(s-\sigma(s))}}{R(s) + \mathrm{e}^{nx(s-\sigma(s))}}\Bigr)\,\mathrm{d}s\,\mathrm{d}t\\
&\quad - \Bigl(\frac{t}{\omega} - \frac{1}{2}\Bigr)\int_0^t \Bigl(p(s) - \frac{Q(s)\mathrm{e}^{nx(s-\sigma(s))}}{R(s) + \mathrm{e}^{nx(s-\sigma(s))}}\Bigr)\,\mathrm{d}s.
\end{aligned}$$

Clearly both QN and $K_P(I-Q)N$ are continuous. By employing the Arzela-Ascoli theorem, it is easy to show that for any open set $\Omega \subset X, K_P(I-Q)N(\overline{\Omega})$ is compact and $QN(\overline{\Omega})$ is bounded. Thus N is L-compact on $\overline{\Omega}$ for any open set $\Omega \subset X$.

Now we are in the position to seek a domain Ω satisfying the requirements given by Lemma 4. Corresponding to the operator equation $Lx = \lambda Nx$, $\lambda \in (0,1)$, we have

$$x'(t) = \lambda\Big(p(t) - \frac{Q(t)e^{nx(t-\sigma(t))}}{R(t) + e^{nx(t-\sigma(t))}}\Big). \tag{18}$$

Suppose that $x(t) \in X$ is a solution of system (18) for a certain $\lambda \in (0,1)$. Integrating (18) over the interval $[0,\omega]$, we have

$$\int_0^\omega \Big(p(t) - \frac{Q(t)e^{nx(t-\sigma(t))}}{R(t) + e^{nx(t-\sigma(t))}}\Big)\,dt = 0.$$

That is,

$$\int_0^\omega p(t)\,dt = \int_0^\omega \frac{Q(t)e^{nx(t-\sigma(t))}}{R(t) + e^{nx(t-\sigma(t))}}\,dt. \tag{19}$$

From (18) and (19) we have

$$\begin{aligned}\int_0^\omega |x'(t)|\,dt &= \lambda \int_0^\omega \Big|p(t) - \frac{Q(t)e^{nx(t-\sigma(t))}}{R(t) + e^{(nx(t-\sigma(t)))}}\Big|\,dt \\ &\leqslant \int_0^\omega p(t)\,dt + \int_0^\omega \frac{Q(t)e^{nx(t-\sigma(t))}}{R(t) + e^{nx(t-\sigma(t))}}\,dt = 2\omega\bar{p} := M_1.\end{aligned} \tag{20}$$

Since $x \in X$, there exist $t_0 \in [0,\omega]$ and a constant M_2 such that $x(t_0) < M_2$. Then

$$x(t) \leqslant x(t_0) + \int_0^\omega |x'(t)|\,dt < M_1 + M_2. \tag{21}$$

By a similar argument, there exist $t_1 \in [0,\omega]$ and $M_3 > 0$ such that $x(t_1) > -M_3$ and

$$x(t) \geqslant x(t_1) - \int_0^\omega |x'(t)|\,dt > -(M_1 + M_3). \tag{22}$$

It is clear that M_i is independent of the choice of λ for $i = 1,2,3$. Take $H = \sum\limits_{i=1}^{4} M_i$, where M_4 is chosen sufficiently large so that the solution u of

$$p(t) - \frac{Q(t)e^{nu}}{R(t) + e^{nu}} = 0$$

satisfies $|\ln u| < M_4$; then $\|x\| < H$.

Let $\Omega = \{x \in X\colon \|x\| < H\}$. Obviously condition (a) of Lemma 4 is satisfied. If $x \in \partial\Omega \cap \operatorname{Ker} L$, $\|x\| = H$, where H is a constant, then

$$QNx = \frac{1}{\omega}\int_0^\omega \Big(P(t) - \frac{Q(t)\mathrm{e}^{nx(t-\sigma(t))}}{R(t) + \mathrm{e}^{nx(t-\sigma(t))}}\Big)\,\mathrm{d}t \neq 0.$$

Therefore, (b) of Lemma 4 holds. Further, by easy calculation,

$$\deg\{JQNx, \Omega \cap \operatorname{Ker} L, 0\} \neq 0,$$

where the isomorphism J is the identity mapping due to $\operatorname{Im} P = \operatorname{Ker} L$. Hence (c) of Lemma 4 is also satisfied. By Lemma 4, we conclude that $Lx = Nx$ has at least one solution in X, i.e., system (17) has at least one ω-periodic solution. That means (5) has at least one ω-periodic solution. By Lemma 1, therefore, system (4) has at least one ω-periodic solution. The proof is complete. $\square$

Remark 1. From the proof of Theorem 1, one can see that the deviating argument $\sigma(t)$ has no effect on the existence of a positive periodic solution of (4). Further, for the case $\sigma(t) = m\omega$, the restricted condition $Q_M > p_m$, $Q_m > p_M$ (Theorem 6 of [10]) ensuring the existence of a positive periodic solution is not needed here. Hence, we have improved and generalized Theorem 6 of [10].

4. Uniqueness and global attractivity

In this section, we obtain an explicit sufficient condition for the uniqueness and global attractivity of a periodic solution with respect to all other positive solutions of (4).

Theorem 3. *Suppose that* (H_1)–(H_4) *hold. Further, let*

$$\limsup_{t\to\infty}\int_t^{t+\sigma^*} \frac{nQ(s)z_2^n\mathrm{e}^{n\mu_2}}{R(s)}\,\mathrm{d}s < \frac{3}{2}, \tag{23}$$

where μ_2 *is defined in* (8), $\sigma^* = \sup\limits_{t\geqslant 0}\sigma(t)$. *Then there exists a unique* ω*-periodic positive solution* $\overline{y}(t)$ *of* (4) *such that*

$$\lim_{t\to\infty} y(t) - \overline{y}(t) = 0 \tag{24}$$

for all other positive solutions $y(t)$ *of* (4).

P r o o f. Obviously, it is immediate that if $y(t)$ satisfies (24), then the positive periodic solution $\overline{y}(t)$ will be unique. Therefore, to complete the proof of Theorem 3, it suffices to prove that $\lim\limits_{t\to\infty} y(t)-\overline{y}(t)=0$. By Lemma 1, we only need to prove that $\lim\limits_{t\to\infty}(z(t)-\overline{z}(t))=0$, where $\overline{z}(t)$ is the periodic positive solution and $z(t)$ is an arbitrary positive solution of (5).

From Theorem 2, under the conditions of Theorem 3, (5) has a positive periodic solution $\overline{z}(t)$. Set $z(t)=\overline{z}(t)\mathrm{e}^{x(t)}$. Then (5) reduces to

$$x'(t)=\frac{Q(t)\overline{z}^n(t-\sigma(t))}{R(t)+\overline{z}^n(t-\sigma(t))}-\frac{Q(t)\overline{z}^n(t-\sigma(t))\mathrm{e}^{nx(t-\sigma(t))}}{R(t)+\overline{z}^n(t-\sigma(t))\mathrm{e}^{nx(t-\sigma(t))}} \tag{25}$$

for almost all $t\geqslant 0$. Let

$$G(t,u)=-\frac{Q(t)\overline{z}^n(t-\sigma(t))\mathrm{e}^{nu}}{R(t)+\overline{z}^n(t-\sigma(t))\mathrm{e}^{nu}}.$$

Then (25) can be rewritten as

$$x'(t)=G(t,x(t-\sigma(t))-G(t,0)).$$

By the mean-value theorem, we have

$$x'(t)-F(t)x(t-\sigma(t))=0, \tag{26}$$

where

$$\begin{aligned}-F(t)&=\frac{-\partial G(t,u)}{\partial u}|_{u=\zeta(t)}=\frac{nR(t)Q(t)\overline{z}^n(t-\sigma(t))\mathrm{e}^{n\zeta(t)}}{(R(t)+\overline{z}^n(t-\sigma(t))\mathrm{e}^{n\zeta(t)})^2}\\&=\frac{nR(t)Q(t)\eta^n(t)}{(R(t)+\eta^n(t))^2}<\frac{nQ(t)\eta^n(t)}{R(t)},\end{aligned}$$

$\eta(t)$ lies between $\overline{z}(t-\sigma(t))$ and $z(t-\sigma(t))$. By (23), using Theorem 1 we have

$$\limsup_{t\to\infty}\int_t^{t+\sigma^*}-F(s)\,\mathrm{d}s\leqslant\limsup_{t\to\infty}\int_t^{t+\sigma^*}\frac{nQ(s)z_2^n\mathrm{e}^{n\mu_2}}{R(s)}\,\mathrm{d}s<\frac{3}{2}.$$

By Lemma 5 then $\lim\limits_{t\to\infty}x(t)=0$, i.e., $\lim\limits_{t\to\infty}z(t)-\overline{z}(t)=0$. In view of Lemma 1, then $\lim\limits_{t\to\infty}y(t)-\overline{y}(t)=0$. The proof is complete. □

5. Discussion

In this paper, by using the comparison theorem, Mawhin's continuation theorem and some analysis techniques, we study the permanence, existence and global attractivity of a positive periodic solution for an impulsive model with periodicity of environment and time-varying delays. Theorem 1 and Theorem 2 show that the delay and impulse have no effect on the permanence and existence of positive periodic solutions, but Theorem 3 implies that they affect the attractivity of the periodic solution. On the other hand, for the special case of $\sigma(t) = m\omega$, the restricted condition $Q_M > p_m$, $Q_m > p_M$ (Theorem 6 of [10]) ensuring the existence of a positive periodic solution is unnecessary. For example, consider the system

$$
(27) \qquad \begin{cases} y'(t) = y(t)\Big(2 + \sin t - \dfrac{(3 + \sin t)y^3(t - 2\pi)}{2 + y^3(t - 2\pi)}\Big), & t \neq t_k, \\ y(t_k^+) = (1 + b_k)y(t_k), \end{cases}
$$

where t_k represents fixed impulsive points with $0 < t_1 < t_2 < \ldots$ and $\lim\limits_{k\to\infty} t_k = \infty$. Suppose $\prod\limits_{0<t_k<t} (1+b_k)$ is 2π-periodic, then by Theorem 2.1 of [8] there exists a $q \in \mathbb{N}$ such that $t_{k+q} = t_k + 2\pi$, $b_{k+q} = b_k$, $\prod\limits_{0<t_k<2\pi} (1 + b_k) = 1$. Let $q = 2$, $b_1 = -0.5$, $b_2 = 1$. It is easy to verify that (H_1)–(H_4) hold. Hence, by Theorem 2, system (27) has at least one positive periodic solution. For $p(t) = 2 + \sin t$, $Q(t) = 3 + \sin t$, $R = 2$, $\sigma(t) = 2\pi$, $q = 2$, $b_1 = -0.5$, $b_2 = 1$, by Matlab we can give the simulation of (27), see Fig. 1. However, by computation, (27) does not satisfy the conditions of Theorem 6 of [10]. It shows that the main results improve and generalize some previously known results [10].

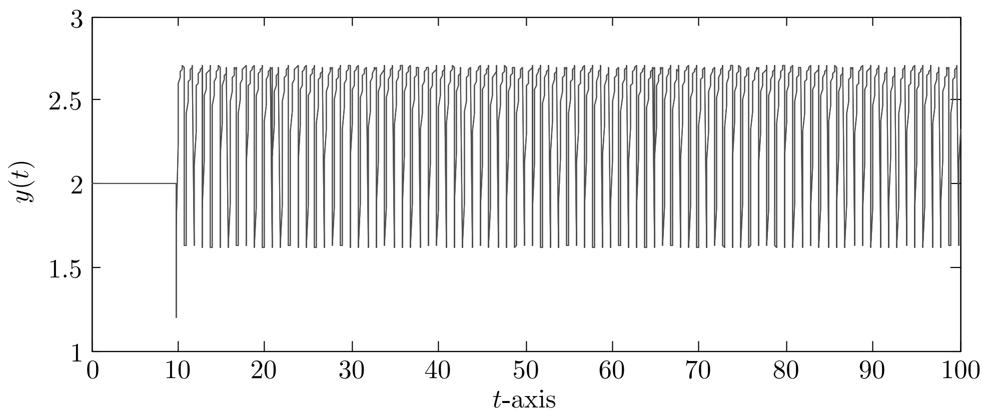

Fig. 1. Dynamics of (27) with $t_{k+2} = t_k + 2\pi$, $p(t) = 2 + \sin t$, $Q(t) = 3 + \sin t$, $\sigma(t) = 2\pi$, $R = 2$, $b_1 = -0.5$, $b_2 = 1$.

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