Gholamreza Hesamian; S. M. Taheri

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# FUZZY EMPIRICAL DISTRIBUTION FUNCTION: PROPERTIES AND APPLICATION

Gholamreza Hesamian and S. M. Taheri

The concepts of cumulative distribution function and empirical distribution function are investigated for fuzzy random variables. Some limit theorems related to such functions are established. As an application of the obtained results, a method of handling fuzziness upon the usual method of Kolmogorov–Smirnov one-sample test is proposed. We transact the $\alpha$-level set of imprecise observations in order to extend the usual method of Kolmogorov–Smirnov one-sample test. To do this, the concepts of fuzzy Kolmogorov–Smirnov one-sample test statistic and p-value are extended to the fuzzy Kolmogorov–Smirnov one-sample test statistic and fuzzy p-value, respectively. Finally, a preference degree between two fuzzy numbers is employed for comparing the observed fuzzy p-value and the given fuzzy significance level, in order to accept or reject the null hypothesis of interest. Some numerical examples are provided to clarify the discussions in this paper.

*Keywords:* fuzzy cumulative distribution function, fuzzy empirical distribution function, Kolmogorov–Smirnov test, fuzzy p-value, convergence with probability one, degree of accept, degree of reject, Glivenko–Cantelli theorem

*Classification:* 93E12, 62A10

## 1. INTRODUCTION

Nonparametric procedures are statistical procedures that makes relatively mild assumptions regarding the distribution and/or the form of underlying functional relationship. The Kolmogorov–Smirnov test is a nonparametric procedure for the equality of continuous, one-dimensional probability distributions that can be used to compare a sample with a reference probability distribution or to compare two samples. In the classical version of such a test, the observations of sample are generally assumed to be crisp (precise) quantities. But real data are usually imprecise. For instance, in hydrology studies in which the water level of a river at a certain time may not turn out to be exact. Typically, such data are expressed in imprecise quantities. For example: "the water level is very low", "it is high", "it is approximately 2.7 (m)", and the like. Similarly, in lifetime analysis in reliability theory, we may not consider a precise/exact value as the lifetime for an item. For instance, an automobile tire may work perfectly for a certain time but be losing in performance for some time, and finally go fail completely at a certain time. In this case, the lifetime data may be reported by imprecise quantities such as: "about 32000 (km)", "approximately 35000 (km)", and so on.

The analysis of such imprecise/fuzzy data requires new statistical methods to be developed. Fuzzy set theory provides necessary tool for modeling the imprecise quantities and for developing statistical procedures to analyze such data. In the present work, we are going to investigate distribution function and empirical distribution function in the fuzzy environment. Moreover, we try to establish some limit theorems for such functions. In addition, based on the obtained results, we extend the Kolmogorov–Smirnov one-sample test to fuzzy environments.

Some previous studies stated below have presented approaches to non-parametric tests for imprecise observations. Kahraman et al. [19], using defuzzification method, proposed some algorithms with crisp results for non-parametric tests when the data are fuzzy. Grzegorzewski [10] introduced a fuzzy confidence interval for the median of a population in the presence of vague data and discussed the problem of testing hypothesis about the median. He demonstrated also a straightforward generalization of some classical non-parametric tests based on fuzzy random variables [13]. Moreover, he studied some non-parametric median tests based on the necessity index of strict dominance suggested by Dubois and Prade [5] for fuzzy observations [11, 12]. In this manner, he obtained a fuzzy test showing a degree of possibility and a degree of necessity for evaluating the underlying hypotheses. He also proposed a modification of the classical sign test to cope with fuzzy data. The proposed test is a so-called bi-robust test, i. e. a test which is both distribution-free and which does not depend so heavily on the shape of the membership functions used for modeling fuzzy data [14]. Denoeux et al. [3] extended some non-parametric rank-based tests for fuzzy data. Their approach relied on the definition of a fuzzy partial ordering based on the necessity index of strict dominance between fuzzy numbers. For evaluating the hypotheses of interest, they employed the concepts of the fuzzy p-value and the degree of rejection of the null hypothesis quantified by a degree of possibility and a degree of necessity, when a given significance level is a crisp number or a fuzzy set. Hryniewicz [17] introduced the fuzzy version of the Goodman-Kruskal's $\gamma$ measure in the contingency tables where observations of the response variable are fuzzy (and observations of the explanatory variable are crisp). Taheri and Hesamian [32] investigated a fuzzy version of the Goodman-Kruskal $\gamma$ measure of association for a two-way contingency table when the observations are crisp but the categories are described by fuzzy sets. They also developed a method for testing of independence in such a two-way contingency table. In addition, they [33] extend the Wilcoxon signed-rank test to the case where the available observations are imprecise quantities, rather than crisp and the significance level is given by a fuzzy number. Moreover, they [15] proposed a generalization of the non-parametric two-sample tests for imprecise observations. They considered mainly location and scale tests that utilized a proposed ranking method based on the credibility measure. For more on statistical methods with fuzzy observations the reader is referred to the relevant literature, for example [6, 26, 35].

This paper is organized as follows: In Section 2, we recall some necessary concepts related to fuzzy numbers and fuzzy random variables. In Section 3, we generalize the concept of cumulative distribution function and empirical distribution function for fuzzy data at a crisp as well as at a fuzzy point. We extend some large sample properties of empirical distribution function to fuzzy environments in Section 4. In Section 5, we provide an approach to test the one-sided Kolmogorov–Smirnov fuzzy hypothesis when the available data are fuzzy. To do this, we introduce the concepts of fuzzy Kolmogorov–Smirnov one-sample test statistic and fuzzy p-value. For making a decision rule to reject or accept the null hypothesis, we use an index for comparing the observed fuzzy p-value with a given fuzzy significance level. We also illustrate a numerical example to clarify the discussions in this paper and to show the possible application in a fuzzy environment. A brief conclusion is presented in Section 6. In addition, a review on the classical Kolmogorov–Smirnov one-sample test is provided in Appendix.

## 2. FUZZY NUMBERS AND FUZZY RANDOM VARIABLES

A fuzzy number [4] is a function $\mu_{\widetilde{A}} : \mathbb{R} \to [0,1]$, satisfying

- $\widetilde{A}$ is normal (i. e. there exist an $x_0 \in \mathbb{R}$, such that $\mu_{\widetilde{A}}(x_0) = 1$.
- $\widetilde{A}$ is fuzzy convex (i. e. $\mu_{\widetilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y)\}$, for every $x, y \in \mathbb{R}$, $\lambda \in [0,1]$.
- $\mu_{\widetilde{A}}$ is upper semi-continuous.
- the closure of $\{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) > 0\}$, denoted by $\text{supp}(\widetilde{A})$, is compact.

These properties imply that for each $\alpha \in (0,1]$, the $\alpha$-cut of $\widetilde{A}$ defined by

$$\widetilde{A}[\alpha] = \{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) \geq \alpha\} = [\widetilde{A}^L_\alpha, \widetilde{A}^U_\alpha]$$

is a closed interval in $\mathbb{R}$, as well as $\text{supp}(\widetilde{A})$, and the following equality holds

$$\widetilde{A}[0] = \text{supp}(\widetilde{A}) = \lim_{\alpha \to 0^+} \widetilde{A}[\alpha].$$

One of the popular forms of a fuzzy number, to be considered in this work, is the so-called triangular fuzzy number $\widetilde{A} = (A^l, A^c, A^r)_T$ whose membership function and $\alpha$-cut are given by

$$\mu_{\widetilde{A}}(x) = \left\{ \begin{array}{ll} 0 & x < A^l, \\ \frac{x-A^l}{A^c-A^l} & A^l \leq x < A^c, \\ \frac{A^r-x}{A^r-A^c} & A^c \leq x \leq A^r, \\ 0 & x > A^r, \end{array} \right\} \quad \forall x \in \mathbb{R},$$

$$\widetilde{A}[\alpha] = [A^l + (A^c - A^l)\alpha, A^r - (A^r - A^c)\alpha],$$

which is typically a formal representation of the concept of "about $A^c$". We denote by $\mathbb{F}(\mathbb{R})$ the set of all fuzzy numbers on $\mathbb{R}$. For more on fuzzy numbers see [23].

Below, we recall some definitions and results related to fuzzy random variables (for more details, see [20, 22, 24, 30]).

Given a probability space $(\Omega, \mathbb{A}, \mathbf{P})$, a fuzzy random variable is defined to be a Borel measurable mapping $\widetilde{X} : \Omega \to \mathbb{F}(\mathbb{R})$ such that for any $\alpha \in (0,1]$, the $\alpha$-cut $\widetilde{X}[\alpha]$ is a random variable, i. e. $\widetilde{X}[\alpha] : \Omega \to \mathfrak{H}(\mathbb{R})$ ($\mathfrak{H}(\mathbb{R})$ is the class of nonempty compact intervals) (see [8] and [30]). It is a Borel measurable function with respect to the Borel $\sigma$-field generated by the topology associated with the Hausdorff metric on $\mathfrak{H}(\mathbb{R})$,

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}|a-b|, \sup_{b\in B}\inf_{a\in A}|a-b|\right\}.$$

Two fuzzy random variables $\widetilde{X}$ and $\widetilde{Y}$ are said to be independent if and only if each random variable in the set $\{\widetilde{X}_\alpha^L,\ \widetilde{X}_\alpha^U : \alpha \in (0,1]\}$ is independent of each random variable in the set $\{\widetilde{Y}_\alpha^L,\ \widetilde{Y}_\alpha^U : \alpha \in (0,1]\}$. They are called identically distributed if $\widetilde{X}_\alpha^L$ and $\widetilde{Y}_\alpha^L$, and also $\widetilde{X}_\alpha^U$ and $\widetilde{Y}_\alpha^U$, are identically distributed for all $\alpha \in (0,1]$. Similar arguments can be used for more than two fuzzy random variables.

We say that $\widetilde{X}_1,\dots,\widetilde{X}_n$ is a fuzzy random sample (with the same distribution as $\widetilde{X}$) if they are independent and identically distributed. We denote by $\widetilde{x}_1,\widetilde{x}_2,\dots,\widetilde{x}_n$ the observed values of a fuzzy random sample.

The aim of this work is to extend the one-sample procedure to the case where the observations are fuzzy rather than crisp.

## 3. FUZZY CUMULATIVE DISTRIBUTION FUNCTION AND FUZZY EMPIRICAL DISTRIBUTION FUNCTION

In this section we extend the concept of c.d.f. and e.d.f. to the case the observations are values of a fuzzy random sample.

**Definition 3.1.** The fuzzy set $\widetilde{F}_{\widetilde{X}}(\widetilde{x})$ is said to be the fuzzy cumulative distribution function (f.c.d.f) of the fuzzy random variable $\widetilde{X}$, whenever its membership function at $\widetilde{x} \in \mathbb{F}(\mathbb{R})$ is given by

$$\mu_{\widetilde{F}_{\widetilde{X}}(\widetilde{x})}(y) = \sup_{\alpha\in[0,1]} \alpha I(y \in [(\widetilde{F}_{\widetilde{X}}(\widetilde{x}))_\alpha^L, (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))_\alpha^U]), \tag{1}$$

where

$$(\widetilde{F}_{\widetilde{X}}(\widetilde{x}))_\alpha^L = \mathbf{P}(\widetilde{X}_\alpha^U \le \widetilde{x}_\alpha^L), \quad (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))_\alpha^U = \mathbf{P}(\widetilde{X}_\alpha^L \le \widetilde{x}_\alpha^U), \tag{2}$$

and $I$ is the indicator function,

$$I(\rho) = \begin{cases} 1 & \text{if } \rho \text{ is true,} \\ 0 & \text{if } \rho \text{ is false.} \end{cases}$$

Let us notice that for $0 < \alpha_1 < \alpha_2 \le 1$ we have $(\widetilde{F}_{\widetilde{X}}(\widetilde{x}))_{\alpha_1}^L \le (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))_{\alpha_2}^L$, and $(\widetilde{F}_{\widetilde{X}}(\widetilde{x}))_{\alpha_2}^U \le (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))_{\alpha_1}^U$. Hence, for $0 < \alpha_1 < \alpha_2 \le 1$ we have $(\widetilde{F}_{\widetilde{X}}(\widetilde{x}))[\alpha_2] \subseteq \widetilde{F}_{\widetilde{X}}(\widetilde{x}))[\alpha_1]$, and so Equation (2) defines the nested set of closed intervals indexed by $\alpha$. Therefore, the membership function of f.c.d.f. can be uniquely determined by (1) due to the Representation Theorem (see, e. g. [23]).

**Remark 3.2.** If the fuzzy point $\widetilde{x}$ reduces to the crisp number $x$, then the $\alpha$-cuts of the f.c.d.f. reduce as follows

$$(\widetilde{F}_{\widetilde{X}}(x))[\alpha] = [\mathbf{P}(\widetilde{X}_\alpha^U \le x), \mathbf{P}(\widetilde{X}_\alpha^L \le x)]. \tag{3}$$

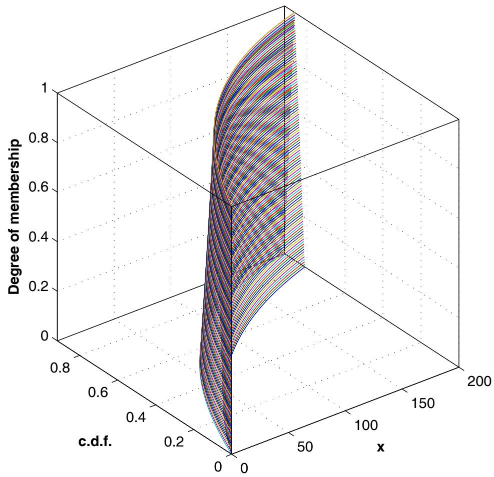


**Fig. 1.** Fuzzy cumulative distribution function in Example 3.5.

In addition, if the fuzzy random sample $\widetilde{X}_1,\ \widetilde{X}_2,\ \ldots,\widetilde{X}_n$ reduce to the crisp random sample $X_1, X_2, \ldots, X_n$, then, for every $\alpha \in (0,1]$

$$\widetilde{F}^L_\alpha(x) = \widetilde{F}^U_\alpha(x) = \mathbf{P}(X \leq x) = F_X(x).$$

Therefore, we obtain the classical c.d.f.

**Definition 3.3.** We say $\widetilde{F}_{\widetilde{X}}(\cdot)$ is continuous on $\mathbb{R}$ if for every $\alpha \in (0,1]$, $(\widetilde{F}_{\widetilde{X}}(\cdot))^L_\alpha$ and $(\widetilde{F}_{\widetilde{X}}(\cdot))^U_\alpha$ are continuous on $\mathbb{R}$.

**Remark 3.4.** As interpreted by some authors [10, 22, 31], a fuzzy random variable $\widetilde{X}$ is a vague perception of an ordinary random variable $X$ called the original of $\widetilde{X}$. Based on the above definition, we can say that the f.c.d.f. $\widetilde{F}_{\widetilde{X}}$ is a vague perception of $F_X$. Now, let $\{F_{X,\underline{\theta}} : \underline{\theta} \in \Theta \subseteq \mathbb{R}^p\}$ be a class of continuous parametric c.d.f. If we consider a vector of fuzzy parameters as a mapping $\underline{\widetilde{\theta}} : \Theta \to (\mathbb{F}(\mathbb{R}))^p, p \geq 1$, then it is natural that we have

$$(\widetilde{F}_{\widetilde{X}})^L_\alpha = \inf_{\underline{\theta} \in \underline{\widetilde{\theta}}[\alpha]} F_{X,\underline{\theta}},\ (\widetilde{F}_{\widetilde{X}})^U_\alpha = \sup_{\underline{\theta} \in \underline{\widetilde{\theta}}[\alpha]} F_{X,\underline{\theta}}. \tag{4}$$

**Example 3.5.** Suppose $\widetilde{X}$ has exponential distribution with fuzzy parameter $\widetilde{\lambda} \in \mathbb{F}(\mathbb{R})$ such that $\mathrm{supp}(\widetilde{\lambda}) \subseteq (0,\infty)$. Therefore, by (4), we obtain

$$\mu_{\widetilde{F}_{\widetilde{X}}(x)}(y) = \sup_{\alpha \in [0,1]} \alpha I \left( y \in \left[ 1 - \exp\left(\frac{-x}{\widetilde{\lambda}^U_\alpha}\right), 1 - \exp\left(\frac{-x}{\widetilde{\lambda}^L_\alpha}\right) \right] \right). \tag{5}$$

For example let $\widetilde{\lambda} = (80, 100, 120)_T$. In this case, 3-dimensional curve of f.c.d.f. is shown in Figure 1. Moreover, $\alpha$-levels of such a f.c.d.f. are shown in Figure 2, for some values of $\alpha$.

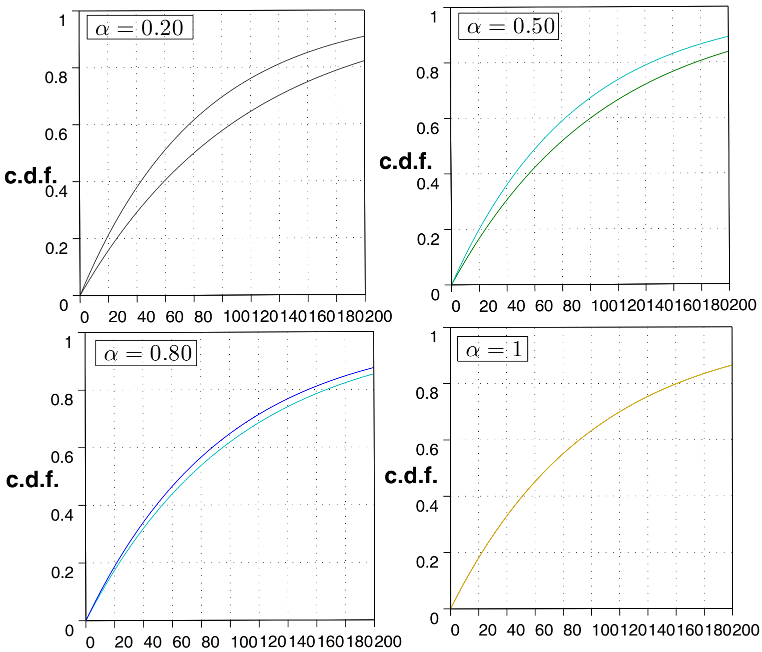


**Fig. 2.** The $\alpha$-levels of f.e.d.f in Example 3.5, for $\alpha = 0.2, 0.5, 0.8, 1$.

**Definition 3.6.** Suppose that $\widetilde{X}_1, \widetilde{X}_1, \ldots, \widetilde{X}_n$ is a fuzzy random sample. Then, the fuzzy empirical distribution function (f.e.d.f.) at $\widetilde{x} \in \mathbb{F}(\mathbb{R})$ is defined to be a fuzzy set with the following membership function

$$\mu_{\widetilde{\widehat{F}}_n(\widetilde{x})}(y) = \sup_{\alpha \in [0,1]} \alpha I \left( y \in \left[ (\widetilde{\widehat{F}}_n(\widetilde{x}))^L_\alpha, (\widetilde{\widehat{F}}_n(\widetilde{x}))^U_\alpha \right] \right),$$

in which,

$$(\widetilde{\widehat{F}}_n(\widetilde{x}))^L_\alpha = \frac{1}{n} \sum_{i=1}^{n} I((\widetilde{X}_i)^U_\alpha \le \widetilde{x}^L_\alpha), \quad (\widetilde{\widehat{F}}_n(\widetilde{x}))^U_\alpha = \frac{1}{n} \sum_{i=1}^{n} I((\widetilde{X}_i)^L_\alpha \le \widetilde{x}^U_\alpha). \tag{6}$$

**Remark 3.7.** If the fuzzy point $\widetilde{x}$ reduces to the crisp number $x$, then the $\alpha$-cuts of the f.e.d.f. reduces as follows

$$(\widetilde{\widehat{F}}_n(x))[\alpha] = \left[ \frac{1}{n} \sum_{i=1}^{n} I((\widetilde{X}_i)^U_\alpha \le x), \frac{1}{n} \sum_{i=1}^{n} I((\widetilde{X}_i)^L_\alpha \le x) \right]. \tag{7}$$

Moreover, if the fuzzy random sample $\widetilde{X}_1$, $\widetilde{X}_2$, $\ldots, \widetilde{X}_n$ reduce to the crisp random sample $X_1, X_2, \ldots, X_n$, then, for every $\alpha \in (0,1]$

$$(\widetilde{\widehat{F_n}}(\widetilde{x}))_\alpha^L = (\widetilde{\widehat{F_n}}(\widetilde{x}))_\alpha^U = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x) = \widehat{F_n}(x),$$

so we obtain the classical e.d.f.

**Remark 3.8.** It should be mentioned that Viertl ([35], p. 44) defined the concept of e.d.f. for fuzzy data. Let $\widetilde{x}_1, \widetilde{x}_1, \ldots, \widetilde{x}_n$ be fuzzy data whose $\alpha$-cuts are given by $\widetilde{x}_i[\alpha] = [(\widetilde{x}_i)_\alpha^L, (\widetilde{x}_i)_\alpha^U]$. At $x \in \mathbb{R}$, the corresponding $\alpha$-cuts of the fuzzy empirical distribution function are defined by

$$(\widetilde{\widehat{F_n}}(x))_\alpha^L = \frac{1}{n}\sum_{i=1}^{n} I((\widetilde{x}_i)_\alpha^U \leq x),$$

$$(\widetilde{\widehat{F_n}}(x))_\alpha^U = \frac{1}{n}\sum_{i=1}^{n} I((\widetilde{x}_i)_\alpha^L \leq x).$$

The above definition for f.e.d.f., although based on a fuzzy sample, is a function of the real argument. However, in Definition 3.6 we define the concept of f.e.d.f. as a function of a fuzzy argument $\widetilde{x}$. Note that, $\widetilde{\widehat{F_n}}(\widetilde{x})$ reduces to the one proposed by Viertl provided $\widetilde{x} = x \in \mathbb{R}$.

It is remarkable that Viertl suggested another concept called by him as "smoothed empirical distribution function" ([35], p. 45). Let $\widetilde{x}_1, \widetilde{x}_1, \ldots, \widetilde{x}_n$ be fuzzy data with the (integrable) membership functions $\mu_{\widetilde{x}_i}$, $i = 1, 2, \ldots, n$. Then, the smoothed empirical distribution function is defined as follows

$$\widehat{F_n}^{sm}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{\int_{-\infty}^{x} \mu_{\widetilde{x}_i}(x)\,\mathrm{d}x}{\int_{-\infty}^{+\infty} \mu_{\widetilde{x}_i}(x)\,\mathrm{d}x}.$$

As mentioned above we use an extension of the classical e.d.f. different from that proposed by Viertl since: 1) In the proposed method, f.c.d.f. is defined on the fuzzy points while in the Viertl's approach it is defined on the real points. 2) We discuss the large sample property of f.e.d.f. as we will see below.

**Example 3.9.** Suppose that, based on a fuzzy random sample of size $n = 30$, we obtain the triangular fuzzy numbers given in Table 1, (data set are taken from [34]). The f.e.d.f. is shown in Figure 3 for two fuzzy points. Note that, the f.e.d.f. can be expressed by "about 0.43" at $\widetilde{x} = (1, 1.5, 2)_T$, and by "about 0.67" at $\widetilde{x} = (1.6, 2, 2.3)_T$. Moreover, as we see in Figure 4, the value of f.e.d.f. increase to "about one" as $\widetilde{x}$ increases.

**Example 3.10.** Consider the data set in Example 3.9. In this case, the 3-dimensional curve of the f.e.d.f. is shown in Figure 5 for $x \in [0, 3]$. Note that, this is a special case of the f.e.d.f. for which the domain is a subset of real numbers. Moreover, the $\alpha$-levels of a such f.e.d.f. are shown in Figure 6, for some values of $\alpha$.

| | | |
|---|---|---|
| $\widetilde{x}_1 = (0.19, 0.23, 0.30)_T$ | $\widetilde{x}_{11} = (0.38, 0.41, 0.49)_T$ | $\widetilde{x}_{21} = (0.53, 0.64, 0.71)_T$ |
| $\widetilde{x}_2 = (0.71, 0.76, 0.78)_T$ | $\widetilde{x}_{12} = (0.78, 0.86, 0.90)_T$ | $\widetilde{x}_{22} = (0.85, 0.94, 0.98)_T$ |
| $\widetilde{x}_3 = (0.86, 0.98, 1.07)_T$ | $\widetilde{x}_{13} = (0.99, 1.02, 1.12)_T$ | $\widetilde{x}_{23} = (0.98, 1.08, 1.14)_T$ |
| $\widetilde{x}_4 = (1.08, 1.14, 1.23)_T$ | $\widetilde{x}_{14} = (1.20, 1.23, 1.37)_T$ | $\widetilde{x}_{24} = (1.29, 1.37, 1.43)_T$ |
| $\widetilde{x}_5 = (1.36, 1.46, 1.53)_T$ | $\widetilde{x}_{15} = (1.40, 1.53, 1.68)_T$ | $\widetilde{x}_{25} = (1.62, 1.64, 1.72)_T$ |
| $\widetilde{x}_6 = (1.64, 1.69, 1.81)_T$ | $\widetilde{x}_{16} = (1.74, 1.78, 1.84)_T$ | $\widetilde{x}_{26} = (1.74, 1.83, 1.88)_T$ |
| $\widetilde{x}_7 = (1.90, 1.95, 2.06)_T$ | $\widetilde{x}_{17} = (1.91, 1.99, 2.08)_T$ | $\widetilde{x}_{27} = (1.93, 2.04, 2.10)_T$ |
| $\widetilde{x}_8 = (2.14, 2.17, 2.22)_T$ | $\widetilde{x}_{18} = (2.21, 2.25, 2.29)_T$ | $\widetilde{x}_{28} = (2.31, 2.36, 2.45)_T$ |
| $\widetilde{x}_9 = (2.32, 2.40, 2.52)_T$ | $\widetilde{x}_{19} = (2.44, 2.45, 2.53)_T$ | $\widetilde{x}_{29} = (2.36, 2.49, 2.54)_T$ |
| $\widetilde{x}_{10} = (2.41, 2.51, 2.65)_T$ | $\widetilde{x}_{20} = (2.50, 2.57, 2.59)_T$ | $\widetilde{x}_{30} = (2.53, 2.61, 2.67)_T$ |

**Tab. 1.** Data set in Example 3.9.

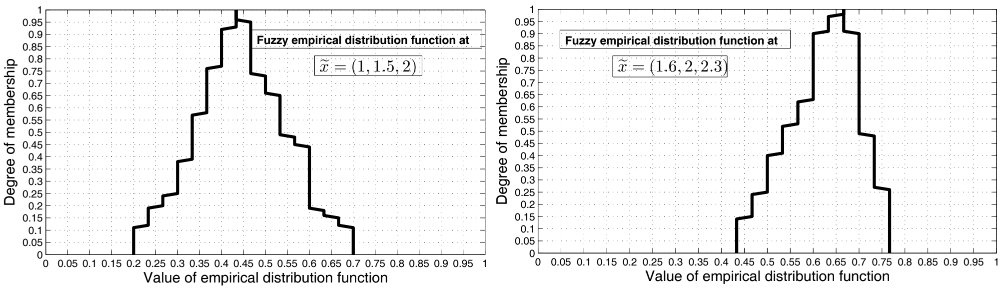


**Fig. 3.** Fuzzy empirical distribution function in two fuzzy points (Example 3.9).

## 4. THE LARGE SAMPLE CASE

First, we extend the Gelivenko-Cantelli theorem for fuzzy random variables.

**Theorem 4.1.** Let $\widetilde{X}$ be a fuzzy random variable with continuous f.c.d.f. $\widetilde{F}_{\widetilde{X}}(x)$. Then, $\widetilde{\widehat{F_n}}(x) \longrightarrow \widetilde{F}_{\widetilde{X}}(x)$ uniformly in $x \in \mathbb{R}$, i. e.

$$\mathbf{P}\left(\sup_{x\in\mathbb{R}}\left\{|(\widetilde{\widehat{F_n}}(x))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha| \vee |(\widetilde{\widehat{F_n}}(x))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^U_\alpha|\right\} \to 0\right) = 1, \text{ for all } \alpha \in (0,1],$$

where the symbol $\vee$ stands for the maximum.

P r o o f. For an arbitrary $\alpha \in (0,1]$, let $D^L_{n,\alpha} = \sup_{x\in\mathbb{R}} |(\widetilde{\widehat{F_n}}(x))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha|$ and $D^U_{n,\alpha} = \sup_{x\in\mathbb{R}} |(\widetilde{\widehat{F_n}}(x))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^U_\alpha|$. By substituting $(\widetilde{\widehat{F_n}}(x))^L_\alpha$ by $F_n(x)$ and $(\widetilde{F}_{\widetilde{X}}(x))^L_\alpha$ by $F_X(x)$, respectively, in Lemma 5.4.1 of [9], we have $\mathbf{P}(D^L_{n,\alpha} \to 0) = 1$ (similarly, $\mathbf{P}(D^U_{n,\alpha} \to 0) = 1$), and therefore, $\mathbf{P}((D^L_{n,\alpha} + D^U_{n,\alpha}) \to 0) = 1$.

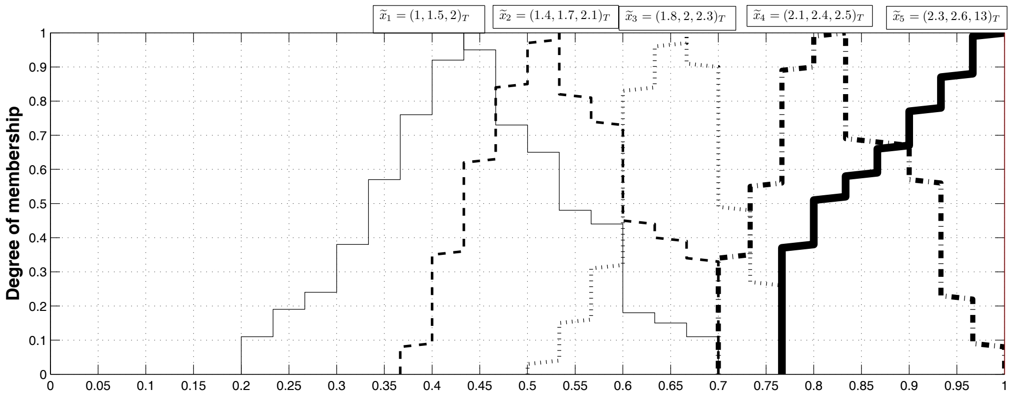


**Fig. 4.** Behavior of f.e.d.f. by increasing $\widetilde{x}$ in Example 3.9.

Now, since $\sup_{x\in\mathbb{R}}\{|(\widetilde{\widehat{F_n}}(x))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha| \vee |(\widetilde{\widehat{F_n}}(x))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^U_\alpha|\} \le D^L_{n,\alpha} + D^U_{n,\alpha}$, we have $\mathbf{P}(\sup_{x\in\mathbb{R}}\{|(\widetilde{\widehat{F_n}}(x))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha| \vee |(\widetilde{\widehat{F_n}}(x))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^U_\alpha|\} \to 0) \ge \mathbf{P}((D^L_{n,\alpha} + D^U_{n,\alpha}) \to 0) = 1$. Consequently, for every $\alpha \in (0,1]$,

$$\mathbf{P}\left(\sup_{x\in\mathbb{R}}\left\{\left|(\widetilde{\widehat{F_n}}(x))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha\right| \vee \left|(\widetilde{\widehat{F_n}}(x))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^U_\alpha\right|\right\} \to 0\right) = 1.$$

□

**Remark 4.2.** If the fuzzy random variables $\{\widetilde{X}_n\}_{n=1}^\infty$ reduce to the crisp random variables $\{X_n\}_{n=1}^\infty$, then the above theorem reduces to $\mathbf{P}(\sup_{x\in\mathbb{R}}|\widehat{F_n}(x) - F_X(x)| \to 0) = 1$, which is the Glivenko–Cantelli theorem for crisp random variables. Therefore, Theorem 4.1 is a generalization of Glivenko–Cantelli theorem for fuzzy random variables.

**Definition 4.3.** For the sequence $\{\widetilde{X}_n\}_{n=1}^\infty$ of fuzzy random variables and a fuzzy number $\widetilde{Z}$, we say $\widetilde{X}_n \to \widetilde{Z}$ with probably one (w.p.1), if

$$\mathbf{P}([|(\widetilde{X}_n)^L_\alpha - \widetilde{Z}^L_\alpha| \vee |(\widetilde{X}_n)^U_\alpha - \widetilde{Z}^U_\alpha|] \to 0) = 1, \text{ for all } \alpha \in (0,1]. \tag{8}$$

**Remark 4.4.** If the fuzzy random variables $\{\widetilde{X}_n\}_{n=1}^\infty$ and $\widetilde{Z}$ reduce to the crisp random variables $\{X_n\}_{n=1}^\infty$ and $Z$, then the above equality reduces to $\mathbf{P}(X_n \to Z) = 1$, which is the definition of convergence w.p.1 of ordinary (non fuzzy) random variables. Definition 4.3, therefore, is a generalization of convergence w.p.1 to the case of fuzzy random variables.

**Theorem 4.5.** Let $\widetilde{X}$ be a fuzzy random variable with continuous f.c.d.f. $\widetilde{F}_{\widetilde{X}}(x)$ and $\widetilde{x}$ be a fuzzy number whose $\alpha$-cuts are $[\widetilde{x}^L_\alpha, \widetilde{x}^U_\alpha]$, $\alpha \in (0,1]$. Then, $\widetilde{\widehat{F_n}}(\widetilde{x}) \to \widetilde{F}_{\widetilde{X}}(\widetilde{x})$ w.p.1, for every $\widetilde{x} \in \mathbb{F}(\mathbb{R})$.

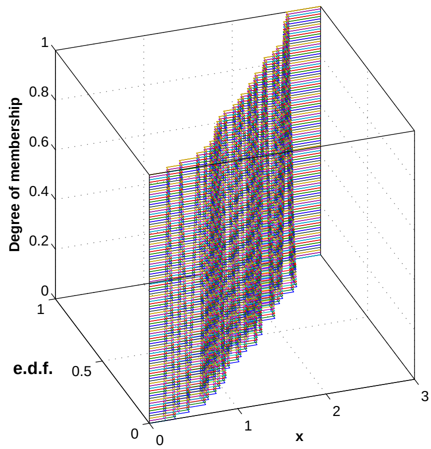


**Fig. 5.** Fuzzy empirical distribution function in Example 3.10.

P r o o f . For every $\alpha \in (0,1]$, we observe that $\widetilde{\widetilde{F_n}}(\widetilde{x})[\alpha]$ is a random variable. Therefore, $\widetilde{\widetilde{F_n}}(\widetilde{x}) : \Omega \longrightarrow \mathbb{F}(\mathbb{R})$ is a fuzzy random variable for any $n \in \mathbb{N}$ (Section 2). Now for a fixed $\alpha \in (0,1]$ and a fixed $\widetilde{x} \in \mathbb{F}(\mathbb{R})$, we have $|(\widetilde{\widetilde{F_n}}(\widetilde{x}))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^L_\alpha| \vee |(\widetilde{\widetilde{F_n}}(\widetilde{x}))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^U_\alpha| \leq |(\widetilde{\widetilde{F_n}}(\widetilde{x}))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^L_\alpha| + |(\widetilde{\widetilde{F_n}}(\widetilde{x}))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^U_\alpha|$. Since for all $\alpha \in (0,1]$, $(\widetilde{X}_i)^L_\alpha$, $i = 1,2,\ldots,n$ (and also $(\widetilde{X}_i)^U_\alpha$, $i = 1,2,\ldots,n$) are independent and identically distributed, $(\widetilde{\widetilde{F_n}}(\widetilde{x}))^L_\alpha$ converges to $F^L_\alpha(\widetilde{x}^L_\alpha) = (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^L_\alpha$, w.p.1 (also $(\widetilde{\widetilde{F_n}}(\widetilde{x}))^U_\alpha$ converges to $F^U_\alpha(\widetilde{x}^U_\alpha) = (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^U_\alpha$, w.p.1) by Strong Law of Large Numbers (see, e. g. [9]). For any fixed $\alpha \in (0,1]$ and any fixed $\widetilde{x} \in \mathbb{F}(\mathbb{R})$, therefore, $\mathbf{P}([|(\widetilde{\widetilde{F_n}}(\widetilde{x}))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^L_\alpha| \vee |(\widetilde{\widetilde{F_n}}(\widetilde{x}))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^U_\alpha|] \to 0) \geq \mathbf{P}((|(\widetilde{\widetilde{F_n}}(\widetilde{x}))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^L_\alpha| + |(\widetilde{\widetilde{F_n}}\ (\widetilde{x}))^U_\alpha - (\widetilde{F}_{\widetilde{X}}\ (\widetilde{x}))^U_\alpha|) \to 0) = 1$. Consequently, for all $\alpha \in (0,1]$, and for all $\widetilde{x} \in \mathbb{F}(\mathbb{R})$

$$\mathbf{P}\left(\left[\left|(\widetilde{\widetilde{F_n}}(\widetilde{x}))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^L_\alpha\right| \vee \left|(\widetilde{\widetilde{F_n}}(\widetilde{x}))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(\widetilde{x}))^U_\alpha\right|\right] \to 0\right) = 1.$$

□

Therefore, we can roughly say that, based on a sequence of independent and identical fuzzy random variables, the f.e.d.f. $\widetilde{\widetilde{F_n}}(\widetilde{x})$ is a strong fuzzy consistent estimator for the c.d.f. $\widetilde{F}_{\widetilde{X}}(\widetilde{x})$.

**Remark 4.6.** If the fuzzy point $\widetilde{x} \in \mathbb{F}(\mathbb{R})$ reduces to the crisp number $x \in \mathbb{R}$, then the result of above theorem reduces as follows

$$\mathbf{P}\left(\left[\left|(\widetilde{\widetilde{F_n}}(x))^L_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha\right| \vee \left|(\widetilde{\widetilde{F_n}}(x))^U_\alpha - (\widetilde{F}_{\widetilde{X}}(x))^U_\alpha)\right|\right] \to 0\right) = 1, \text{ for all } \alpha \in (0,1].$$

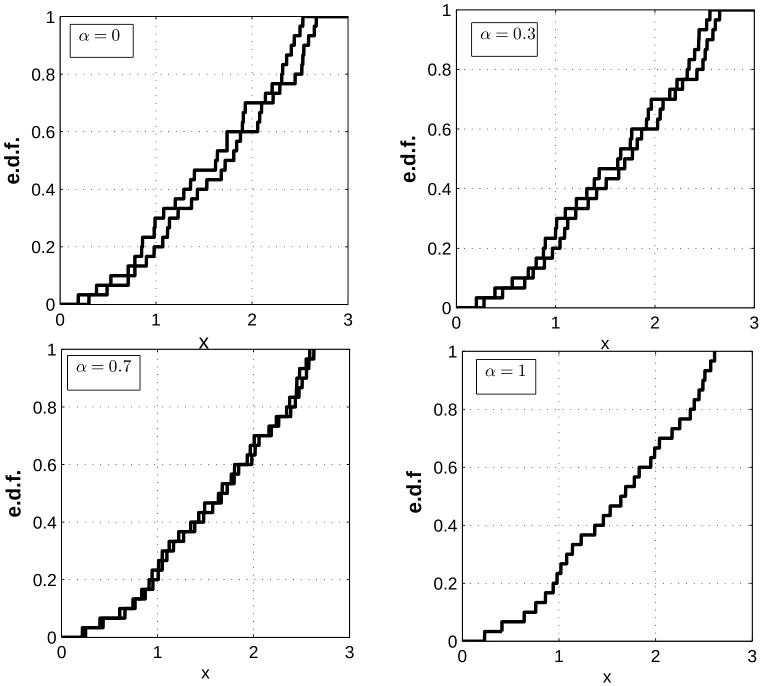


**Fig. 6.** The $\alpha$-levels of f.e.d.f in Example 3.10, for $\alpha = 0, 0.3, 0.7, 1$.

In other words, $\widetilde{\widehat{\widetilde{F}}}_n(x) \to \widetilde{F}_{\widetilde{X}}(x)$ w.p.1, at $x \in \mathbb{R}$. In addition, if the fuzzy observations and fuzzy point reduce to the crisp observations and crisp point, then the above theorem reduces to the ordinary convergence w.p.1 of the classical one (i. e. $\widehat{F}_n(x) \to F_X(x)$ w.p.1, at $x \in \mathbb{R}$ (see [9], p. 125)).

**Remark 4.7.** Assume that $\widetilde{\widehat{\widetilde{F}}}_n(x) \longrightarrow \widetilde{F}_{\widetilde{X}}(x)$ w.p.1, uniformly in $x \in \mathbb{R}$. Then it is easy to verify that $\widetilde{\widehat{F}}_n(x) \longrightarrow \widetilde{F}_{\widetilde{X}}(x)$ w.p.1, at $x \in \mathbb{R}$. Hence, we have a relationship between convergence with probability one and the Glivenko–Cantelli theorem for fuzzy random variables.

**Remark 4.8.** It is notable that, Parthasarathy [29] and Nguyen et al. [27] investigated the Glivenko–Cantelli theorem in a general setting. However, in our proposed approach, we investigated the basic definitions and the Gelivenko–Cantelli theorem in fuzzy points rather than crisp ones. Meanwhile, Krätschmer [21] established this theorem in a essentially different framework. His method is based on counting the number of observations which are "equal to" any specific fuzzy set. But, our method handles a vague concept of "less than or equal" to compare the observation of a fuzzy random variable and a given fuzzy number $\widetilde{x}$ (or $x \in \mathbb{R}$) based on alpha-cuts of the fuzzy observations, as we proposed in Definition 3.1. Moreover, we use a concept of strong convergence, while he established the result by using some metrics on the space of fuzzy numbers. Finally, it should be mentioned that, Bzowski and Urbanski [2] stated and proved a fuzzy version of Glivenko–Cantelli theorem for fuzzy variables (i. e. the variables with some possibility distributions rather than probability distributions), which it is established in a completely different context.

## 5. KOLMOGOROV–SMIRNOV ONE-SAMPLE TEST IN FUZZY ENVIRONMENT

Now, suppose that we have a fuzzy random sample $\widetilde{X}_1, \widetilde{X}_1, \ldots, \widetilde{X}_n$ with observed values $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$ from a population with continuous f.c.d.f. $\widetilde{F}_{\widetilde{X}}$. In this section we generalize the classical one-sided Kolmogorov–Smirnov one-sample test to a fuzzy environment. In fact, based on the observations of a fuzzy random sample, we are going to test the following fuzzy hypothesis

$$\begin{cases} \widetilde{H}_0: & \widetilde{F}_{\widetilde{X}}(x) = \widetilde{F}^0(x), \ \forall x, \\ \widetilde{H}_1: & \widetilde{F}_{\widetilde{X}}(x) \succ \widetilde{F}^0(x), \ \text{for some } x. \end{cases} \tag{9}$$

in which it means that

$$\begin{cases} \widetilde{H}_0 : (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha = (\widetilde{F}^0(x))^L_\alpha, \ (\widetilde{F}_{\widetilde{X}}(x))^U_\alpha = (\widetilde{F}^0(x))^U_\alpha, \ \forall \alpha \in (0,1] \ , \ \forall x. \\ \widetilde{H}_1 : (\widetilde{F}_{\widetilde{X}}(x))^L_\alpha > (\widetilde{F}^0(x))^L_\alpha, \ (\widetilde{F}_{\widetilde{X}}(x))^U_\alpha > (\widetilde{F}^0(x))^U_\alpha, \ \forall \alpha \in (0,1], \ \text{for some } x. \end{cases} \tag{10}$$

To interpret the alternative hypothesis $\widetilde{H}_1$, consider the following example.

**Example 5.1.** Let $\widetilde{X}_0$ and $\widetilde{X}$ have the exponential distribution with fuzzy parameters $\widetilde{\lambda}_0 = (100, 120, 140)_T$ and $\widetilde{\lambda} = (50, 70, 90)_T$, respectively. As we can observe in Figure 7, $\widetilde{F}_{\widetilde{X}}(x) \succ \widetilde{F}^0(x)$, for some $x$. In addition, the $\alpha$-levels of a such f.e.d.f. are shown in Figure 8, for some values of $\alpha$.

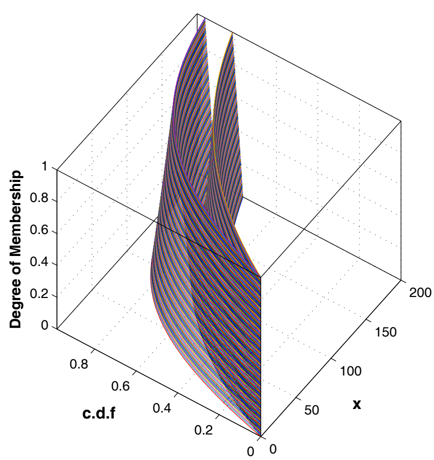


**Fig. 7.** Fuzzy cumulative distribution functions in Example 5.1.

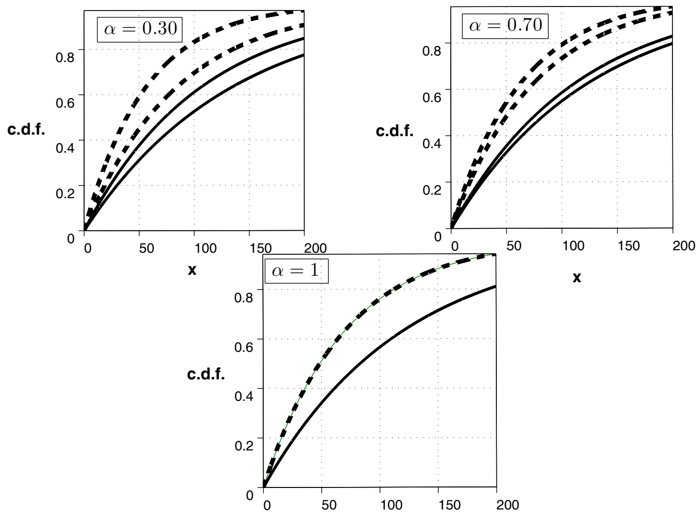


**Fig. 8.** The $\alpha$-levels of f.e.d.f in Example 5.1, for $\alpha = 0.3, 0.7, 1$.

**Definition 5.2.** For a fuzzy random sample $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$, the fuzzy Kolmogorov–Smirnov one-sample test statistic is a fuzzy set $\widetilde{\sqrt{n}D_n^+}$ on $[0, \infty)$ with the following membership function

$$\mu_{\widetilde{\sqrt{n}D_n^+}}(y) = \sup_{\alpha \in [0,1]} \alpha I\left(y \in \left[(\widetilde{\sqrt{n}D_n^+})_\alpha^L, (\widetilde{\sqrt{n}D_n^+})_\alpha^U\right]\right), \tag{11}$$

where

$$\begin{aligned}
(\widetilde{\sqrt{n}D_n^+})_\alpha^L &= \inf_{\beta \geq \alpha} \sqrt{n} \max\left\{ \max_{1 \leq i \leq n} \left\{ i/n - (\widetilde{F}^0((\widetilde{x}_i)_\beta^L))_\beta^U \right\}, 0 \right\}, \\
&= \sqrt{n} \max\left\{ \max_{1 \leq i \leq n} \left\{ i/n - (\widetilde{F}^0((\widetilde{x}_i)_\alpha^L))_\alpha^U \right\}, 0 \right\},
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{\sqrt{n}D_n^+})_\alpha^U &= \inf_{\beta \geq \alpha} \sqrt{n} \max\left\{ \max_{1 \leq i \leq n} \left\{ i/n - (\widetilde{F}^0((\widetilde{x}_i)_\beta^U))_\beta^L \right\}, 0 \right\}, \\
&= \sqrt{n} \max\left\{ \max_{1 \leq i \leq n} \left\{ i/n - (\widetilde{F}^0((\widetilde{x}_i)_\alpha^U))_\alpha^L \right\}, 0 \right\}.
\end{aligned}$$

**Remark 5.3.** If the observed fuzzy random sample $\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_n$ reduce to the crisp random sample $X_1, X_2, \ldots, X_n$, then, for every $\alpha \in (0, 1]$,

$$(\widetilde{\sqrt{n}D_n^+})_\alpha^L = \widetilde{\sqrt{n}D_n^+})_\alpha^U = \sqrt{n} \max\left\{ \max_{1 \leq i \leq n} \left\{ i/n - F_X^0(X_i) \right\}, 0 \right\},$$

which is the classical Kolmogorov–Smirnov one-sample test statistic (see Appendix).

Now we define the fuzzy p-value for testing fuzzy hypothesis (9) (i.e. (10)).

**Definition 5.4.** A fuzzy p-value for the Kolmogorov–Smirnov for testing (9) (equivalently (10)) based on a fuzzy random sample is given by

$$\mu_{\widetilde{p}-value}(x) = \sup_{\alpha\in[0,1]} \alpha I(x \in [(\widetilde{p} - value)^L_\alpha, (\widetilde{p} - value)^U_\alpha]), \tag{12}$$

where

$$(\widetilde{p} - value)^L_\alpha = \inf \left\{ \mathbf{P}_{H_0^{L\beta}}((\sqrt{n}D_n^+)^{L\beta} \geq d) : \beta \geq \alpha, d \in \widetilde{\sqrt{n}d_n^+}[\beta] \right\},$$

$$(\widetilde{p} - value)^U_\alpha = \sup \left\{ \mathbf{P}_{H_0^{U\beta}}((\sqrt{n}D_n^+)^{U\beta} \geq d) : \beta \geq \alpha, d \in \widetilde{\sqrt{n}d_n^+}[\beta] \right\},$$

$\widetilde{\sqrt{n}d_n^+}$ denotes the observed fuzzy statistic $\widetilde{\sqrt{n}D_n^+}$ and

$$H_0^{L\beta} : \left(\widetilde{F}_{\widetilde{X}}\right)^L_\beta = \left(\widetilde{F}^0\right)^L_\beta, \quad H_0^{U\beta} : \left(\widetilde{F}_{\widetilde{X}}\right)^U_\beta = \left(\widetilde{F}^0\right)^U_\beta,$$

$$\left(\sqrt{n}D_n^+\right)^{L\beta} = \sqrt{n} \max \left\{ \max_{1\leq i\leq n} \left\{ i/n - (\widetilde{F}_{\widetilde{X}}((\widetilde{X}_i)^U_\beta))^L_\beta \right\}, 0 \right\},$$

$$\left(\sqrt{n}D_n^+\right)^{U\beta} = \sqrt{n} \max \left\{ \max_{1\leq i\leq n} \left\{ i/n - (\widetilde{F}_{\widetilde{X}}((\widetilde{X}_i)^L_\beta))^U_\beta \right\}, 0 \right\}.$$

It is noticeable that for any $\beta \in (0,1]$, $\mathbf{P}_{H_0^{L\beta}}((\sqrt{n}D_n^+)^{L\beta} \geq (\sqrt{n}d_n^+)^{L\beta})$ ($\mathbf{P}_{H_0^{U\beta}}((\sqrt{n}D_n^+)^{U\beta} \geq (\sqrt{n}d_n^+)^{U\beta})$) is the classical p-value for testing the null hypothesis $H_0^{L\beta} : (\widetilde{F}_{\widetilde{X}})^L_\beta = (\widetilde{F}^0)^L_\beta$ ($H_0^{U\beta} : (\widetilde{F}_{\widetilde{X}})^U_\beta = (\widetilde{F}^0)^U_\beta$) based on a crisp random sample $(\widetilde{X}_1)^L_\beta, (\widetilde{X}_2)^L_\beta, \ldots, (\widetilde{X}_n)^L_\beta$ ($(\widetilde{X}_1)^U_\beta, (\widetilde{X}_2)^U_\beta, \ldots, (\widetilde{X}_n)^U_\beta$). Therefore $\widetilde{p} - value$ is a natural extension of the classical p-value (see Appendix) for Kolmogorov–Smirnov one-sample test with fuzzy observations and fuzzy hypothesis.

**Remark 5.5.** Based on Remark 3.8, for every $\beta \in (0,1]$, note that the distribution of $(\sqrt{n}D_n^+)^{L\beta}$ ($(\sqrt{n}D_n^+)^{U\beta}$) under the null hypothesis $H_0^{L\beta}$ ($H_0^{U\beta}$) is the same as the distribution $\sqrt{n}D_n^+$ under $H_0 : F_X(x) = F_{X^0}(x)$. Therefore, the membership function of $\widetilde{p}$-value reduces to

$$\mu_{\widetilde{p}-value}(x) = \sup_{\alpha\in[0,1]} \alpha I \left( x \in \left[ \mathbf{P}_{H_0}\left(\sqrt{n}D_n^+ \geq \left(\widetilde{\sqrt{n}d_n^+}\right)^U_\alpha\right), \mathbf{P}_{H_0}\left(\sqrt{n}D_n^+ \geq \left(\widetilde{\sqrt{n}d_n^+}\right)^L_\alpha\right) \right] \right). \tag{13}$$

**Remark 5.6.** It is easy to show that the fuzzy p-value is a fuzzy number on $[0,1]$. In addition, if the fuzzy random sample $\widetilde{X}_1, \widetilde{X}_1, \ldots, \widetilde{X}_n$ reduce to the crisp random sample, then the fuzzy p-value reduces to the classical p-value.

### 5.1. Method of decision making

Finally, a decision is made to accept or reject the hypothesis of interest, by comparing the observed fuzzy p-value with the given significance level. Since the p-value is defined as a fuzzy set, it is natural to consider the significance level as a fuzzy set, too. Based on [16], any set $\widetilde{\delta}$ on $(0,1)$ can be considered as a fuzzy significance level.

In this situation we need a method for comparing the obtained fuzzy p-value with a given fuzzy significance level. There are several ways to carry out this comparison (see, e. g. [5, 36]). We use a method for ranking fuzzy numbers based on the fuzzy preference relation [38], and then, by inception of the work by Parchami et al. [28], a method will be defined to test the hypothesis of interest.

**Definition 5.7.** Let $\mathbb{F}_{\mathcal{C}}(\mathbb{R})$ be the class of fuzzy numbers with continuous membership functions. For $\widetilde{A},\ \widetilde{B} \in \mathbb{F}_{\mathcal{C}}(\mathbb{R})$, let

$$\Delta_{\widetilde{A}\widetilde{B}} = \int_{\alpha:\widetilde{A}^U_\alpha > \widetilde{B}^L_\alpha} (\widetilde{A}^U_\alpha - \widetilde{B}^L_\alpha)\,\mathrm{d}\alpha + \int_{\alpha:\widetilde{A}^L_\alpha > \widetilde{B}^U_\alpha} (\widetilde{A}^L_\alpha - \widetilde{B}^U_\alpha)\,\mathrm{d}\alpha. \tag{14}$$

Then the degree of truth of "$\widetilde{A}$ is greater than $\widetilde{B}$", is defined to be

$$\Delta(\widetilde{A} \succ \widetilde{B}) = \frac{\Delta_{\widetilde{A}\widetilde{B}}}{\Delta_{\widetilde{A}\widetilde{B}} + \Delta_{\widetilde{B}\widetilde{A}}}. \tag{15}$$

The relation $\Delta$ is called the fuzzy preference relation.

**Proposition 5.8.** (a) The fuzzy preference relation $\Delta$ is reciprocal, i. e., for two fuzzy numbers $\widetilde{A}$ and $\widetilde{B}$

$$\Delta(\widetilde{B} \succ \widetilde{A}) = 1 - \Delta(\widetilde{A} \succ \widetilde{B}),$$

and especially

$$\Delta(\widetilde{A} \succ \widetilde{A}) = 0.5.$$

(b) The fuzzy preference relation $\Delta$ is transitive, in the sense that for three fuzzy numbers $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$, if $\Delta(\widetilde{A} \succ \widetilde{B}) \geq 0.5$ and $\Delta(\widetilde{B} \succ \widetilde{C}) \geq 0.5$, then $\Delta(\widetilde{A} \succ \widetilde{C}) \geq 0.5$.

**Definition 5.9.** Consider the problem of testing fuzzy hypothesis(9) based on a fuzzy random sample. Then, at fuzzy significance level of $\widetilde{\delta}$, the fuzzy test is defined to be a fuzzy set as follows

$$\widetilde{\varphi}_{\widetilde{\delta}}[\widetilde{x}_1, \dots, \widetilde{x}_n] = \left\{ \frac{\widetilde{\varphi}_{\widetilde{\delta}}(1)}{1}, \frac{\widetilde{\varphi}_{\widetilde{\delta}}(0)}{0} \right\}, \tag{16}$$

where $\widetilde{\varphi}_{\widetilde{\delta}}(1) = \Delta(\widetilde{p} - value \succ \widetilde{\delta})$ is called the degree of acceptance of $\widetilde{H}_0$ and $\widetilde{\varphi}_{\widetilde{\widetilde{\delta}}}(0) = 1 - \widetilde{\varphi}_{\widetilde{\delta}}(1)$ is the degree of rejection of $\widetilde{H}_0$.

**Remark 5.10.** If the decision maker is willing to do the test at the exact significance level, then equation (14) reduces as follows

$$\Delta_{\widetilde{A}\delta} = \int_{\alpha:\widetilde{A}^U_\alpha > \delta} (\widetilde{A}^U_\alpha - \delta)\,\mathrm{d}\alpha + \int_{\alpha:\widetilde{A}^L_\alpha > \delta} (\widetilde{A}^L_\alpha - \delta)\,\mathrm{d}\alpha. \tag{17}$$

To demonstrate the application of the proposed method, we provide a practical example using a real data set given in [37].

**Example 5.11.** A tire and rubber company is interested in the quality of a tire it has recently developed. Only 24 new tires were tested because the tests were destructive and took considerable time to complete. Six cars, all the same model and brand, were used to test the tires. Car model and brand were alike so that the car effects were not considered. Since, under some unexpected situations, we cannot measure the tire lifetime precisely, we can just mention the tire lifetime using terms of approximate. Therefore, the tire lifetimes are taken to be triangular fuzzy numbers as shown in Table 2.

| | |
|---|---|
| $\widetilde{x}_1 = (33262, 33978, 34889)_T$ | $\widetilde{x}_{13} = (32093, 32617, 33255)_T$ |
| $\widetilde{x}_2 = (32585, 33052, 33787)_T$ | $\widetilde{x}_{14} = (31720, 32611, 33497)_T$ |
| $\widetilde{x}_3 = (32806, 33418, 33908)_T$ | $\widetilde{x}_{15} = (31977, 32455, 33034)_T$ |
| $\widetilde{x}_4 = (33065, 33463, 34131)_T$ | $\widetilde{x}_{16} = (31943, 32466, 33212)_T$ |
| $\widetilde{x}_5 = (30743, 31624, 32460)_T$ | $\widetilde{x}_{17} = (32169, 33070, 33968)_T$ |
| $\widetilde{x}_6 = (32415, 33127, 34072)_T$ | $\widetilde{x}_{18} = (32900, 33543, 34335)_T$ |
| $\widetilde{x}_7 = (32687, 33224, 33908)_T$ | $\widetilde{x}_{19} = (30327, 30881, 31455)_T$ |
| $\widetilde{x}_8 = (32185, 32597, 33186)_T$ | $\widetilde{x}_{20} = (31187, 31565, 32237)_T$ |
| $\widetilde{x}_9 = (33423, 34036, 34771)_T$ | $\widetilde{x}_{21} = (33208, 34053, 34876)_T$ |
| $\widetilde{x}_{10} = (31639, 32584, 33542)_T$ | $\widetilde{x}_{22} = (30945, 31838, 32739)_T$ |
| $\widetilde{x}_{11} = (31511, 32290, 33064)_T$ | $\widetilde{x}_{23} = (31934, 32800, 33445)_T$ |
| $\widetilde{x}_{12} = (33060, 33844, 34449)_T$ | $\widetilde{x}_{24} = (33464, 34157, 34974)_T$ |

**Tab. 2.** Data set in Example 5.11.

Now, suppose that we wish to test fuzzy hypothesis (9) in which $\widetilde{F}^0$ denotes the c.d.f. of the normal distribution with the following fuzzy parameters

$$\widetilde{\mu} = (30000, 32000, 34000)_T,\ \widetilde{\sigma^2} = ((1500)^2, (2000)^2, (2500)^2)_T.$$

To compute $\widetilde{p}$-value, we should calculate the $\alpha$-cuts of fuzzy test statistic for every $\alpha \in (0, 1]$. For example, at level of $\alpha = 0.5$, from Equation (11),

$$\left(\widetilde{\sqrt{n}d_n^+}\right)_{0.5}^L = \sqrt{n}\max\left\{\max_{1\le i\le n}\left\{i/n - (\widetilde{F}^0((\widetilde{x}_i)_{0.5}^L))_{0.5}^U\right\}, 0\right\},$$

$$\left(\widetilde{\sqrt{n}d_n^+}\right)_{0.5}^U = \sqrt{n}\max\left\{\max_{1\le i\le n}\left\{i/n - (\widetilde{F}^0((\widetilde{x}_i)_{0.5}^U))_{0.5}^L\right\}, 0\right\},$$

in which

$$(\widetilde{F}^0(x))_{0.5}^L = \inf_{\mu\in\widetilde{\mu}[0.5],\sigma^2\in\widetilde{\sigma}^2[0.5]} F_{\mu,\sigma^2}(x) = \Phi\left(\frac{x - \widetilde{\mu}_{0.5}^U}{\sqrt{\widetilde{\sigma^2}_{0.5}^U}}\right),$$

$$(\widetilde{F}^0(x))_{0.5}^U = \sup_{\mu\in\widetilde{\mu}[0.5],\sigma^2\in\widetilde{\sigma}^2[0.5]} F_{\mu,\sigma^2}(x) = \Phi\left(\frac{x - \widetilde{\mu}_{0.5}^L}{\sqrt{\widetilde{\sigma^2}_{0.5}^L}}\right),$$

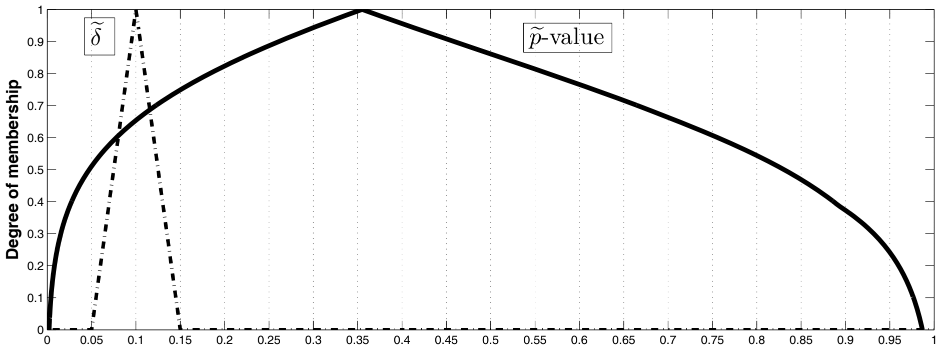


**Fig. 9.** Fuzzy p-value and fuzzy significance level in Example 5.11.

where $F_{\mu,\sigma^2}$ denotes the c.d.f. of the normal distribution with mean $\mu$ and variance $\sigma^2$ such that $\mu \in \widetilde{\mu}[0.5] = [31000, 33000]$, $\sigma^2 \in \widetilde{\sigma}^2[0.5] = [3125000, 5125000]$, and $\Phi$ denotes the c.d.f. of the standard normal distribution. So, we obtain $\widetilde{\sqrt{n}d_n^+}[0.5] = [0.274, 1.198]$, and from (13) we obtain $\widetilde{p} - value[0.5] = [0.047, 0.830]$. By continuing this procedure for other values of $\alpha$, the membership function of the observed $\widetilde{p} - value$ is drawn point-by-point based on values of $\alpha \in \{0.001, 0.002, \ldots, 1\}$ versus $\{(\widetilde{p} - value)^L_\alpha, (\widetilde{p} - value)^U_\alpha\} \subseteq [0, 1]$ as we observed in Figure 9. This function can be divided in two functions, say left and right. So we may estimate them by two quadratic functions as $\alpha = a((\widetilde{p} - value)^L_\alpha)^2 + b(\widetilde{p} - value)^L_\alpha + c$ and $\alpha = a^{'}((\widetilde{p} - value)^U_\alpha)^2 + b^{'}(\widetilde{p} - value)^U_\alpha + c^{'}$. Among many different methods to evaluate the raw data to find the curve fitting model parameters, we apply "polynomial method" using "MATLAB" software to estimate the functional forms of the left and right. Finally, the membership function of $\widetilde{p} - value$ is derived "about 0.355" with the following membership function

$$\mu_{\widetilde{p}-value}(x) = \begin{cases} 0 & 0 \le x < 0.002, \\ -10.572x^2 + 5.7313x + 0.297 & 0.002 \le x < 0.355, \\ -2.0049x^2 + 1.405x + 0.753 & 0.355 \le x < 0.986, \\ 0 & 0.986 \le x \le 1. \end{cases}$$

For the fuzzy significance level $\widetilde{\delta} = (0.05, 0.10, 0.15)_T$ by (16) we get

$$\widetilde{\varphi}_{\widetilde{\delta}}[\widetilde{x}_1, \ldots, \widetilde{x}_{24}] = \left\{ \frac{0.02}{1}, \frac{0.98}{0} \right\}.$$

Therefore, the null hypothesis of $\widetilde{H}_0$ is accepted with degree of acceptance 0.98.

**Example 5.12.** Consider the previous example. Assume that the significance level is given by a crisp number $\delta = 0.10$. Now using Remark 5.10 we obtain $\Delta(\widetilde{p} - value \succ \delta) = 1$. So the fuzzy test is obtained as follows

$$\widetilde{\varphi}_{\delta}[\widetilde{x}_1, \ldots, \widetilde{x}_{24}] = \left\{ \frac{0}{1}, \frac{1}{0} \right\}.$$

Therefore, we accept the null hypothesis of $\widetilde{H}_0$ with degree of acceptance 1.

## 6. CONCLUSION

We extended the concepts of fuzzy distribution function and fuzzy empirical distribution function at a crisp number and at a fuzzy number. Then, we extended the Glivenko–Cantelli theorem to a fuzzy environment. Based on the obtained results, the Kolmogorov–Smirnov one-sample test statistic was extended to the fuzzy environment. Then, a new method was introduced to compute the so-called fuzzy p-value for testing imprecise hypothesis. Finally, for evaluating the null hypothesis, we employ a fuzzy preference to compare the observed fuzzy p-value and the given fuzzy significance level.

The problem of Kolmogorov–Smirnov two-sided test and other non-parametric Goodness-of-Fit tests in fuzzy environment are potential subjects for future research.

## APPENDIX: KOLMOGOROV–SMIRNOV ONE-SAMPLE TEST

Let $X_1, X_2, \ldots, X_n$ be a random sample from a population with a continuous, but unknown cumulative distribution function (c.d.f.) $F_X(x)$. The Kolmogorov–Smirnov one-sample test statistic is based on the differences between the c.d.f. $F_X(x)$ and the empirical distribution function (e.d.f.) $\widehat{F}_n(x)$. Due to the Strong Law of Large Numbers, $\widehat{F}_n(x) \to F_X(x)$ with probability one for each $x$. Concerning the large sample behavior of the $\widehat{F}_n(x)$, we have also the following (see, e. g. [9]).

**Lemma 6.1.** (Glivenko–Cantelli theorem)

$$\mathbf{P}\left(\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|\widehat{F}_n(x) - F_X(x)| = 0\right) = 1.$$

In other words, with probability one, $\widehat{F}_n(x) \to F_X(x)$ uniformly in $x$.

So, as $n$ increases, the step function $\widehat{F}_n(x)$ approaches to the true distribution $F_X(x)$ for all $x$, so that for large $n$, $\mid \widehat{F}_n(x) - F_X(x) \mid$ should be small for all values of $x$. If $n$ is large enough then the so-called Kolmogorov–Smirnov one-sample statistic is defined as

$$\sqrt{n}D_n = \sqrt{n}\sup_{x\in\mathbb{R}} \mid \widehat{F}_n(x) - F_X(x) \mid \tag{18}$$

In addition, the directional deviations defined as

$$\sqrt{n}D_n^+ = \sqrt{n}\sup_{x\in\mathbb{R}}[\widehat{F}_n(x) - F_X(x)] = \sqrt{n}\max\left\{\max_{1\le i\le n}\{i/n - F_X(X_i)\}, 0\right\}, \tag{19}$$

$$\sqrt{n}D_n^- = \sqrt{n}\sup_{x\in\mathbb{R}}[F_X(x) - \widehat{F}_n(x)], \tag{20}$$

are called the one-sided Kolmogorov–Smirnov test statistics. Obviously, $\sqrt{n}D_n = \max\{\sqrt{n}D_n^-, \sqrt{n}D_n^+\}$ and also $\sqrt{n}D_n^+$ and $\sqrt{n}D_n^-$ have identical distributions because of symmetry. The sampling distribution of $\sqrt{n}D_n$ is given below

$$\mathbf{P}(\sqrt{n}D_n < \frac{1}{2\sqrt{n}} + c) = \begin{cases} 0 & v \le 0, \\ p(n) & 0 < c < \frac{2n-1}{2\sqrt{n}}, \\ 1 & c \ge \frac{2n-1}{2\sqrt{n}}, \end{cases} \tag{21}$$

where

$$p(n)=\int_{1/2n-\frac{c}{\sqrt{n}}}^{1/2n+\frac{c}{\sqrt{n}}}\dots\int_{(2n-1)/2n-\frac{c}{\sqrt{n}}}^{(2n-1)/2n+\frac{c}{\sqrt{n}}} f(u_1,\dots,u_n)\,\mathrm{d}u_1\dots\,\mathrm{d}u_n,$$

in which, $f(u_1,u_2,\dots,u_n)=n!\ I(u_1<u_2<\dots u_n)$, and

$$\mathbf{P}(\sqrt{n}D_n\le c)=1-2\sum_{i=1}^{\infty}(-1)^{i-1}e^{-2i^2c^2},\ c>0, \tag{22}$$

for large sample $n$ ([7], p. 117). Moreover, the sampling distributions $\sqrt{n}D_n^+$ under $H_0$ is given below

$$\mathbf{P}_{H_0}(\sqrt{n}D_n^+>c)=\begin{cases}1 & c\le 0,\\ h(c) & 0<c<\sqrt{n},\\ 0 & c\ge\sqrt{n},\end{cases} \tag{23}$$

where

$$\begin{aligned}h(c)=(1-c/\sqrt{n})^n+&\\ &(c/\sqrt{n})\textstyle\sum_{j=1}^{[n(1-c/\sqrt{n})]}\begin{pmatrix}n\\ j\end{pmatrix}(1-c/\sqrt{n}-\frac{j}{n})^{n-j}(c/\sqrt{n}+\frac{j}{n})^{j-1},\end{aligned} \tag{24}$$

in addition,

$$\mathbf{P}_{H_0}(\sqrt{n}D_n^+<c)=1-e^{-2c^2} \tag{25}$$

for $c\ge 0$ as $n\to\infty$ (see [7], p. 119).

In Kolmogorov–Smirnov one-sample test, we wish to test $H_0: F_X(x)=F_X^0(x)$ for all $x$, where $F_X^0(x)$ is a specified continuous distribution function. If $(\sqrt{n}d_n)$, $(\sqrt{n}d_n^+)$ and $(\sqrt{n}d_n^-)$ denote the observed value of $\sqrt{n}D_n$, $\sqrt{n}D_n^+$ and $\sqrt{n}D_n^-$, respectively, then for the alternative

$$H_{1,+}: F_X(x)>F_X^0(x)\ \text{ for some } x,$$

we reject the null hypothesis, at significance level of $\delta$, when $p-value=\mathbf{P}_{H_0}(\sqrt{n}D_n^+>(\sqrt{n}d_n^+))<\delta$, and for the alternative

$$H_{1,-}: F_X(x)<F_X^0(x)\ \text{ for some } x,$$

we reject the null hypothesis when $p-value=\mathbf{P}_{H_0}(\sqrt{n}D_n^->(\sqrt{n}d_n^-))<\delta$, and for the alternative

$$H_1: F_X(x)\neq F_X^0(x)\ \text{ for some } x,$$

we reject the null hypothesis when $p-value=\mathbf{P}_{H_0}(\sqrt{n}D_n>(\sqrt{n}D_n)_{Ob.})<\delta$ ([7, 25]).

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*Gholamreza Hesamian, Corresponding author. Department of Statistics, Payame Noor University, 19395-3697, Tehran. Iran.*
*e-mail: gh.hesamian@chb.pnu.ac.ir*

*S. M. Taheri, Department of Engineering Science, College of Engineering, University of Tehran, Tehran, Iran, and Department of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111. Iran.*
*e-mail: sm_taheri@ut.ac.ir*