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DEGENERATE HOPF BIFURCATIONS AND THE FORMATION MECHANISM OF CHAOS IN THE QI 3-D FOUR-WING CHAOTIC SYSTEM

Hongtao Liang, Yanxia Tang, Li Li, Zhouchao Wei and Zhen Wang

In order to further understand a complex 3-D dynamical system proposed by Qi et al, showing four-wing chaotic attractors with very complicated topological structures over a large range of parameters, we study degenerate Hopf bifurcations in the system. It exhibits the result of a period-doubling cascade to chaos from a Hopf bifurcation point. The theoretical analysis and simulations demonstrate the rich dynamics of the system.

Keywords: four-wing chaotic attractors, Lyapunov coefficient, degenerate Hopf bifurcations, period-doubling cascade

Classification: 34H10, 34H20

1. INTRODUCTION

Chaos theory is a field of study in applied mathematics, and it has applications in several disciplines including physics, economics, biology and philosophy. The now-classic Lorenz system [3] has motivated a great deal of interest and investigation of 3-D autonomous chaotic systems with simple nonlinearities (see, for instance, those found in [16-18]). In the continuous case, however, intentionally constructing a new chaotic system is still a challenging task.

Many 3-D chaotic systems have been found in recent years. In 2004, Lü, Chen and Cheng discussed the important problems of classification and normal form of three-dimensional quadratic autonomous chaotic systems [1]. It is noted that some classical 3-D autonomous chaotic systems have three particular fixed points: one unstable node and two unstable saddle-foci (for example, Lorenz system [3], Chen system [1], Lü system [4], the conjugate Lorenz-type system [5] et. al.). The other 3-D chaotic systems, such as the original Rölsser system [12], DLS [20], Burke–Shaw system [13], have two unstable saddle-foci. Yang and Chen found another 3-D chaotic system with three fixed points: one saddle and two stable node-foci [27]. In 2010, Yang, Wei and Chen [28] introduced and analyzed a new 3-D chaotic system with six terms including only two quadratic terms in a form very similar to the Lorenz, Chen, Lü and Yang-Chen systems, but it has only two fixed points: two stable node-foci. Some questions about periodic, homoclinic and heteroclinic orbits and classification of chaos, are related to the dynamics of some dynamical systems. Recently, Wei and Yang [28] proposed a new 3-D chaotic system with six terms including only one exponential quadratic nonlinear term, which can generate a double-scroll chaotic attractor when all of equilibria are stable. Many theoretical analysis and numerical simulation about this kinds of systems are showed in [21–23].

On the other hand, the topic on generating multi-wing chaotic attractors from a 3-D smooth autonomous quadratic system deserves further detailed investigation. Qi et al. proposed a new 3-D quadratic autonomous system [11] ranging from one or more stationary points to periodic motion and even four-wing chaotic attractor with very complicated topological structures over a large range of parameters. The chaotic system is described by

$$\begin{cases} \dot{x} = a(y - x) + eyz \\ \dot{y} = cx + dy - xz \\ \dot{z} = -bz + xy, \end{cases} \tag{1}$$

where a, b, d are all real positive constant parameters and c, e are real constant parameters. The four-wing chaotic attractor and its projection are shown in Figure 1, respectively.

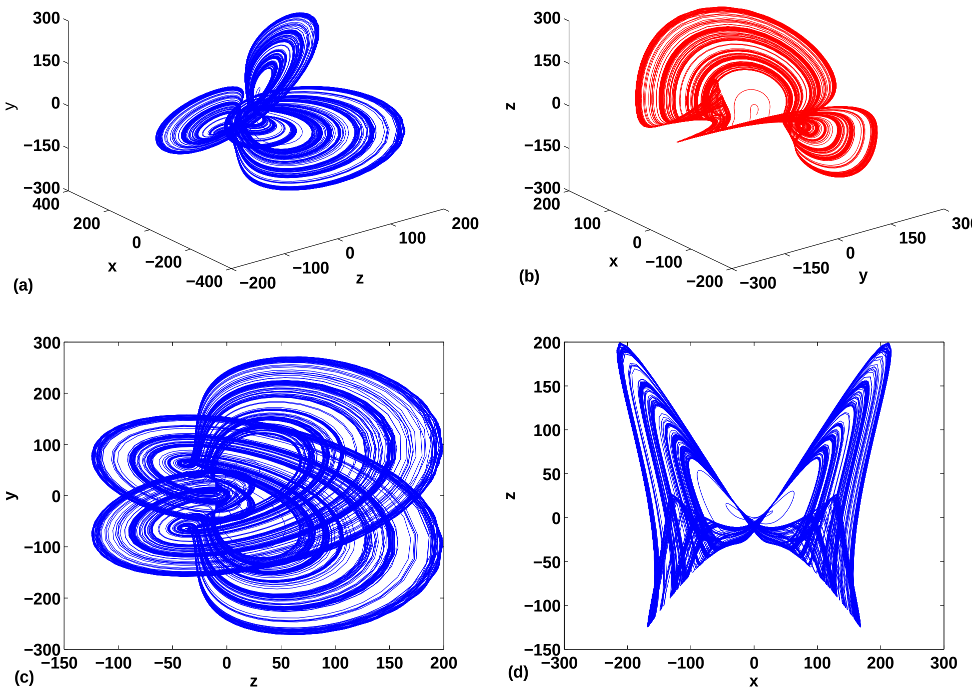

Fig. 1. Parameter values $(a, b, c, d, e) = (16, 102.8, -16.6, 20.52, 0.5)$ and initial values $(0.001, 0.001, 0.001)$: (a) Four-wing chaotic attractor of system (1) in z-x-y space; (b) Four-wing chaotic attractor of system (1) in y-x-z space; (c) Projection of (a) into $y - z$ plane; (d) Projection of (a) into $x - z$ plane.

As far as we know, the simplest way of the regime about the existence of a periodic orbit is through the Hopf bifurcations. The analysis of the codimension one Hopf bifurcation about equilibrium $O = (0, 0, 0)$ using the center manifold theorem are presented in [18]. In this paper, we study the complicated dynamics as degenerate Hopf bifurcations in the Qi 3-D four-wing system. It exhibits the result of a period-doubling cascade to chaos from a Hopf bifurcation point. The influence of system parameters on other bifurcations are also investigated. The theoretical analysis and simulations demonstrate the rich dynamics of the system. By using the calculation of the Lyapunov coefficients associated to the Hopf bifurcations, we study all possible bifurcations (generic and degenerate ones) which occur at the equilibrium O of system (1). In this way the analysis presented in [19] are extended. More precisely, for the equilibrium O, the Hopf surface is obtained in the space of parameters and the first Lyapunov coefficient l_1 is calculated. It is shown that this coefficient vanishes along a curve on the Hopf surface, giving rise to codimension two bifurcations, and the second Lyapunov coefficient l_2 is calculated. In particular, we obtain the result of a period-doubling cascade from a Hopf bifurcation point.

The paper is organized as follows. In Section 2, we present the outline of the Hopf bifurcation methods about codimension one, two and three Hopf bifurcations, in particular, how to calculate the Lyapunov coefficients related to the stability of the equilibrium $O = (0, 0, 0)$. In Section 3, we obtain the main results of this paper, described in Theorems 3.1-3.3. In Section 4, numerical simulations demonstrate the rich dynamics of the system. Finally, in Section 5, we make some concluding remarks.

2. LINEAR ANALYSIS AND AN OUTLINE OF THE HOPF BIFURCATION METHODS

System (1) has the equilibrium $O = (0, 0, 0)$, which exists for any parameter values. The Jacobian matrix of system (1) at the equilibrium O is

$$J(O) = \begin{pmatrix} -a & a & 0 \\ c & d & 0 \\ 0 & 0 & -b \end{pmatrix},$$

and its corresponding characteristic equation

$$(\lambda + b)[\lambda^2 + (a - d)\lambda - a(c + d)] = 0. \tag{2}$$

According to the Routh–Hurwitz criterion and a, b, d are all real positive parameters, the characteristic polynomial (2) has three roots with negative real parts under the following condition:

$$d > a > 0, \quad c < -d. \tag{3}$$

Suppose that the characteristic equation of system (1) has a pair of pure imaginary roots $\pm i\omega$ ($\omega \in R^+$). It is easy to show that when $d = d_0 = a$, (2) yields

$$\lambda_1 = -b < 0, \quad \lambda_{2,3} = \pm\sqrt{-a(a + c)}\, i,$$

where $a + c < 0$. For convenience, we mark $k = -(a + c)$. Summarizing, we have the following proposition.

Proposition 2.1. Define

$$T = \{(a, b, d, e, k) | a > 0, b > 0, d = d_0 = a, e \in R, k > 0\},$$

then Jacobian matrix of system (2) at $O(0, 0, 0)$ has one negative real eigenvalue $-b$ and a pair of purely imaginary eigenvalues $\pm\sqrt{ak}\, i$.

Taking d as the Hopf bifurcation parameter, the transversally condition

$$\mathrm{Re}(\lambda'(d_0))|_{\lambda=\sqrt{ak}i} = \frac{1}{2} > 0$$

is also satisfied. Therefore, we have the following theorem.

Theorem 2.1. (Existence of Hopf bifurcation) If $(a, b, d, e, k) \in T$ and d varies and passes through the critical value $d_0 = a$, system (1) undergoes a Hopf bifurcation at the equilibrium $O(0, 0, 0)$.

The rest of this section is showing the projection method described in [2,14,19] for the calculation of the first, second and third Lyapunov coefficients associated to the Hopf bifurcations, denoted by l_1, l_2, and l_3 respectively. The method has been applied in some systems [10,14,15]. Consider the differential equation

$$\dot{X} = f(X, \mu), \tag{4}$$

where $X \in R^3$ and $\mu \in R^5$ are respectively vectors representing phase variables and control parameters. Assume that f is a class of C^∞ in $R^3 \times R^5$. Suppose that (4) has an equilibrium point $X = X_0$ at $\mu = \mu_0$, and denoting the variable $X - X_0$ also by X, write

$$F(X) = f(X, \mu_0), \tag{5}$$

as

$$\begin{aligned} F(X) &= AX + \frac{1}{2}B(X, X) + \frac{1}{6}C(X, X, X) + \frac{1}{24}D(X, X, X, X) && (6)\\ &\quad + \frac{1}{120}E(X, X, X, X, X) + \frac{1}{720}K(X, X, X, X, X, X) && (7)\\ &\quad + \frac{1}{5040}L(X, X, X, X, X, X, X) + O(\| X \|^8), && (8) \end{aligned}$$

where $A = f_x(0, \mu_0)$ and, for $i = 1, 2, 3$,

$$B(X, Y) = \sum_{j,\,k=1}^{3} \frac{\partial^2 F_i(\xi)}{\partial \xi_j \partial \xi_k}\Big|_{\xi=0} X_j Y_k,\ C(X, Y, Z) = \sum_{j,\,k,\,l=1}^{3} \frac{\partial^3 F_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}\Big|_{\xi=0} X_j Y_k Z_l,$$

and so on for D_i, E_i, K_i and L_i. Suppose that A has a pair of complex eigenvalues on the imaginary axis: $\lambda_{2,3} = \pm i w_0 \, (w_0 > 0)$, and these eigenvalues are the only eigenvalues with $\mathrm{Re}\lambda = 0$. Let T^c be the generalized eigenspace of A corresponding to $\lambda_{2,3}$. Let $p, q \in C^3$ be vectors such that

$$Aq = iw_0 q, \quad A^T p = -iw_0 p, \quad \langle p, q \rangle = 1, \tag{9}$$

where A^T is the transposed of the matrix A. Any vector $y \in T^c$ can be represented as $y = wq + \bar{w}\bar{q}$, where $w = \langle q, y \rangle \in C$. The two-dimensional center manifold associated to the eigenvalues $\lambda_{2,3}$ can be parameterized by w and $\bar{w}$, by means of an immersion of the form $X = H(w, \bar{w})$, where $H : C^2 \to R^3$ has a Taylor expansion of the form

$$H(w, \bar{w}) = wq + \bar{w}\bar{q} + \sum_{2 \le j+k \le 5} \frac{1}{j!k!} h_{jk} w^j \bar{w}^k + O(|w|^6),$$

with $h_{jk} \in C^3$ and $h_{jk} = \bar{h_{kj}}$. Substituting this expression into (5) we obtain the following differential equation

$$H_w w' + H_{\bar{w}} \bar{w}' = F(H(w, \bar{w})),$$

where F is give by (5). The complex vectors h_{ij} are obtained solving the system of linear equations defined by the coefficients of (5), taking into account the coefficients of F, so that system (5), on the chart w for a central manifold, writes as follows

$$\dot{w} = iw_0 w + \frac{1}{2} G_{21} w |w|^2 + \frac{1}{12} G_{32} w |w|^4 + O(|w|^6),$$

where $G_{ij} \in C$. The first Lyapunov coefficient can be written as

$$l_1 = \frac{1}{2} \mathrm{Re}\, \mathrm{G}_{21}, \tag{10}$$

where $G_{21} = \langle p, C(q, q, \bar{q}) + B(\bar{q}, h_{20}) + 2B(q, h_{11}) \rangle$. Defining $\mathcal{H}_{32}$ as

$$\begin{aligned}
\mathcal{H}_{32} \quad = \quad & 6B(h_{11}, h_{21}) + B(\bar{h}_{20}, h_{30}) + 3B(\bar{h}_{21}, h_{20}) + 3B(q, h_{22}) \\
& + 2B(\bar{q}, h_{31}) + 6C(q, h_{11}, h_{11}) + 3C(q, \bar{h}_{20}, h_{20}) + 3C(q, q, \bar{h}_{21}) \\
& + 6C(q, \bar{q}, h_{21}) + 6C(\bar{q}, h_{20}, h_{11}) + C(\bar{q}, \bar{q}, h_{30}) \\
& + D(q, q, q, \bar{h}_{20}) + 6D(q, q, \bar{q}, \bar{h}_{11}) + 3D(q, \bar{q}, \bar{q}, h_{20}) \\
& + E(q, q, q, \bar{q}, \bar{q}) - 6G_{21} h_{21} - 3\bar{G}_{21} h_{21}
\end{aligned}$$

and $G_{32} = \langle p, \mathcal{H}_{32} \rangle$, the second Lyapunov coefficient l_2 is given by

$$l_2 = \frac{1}{12} \mathrm{Re}\, \mathrm{G}_{32}. \tag{11}$$

The third Lyapunov coefficient is defined by

$$l_3 = \frac{1}{144} \mathrm{Re}\, \mathrm{G}_{43}, \tag{12}$$

where $G_{43} = \langle p, \mathcal{H}_{43} \rangle$. The expression for $\mathcal{H}_{43}$ is too large to be put in print and can be found in [19].

3. HOPF BIFURCATION OF SYSTEM (1)

In this section, we study the stability of O under the conditions $d = d_0$. Then, using the notation of the previous section, the multilinear symmetric functions can be written as

$$B(X,Y) = (eX_2Y_3 + eX_3Y_2, -X_1Y_3 - X_3Y_1, X_1Y_2 + X_2Y_1), \tag{13}$$
$$C(X,Y,Z) = (0,0,0). \tag{14}$$

From (9), one has

$$q = \left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{k}i}, 1, 0\right), \; p = \left(-\frac{a+k}{2\sqrt{ak}}i, \frac{1}{2}\left(1+\sqrt{\frac{a}{k}}\right), 0\right)$$

The complex vectors h_{11} and h_{20} are

$$h_{11} = \left(0, 0, \frac{2a}{ab+k}\right), \quad h_{20} = \left(0, 0, \frac{2\sqrt{a}}{(\sqrt{a}+\sqrt{k}i)(b+2\sqrt{ak}i)}\right)$$

The complex number G_{21} defined in (4) has the form

$$G_{21} = \frac{u_1(a,b,e,k)}{(a+k)(b^2+4ak)} + \frac{\sqrt{a}u_2(a,b,e,k)}{\sqrt{k}b(a+k)(b^2+4ak)}i,$$

where

$$\begin{aligned} u_1(a,b,e,k) &= 2a^2 - ab + 2a^2e + abe + 2aek + bek, \\ u_2(a,b,e,k) &= (3+e)ab^2 + (8a^2+2ab)k + (8a^2-2ab+3b^2)ek \\ &\quad +(8a-2b)ek^2. \end{aligned}$$

Defining $b_0 = -\frac{2a(ae+ek+a)}{ae+ek-a}$ and the following subsets of the Hopf surface T

$$\begin{aligned} U_1 &= \left\{(a,b,d,e,k)|a>0, b>0, d=d_0, e \geq \frac{a}{k+a}, k>0\right\}, \\ U_2 &= \left\{(a,b,d,e,k)|a>0, 0<b<b_0, d=d_0, |e| < \frac{a}{k+a}, k>0\right\}, \\ S_1 &= \left\{(a,b,d,e,k)|a>0, b>b_0, d=d_0, |e| < \frac{a}{k+a}, k>0\right\}, \\ S_2 &= \left\{(a,b,d,e,k)|a>0, b>0, d=d_0, e \leq -\frac{a}{k+a}, k>0\right\}, \end{aligned}$$

we have the following theorem.

Theorem 3.1. Consider the five-parameter family of differential equations (1). The first Lyapunov coefficient associated with the equilibrium O is given by

$$l_1 = \frac{2a^2 - ab + 2a^2e + abe + 2aek + bek}{2(a+k)(b^2+4ak)}. \tag{15}$$

If

$$h(a, b, e, k) = 2a^2 - ab + 2a^2e + abe + 2aek + bek$$

is different from zero then the three-parameter family of differential equations (1) has a transversal Hopf point at O for $d = d_0 = a$ and $k = -(a+c) > 0$. More specifically, if $(a, b, d, e, k) \in U_1 \cup U_2$ then the Hopf point at O is unstable (weak repelling focus) and for each $d < d_0$, but close to d_0, there exists an unstable limit cycle near the asymptotically stable equilibrium O; if $(a, b, d, e, k) \in S_1 \cup S_2$ then the Hopf point at O is asymptotically stable (weak attractor focus) and for each $d > d_0$, but close to d_0, there exists a stable limit cycle near the unstable equilibrium O.

The sign of the first Lyapunov coefficient is determined by the sign of the numerator of (9) since the denominator is positive. Observe that the first Lyapunov coefficient vanishes on the straight line

$$D = \left\{(a, b, d, e, k)|a > 0, b = b_0, d = d_0, -\frac{a}{k+a} < e < \frac{a}{k+a}, k > 0\right\}.$$

In the following theorem we study the sign of the second Lyapunov coefficient on the straight line D where the first coefficient vanishes.

Theorem 3.2. Consider the system (1). The second Lyapunov coefficient at O for parameter values in D is given by

$$l_2|_D = \frac{(ae + ek - a)^2}{16ak(a+k)^2}.$$

As $e \neq \frac{a}{k+a}$ then system (1) has a transversal Hopf point of codimension 2 at O for parameters in D. Moreover, the Hopf point at O is unstable since $l_2 > 0$. There are two limit cycles, one stable and the other unstable, near the equilibrium O for suitable values of the parameters. The bifurcation diagram at typical points P (on the straight line D) is illustrated in Figure 2.

Proof. As the function $B(X, Y)$ and $C(X, Y, Z)$ in (14), the second Lyapunov coefficient can be obtained for the parameters on the straight line D. One has

$$G_{21} = -\frac{3\sqrt{a}(ek + ea - a)}{2\sqrt{k}b(a+k)}i,$$

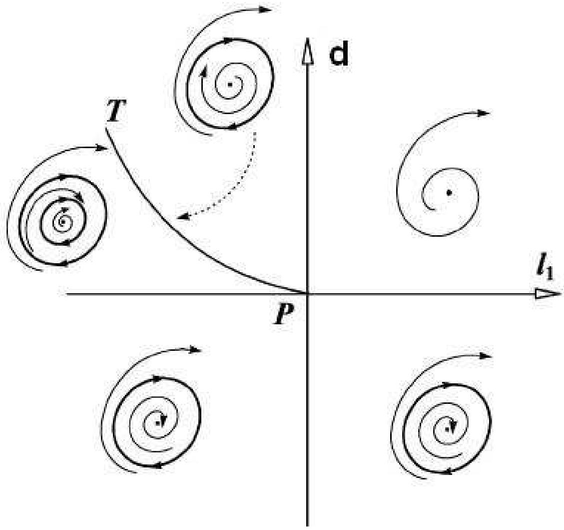

Fig. 2. Bifurcation diagrams of system (1) at typical points P on D. The curves T correspond to the fold limit cycle bifurcations.

$$h_{21} = \Bigg(\frac{(ek+ea-a)(3a^2(1+e)^2 + 2ae(3e-1)k + 3e^2k^2)}{4\sqrt{a}k(a+k)(\sqrt{a}+\sqrt{k}i)(a+ae-ek-2\sqrt{ak}ei)(ek+ea+a)}, \frac{(ek+ea-a)(3a^2(1+e)^2 + 2ae(3e-1)k + 3e^2k^2)}{4ak(\sqrt{a}+\sqrt{k}i)^2(a+ae-ek-2\sqrt{ak}ei)(ek+ea+a)}, 0 \Bigg),$$

$$h_{30} = \Bigg(-\frac{3(ek+ea-a)(a+ae+3ek-2e\sqrt{ak}i)}{8\sqrt{a}k(\sqrt{a}+\sqrt{k}i)^3(a+ae-ek-2e\sqrt{ak}i)}, -\frac{3(ek+ea-a)(a^{3/2}(1+e)+\sqrt{a}ek+a(3+e)\sqrt{k}+iek^{3/2})}{8ak(\sqrt{a}+\sqrt{k}i)^3(a+ae-ek-2\sqrt{ak}ei)}, 0 \Bigg),$$

$$h_{31} = \Bigg(0, 0, \frac{3(ek+ea-a)(5a^{3/2}(1+e) - 3\sqrt{a}ek + iek^{3/2} + ia(1+9e)\sqrt{k})}{4k(\sqrt{a}+\sqrt{k}i)^3(ek+ea+a)(a-i\sqrt{ek})} \Bigg),$$

$$h_{22} = \Bigg(0, 0, -\frac{(ek+ea-a)^2(3a^2(1+e)^2 + 2ae(3e-1)k + 3e^2k^2))}{ak(1+k)^2(ek+ea+a)(a^2(1+e)^2 + 2a(e-1)ek + e^2k^2)} \Bigg),$$

$$G_{32} = \frac{3(ek+ea-a)^2}{4ak(a+k)^2} + \frac{3g_1(a,e,k)}{g_2(a,e,k)}i,$$

where

$$
\begin{aligned}
g_1(a,e,k) = {} & a^6(1+e)^3[-41+e^2(41-104k)+a^2e^4k^4(-87+246e-64ke \\
& -615e^2+120e^2k)+20a^7(-1+e)(1+e)^5-41e^6k^6 \\
& +2ae^5k^5(41-123e+10ke)+e(-23+17k)] \\
& +2a^5e(1+e)k[41-34ek-38e^3k+e^2(82+34k)+3e^4(-41+50k)] \\
& +a^4e^2k^2[-87+92e-4e^3(41+16k)+2e^2(59+24k) \\
& +5e^4(-123+80k)]+4a^3e^3k^3[23+e^2(41-44k)+5e^3(-41+15k) \\
& +e(41+64k)+e^3(-41+120k)], \\
g_2(a,e,k) = {} & 16a^{3/2}k^{3/2}(a+k)^2(a+ae+ek)^2(a^2(1+e)^2+2ae(e-1)k+e^2k^2).
\end{aligned}
$$

By the above theorem and calculation, one has

$$
l_1 = \frac{1}{2}\mathrm{Re}G_{21} = 0, \quad l_2|_D = \frac{1}{12}\mathrm{Re}G_{32} = \frac{(ek+ea-a)^2}{16ak(a+k)^2} > 0.
$$

Therefore, the theorem 3.2 is proved. □

The largest number of small periodic orbits which can be created via Hopf bifurcation is determined by its codimension, which is directly related to the Lyapunov coefficients. Thus, the codimension of a Hopf point plays a key role in determining the number of small periodic orbits of the system.

4. NUMERICAL SIMULATIONS

In this section we present some numerical simulations of system (1) for several values of the parameters. The main purpose is to illustrate the creation of stable limit cycles through the Hopf bifurcations at the equilibrium O, proved to occur in the previous sections, and demonstrate the existence of the four-wing chaotic attractor.

For $a = 16$, $c = -16.6$ (i. e. k=0.6), $b = 102.8$, $e = 0.5$, system (1) has three equilibria and the origin $O = (0,0,0)$ as its equilibrium. Note that for these parameter values, we have the bifurcation value $d_0 = a = 16$ and $(a,b,d,e,k) \in S_1$. According to Theorem 3.1, the system (1) undergoes a Hopf bifurcation when the parameter d crosses the critical value $d = d_0$, and a stable periodic orbit emerges from O with $d > a$ in the neighborhood $d = a$. Choosing initial values $(0.001, 0.001, 0.001)$ near the equilibrium O, we take $d = d_0 + 0.4$ in Figure 3(a), a stable periodic orbit exists near the unstable equilibrium O. Furthermore, we take $d = d_0 + 0.98$ in Figure 3(b), $d = d_0 + 1.32$ in Figure 3(c) and $d = d_0 + 4.52$ in Figure 3(d). To better characterize the dynamic behavior of the system, we give the Poincaré mappings of the system about these parameters values (Figures 4(a-d)). It shows that when the parameter d moves away from the critical value $d = d_0$, a cascade of period doubling bifurcations occurs from the limit cycles that arose in the Hopf bifurcation. Finally, a four-wing chaotic attractor is generated.

In the limit of this period doubling bifurcations, after this, these two attractors merge into an strange attractor (see Figure 3(c)). This is one of the mechanisms through which

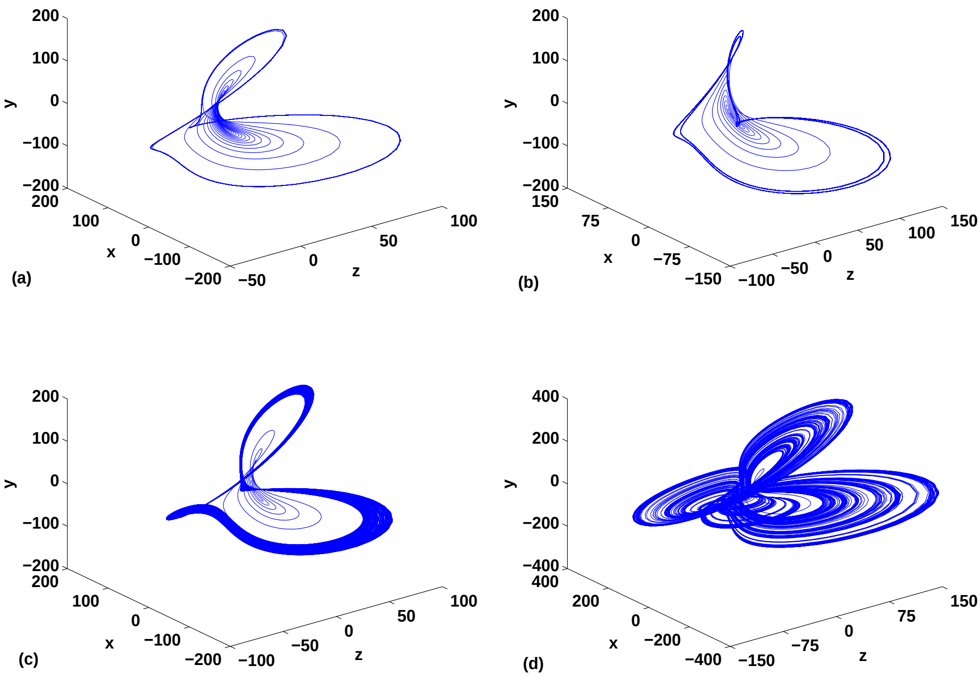

Fig. 3. Orbits of system (1) with parameter values $(a, b, c, e) = (16, 102.8, -16.6, 0.5)$, starting initial values $(0.001, 0.001, 0.001)$: (a) $d = 16 + 0.4$; (b) $d = 16 + 0.98$; (c) $d = 16 + 1.32$; (d) $d = 16 + 4.52$.

system (1) enters into chaotic regimes. Observe that it begins with the creation of the limit cycles in the Hopf bifurcations which take place at the points O for the critical parameter value $d = d_0$. It is an interesting and a tough task to determinate of the basins of attraction of strange attractor shown in Figure 3.

5. CONCLUSION

In this paper, we analyze the Lyapunov stability of the equilibrium O of system (1). Through the analysis we obtain the surfaces for which the system presents Hopf bifurcations at the equilibrium. Then we make an extension of the analysis to the degenerate cases, happening in the locus on the Hopf surfaces where the Lyapunov coefficient vanishs. The second Lyapunov coefficient makes possible the determination of the Lyapunov stability. Moreover, numerical simulations were performed for several values of the parameters, which illustrate and corroborate some of the analytical results stated.

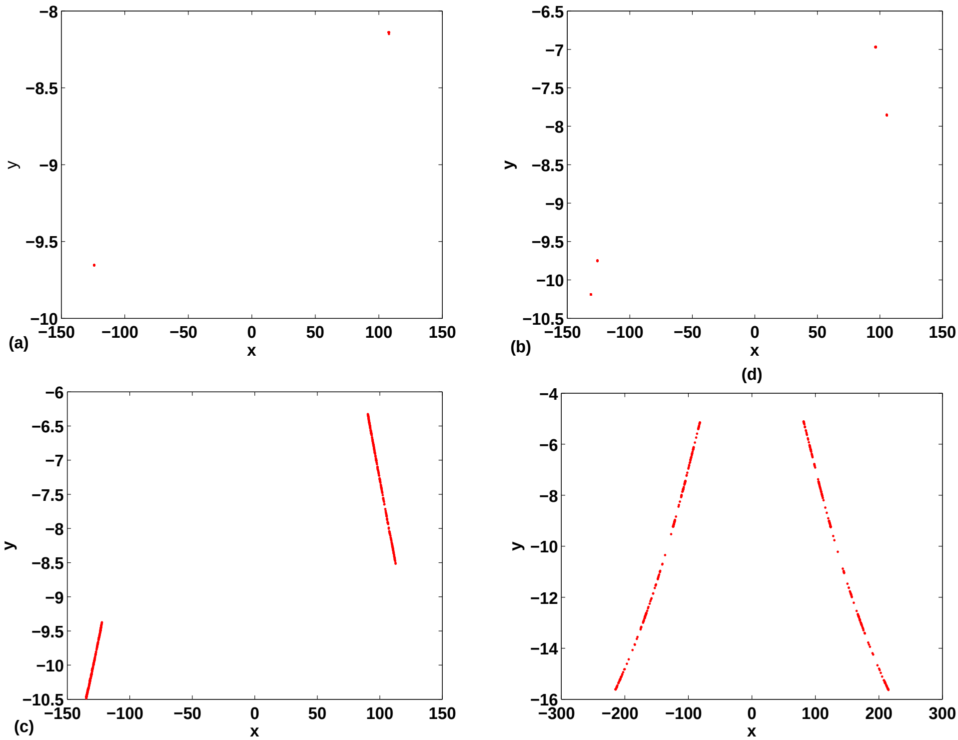

Fig. 4. Poincaré mapping on $x = 0$ of the solutions of system (1) with parameter values $(a, b, c, e) = (16, 102.8, -16.6, 0.5)$, starting initial values $(0.001, 0.001, 0.001)$: (a) $d = 16 + 0.4$; (b) $d = 16 + 0.98$; (c) $d = 16 + 1.32$; (d) $d = 16 + 4.52$.

Cascade of period doubling bifurcations and the existence of four-wing attractors are in some sense related to the Hopf bifurcations which occur at the equilibrium O.

In future works, we will use the proposed analysis method to investigate some complex chaotic systems, such as the typical multi-scroll chaotic systems by some effective design methods using piecewise-linear functions, cellular neural networks, nonlinear modulating functions, circuit component design, switching manifolds, etc. [6,7,29]. It is expected that more detailed theory analysis will be provided in a forthcoming paper.

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