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Some evolution equations under the List's flow and their applications

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Abstract. In this paper, we consider some evolution equations of generalized Ricci curvature and generalized scalar curvature under the List's flow. As applications, we obtain L^2 -estimates for generalized scalar curvature and the first variational formulae for non-negative eigenvalues with respect to the Laplacian.

Keywords: List's flow; eigenvalue; scalar curvature

Classification: Primary 53C44; Secondary 53C21

1. Introduction

Let $(M^n, g(t))$, $(t \in [0, T])$ be a solution to the following List's flow which was introduced by B. List [5]:

$$(1.1) \quad \begin{cases} \frac{\partial}{\partial t} g = -2\text{Ric} + 2\alpha d\varphi \otimes d\varphi, \\ \varphi_t = \Delta\varphi, \end{cases}$$

where $\alpha > 0$ is a constant and φ is a smooth function on M^n . Δ denotes the Laplace-Beltrami operator given by $g(t)$. Throughout this paper, we always assume that M^n is compact without boundary. The motivation to study the system (1.1) stems from its connection to general relativity. In [7], Lott and Sesum obtain some long time behavior of the List's flow when $\alpha = 2$. Li [4] studied the eigenvalues and entropies under the harmonic-Ricci flow, he derived some monotonicity formulas for eigenvalues of Laplacian. Wang [10] proved some differential Harnack inequalities about the coupled Ricci flow which is closely related to List's flow. In [2], Fang also obtained some differential Harnack inequalities associated with the List's flow which generalize the results of Cao and Hamilton in [1]. For the study of List's flow and some developments, see [6], [5], [8], [3] and the references therein.

Let $h_{ij} = R_{ij} - \alpha \varphi_i \varphi_j$ be a symmetric two-tensor which we call the *generalized Ricci curvature*. Then (1.1) becomes

$$(1.2) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} = -2h_{ij}, \\ \varphi_t = \Delta\varphi. \end{cases}$$

In this paper, we consider the following evolution equation:

$$(1.3) \quad \begin{cases} \frac{\partial}{\partial t} g_{ij} = -2\tilde{h}_{ij}, \\ \varphi_t = \Delta\varphi, \end{cases}$$

where $\tilde{h}_{ij} = h_{ij} - (ar + b)g_{ij}$. Here a, b are two constants,

$$r = \frac{\int_M (\text{tr}_g h) dV_g}{\int_M dV_g}$$

is the average of the trace of two-tensor h_{ij} and $\text{tr}_g h = g^{ij}h_{ij} = R - \alpha|\nabla\varphi|^2$ which is called the *generalized scalar curvature*. Note that if $a = \frac{1}{n}$ and $b = 0$, then the volume of M^n is a constant for all t . In this case, the equation (1.3) with $a = \frac{1}{n}$ and $b = 0$ is called the normalized List's flow.

The rest of this paper is organized as follows: In Section 2, we first derive some evolution equations of generalized Ricci curvature and generalized scalar curvature under the List's flow. As applications, we obtain L^2 -estimates for generalized scalar curvature and the first variational formulae for non-negative eigenvalues with respect to the Laplacian.

2. Some evolution equations

We first recall the following lemma (see Lemma 1.4 in [5]):

Lemma 2.1 ([5]). *Let $g(t)$ be a solution to the evolution equation*

$$\frac{\partial}{\partial t} g_{ij} = v_{ij}$$

on M^n . Then the following evolution equations hold:

- (1) $\frac{\partial}{\partial t} g^{ij} = -v^{ij}$, where $v^{ij} = g^{ik}g^{jl}v_{kl}$;
- (2) $\frac{\partial}{\partial t} R_{ij} = -\frac{1}{2}\Delta v_{ij} - \frac{1}{2}(\text{tr}_g v)_{ij} + \frac{1}{2}g^{kl}(v_{ik,jl} + v_{jk,il})$, where $\text{tr}_g v = g^{ij}v_{ij}$;
- (3) $\frac{\partial}{\partial t} R = -\Delta(\text{tr}_g v) + g^{ij}g^{kl}(v_{ik,lj} - R_{ik}v_{jl})$;
- (4) $\frac{\partial}{\partial t} dV_g = \frac{1}{2}(\text{tr}_g v) dV_g$,

where \cdot_i denotes the covariant derivative in the direction of i .

Using Lemma 2.1 above, we can prove the following results:

Lemma 2.2. *Let $g(t)$ be a solution to the evolution equation (1.3) on M^n . Then the following evolution equations hold:*

- (1) $(\frac{\partial}{\partial t} - \Delta)h_{ij} = -g^{kl}R_{ik}h_{jl} - g^{kl}R_{jk}h_{il} - 2R_{lij}h^{lk} + 2\alpha(\Delta\varphi)\varphi_{ij}$;
- (2) $(\frac{\partial}{\partial t} - \Delta)(\text{tr}_g h) = 2|h_{ij}|^2 + 2\alpha(\Delta\varphi)^2 - 2(ar + b)\text{tr}_g h$.

PROOF: Using Lemma 2.1, we have

$$(2.1) \quad \frac{\partial}{\partial t} R_{ij} = -\frac{1}{2}\Delta v_{ij} - \frac{1}{2}(\text{tr}_g v)_{ij} + \frac{1}{2}g^{kl}(v_{ik,jl} + v_{jk,il}).$$

Taking $v_{ij} = -2\tilde{h}_{ij} = -2[h_{ij} - (ar + b)g_{ij}]$, we have

$$\begin{aligned} -\frac{1}{2}\Delta v_{ij} &= \Delta h_{ij} = \Delta R_{ij} - \alpha \Delta(\varphi_i \varphi_j) \\ &= \Delta R_{ij} - \alpha(\Delta \varphi_i)\varphi_j - \alpha \varphi_i(\Delta \varphi_j) - 2\alpha \varphi_{ik}\varphi_{jk}. \end{aligned}$$

Here and hereafter we use moving frames in all calculations. The second term of the right hand side in (2.1) gives

$$-\frac{1}{2}(\text{tr}_g v)_{ij} = [R - \alpha|\nabla\varphi|^2 - n(ar + b)]_{ij} = R_{,ij} - 2\alpha\varphi_{ki}\varphi_{kj} - 2\alpha\varphi_k\varphi_{kji}.$$

Note that

$$\begin{aligned} \frac{1}{2}v_{ik,jk} &= -h_{ik,jk} \\ &= -R_{ik,jk} + \alpha(\varphi_i\varphi_k)_{,jk} \\ &= -(R_{ik,kj} + R_{lk}R_{lij} + R_{il}R_{lkj}) \\ &\quad + \alpha[\varphi_{ijk}\varphi_k + (\Delta\varphi)\varphi_{ij} + \varphi_{ik}\varphi_{jk} + \varphi_i\varphi_{jkk}] \\ &= -\frac{1}{2}R_{,ij} - R_{lk}R_{lij} - R_{ik}R_{jk} \\ &\quad + \alpha[\varphi_{ikj}\varphi_k + \varphi_k\varphi_l R_{lij} + (\Delta\varphi)\varphi_{ij} + \varphi_{ik}\varphi_{jk} + \varphi_i\varphi_{jkk}]. \end{aligned}$$

Hence, we obtain from (2.1) that

$$(2.2) \quad \begin{aligned} \frac{\partial}{\partial t}R_{ij} &= \Delta R_{ij} - 2R_{ik}R_{jk} - 2R_{lk}R_{lij} + 2\alpha\varphi_k\varphi_l R_{lij} \\ &\quad + 2\alpha(\Delta\varphi)\varphi_{ij} - 2\alpha\varphi_{ki}\varphi_{kj}. \end{aligned}$$

By a direct calculation, we have

$$\begin{aligned} \frac{\partial}{\partial t}(\varphi_i\varphi_j) &= (\varphi_t)_i\varphi_j + \varphi_i(\varphi_t)_j \\ &= (\Delta\varphi)_i\varphi_j + \varphi_i(\Delta\varphi)_j \\ &= \Delta(\varphi_i)\varphi_j - R_{ik}\varphi_k\varphi_j + \varphi_i\Delta(\varphi_j) - R_{jk}\varphi_k\varphi_i \\ &= \Delta(\varphi_i\varphi_j) - 2\varphi_{ik}\varphi_{jk} - R_{ik}\varphi_k\varphi_j - R_{jk}\varphi_k\varphi_i. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}R_{ij} - \alpha\frac{\partial}{\partial t}(\varphi_i\varphi_j) \\ &= \Delta R_{ij} - 2R_{ik}R_{jk} - 2R_{lk}R_{lij} + 2\alpha\varphi_k\varphi_l R_{lij} \\ &\quad + 2\alpha(\Delta\varphi)\varphi_{ij} - 2\alpha\varphi_{ki}\varphi_{kj} - \alpha\Delta(\varphi_i\varphi_j) + 2\alpha\varphi_{ik}\varphi_{jk} \\ &\quad + \alpha R_{ik}\varphi_k\varphi_j + \alpha R_{jk}\varphi_k\varphi_i \\ &= \Delta h_{ij} - R_{ik}h_{jk} - R_{jk}h_{ik} - 2R_{lij}h_{lk} + 2\alpha(\Delta\varphi)\varphi_{ij}. \end{aligned}$$

On the other hand, using Lemma 2.1, it holds that

$$\begin{aligned}
\frac{\partial}{\partial t}(\operatorname{tr}_g h) &= \frac{\partial}{\partial t} R - \alpha \frac{\partial}{\partial t} (|\nabla \varphi|^2) \\
&= 2\Delta R - 4\alpha |\nabla^2 \varphi|^2 - 4\alpha \varphi_j \varphi_{jii} \\
&\quad - \Delta R + 2\alpha [2\varphi_j \varphi_{jii} + (\Delta \varphi)^2 + |\nabla^2 \varphi|^2 - R_{ij} \varphi_i \varphi_j] \\
&\quad + 2R_{ik} h_{ik} - 2(ar + b)R \\
&\quad - \alpha [\Delta (|\nabla \varphi|^2) - 2|\nabla^2 \varphi|^2 - 2\alpha |\nabla \varphi|^4 - 2(ar + b)|\nabla \varphi|^2] \\
&= \Delta(\operatorname{tr}_g h) + 2|h_{ij}|^2 + 2\alpha(\Delta \varphi)^2 - 2(ar + b)\operatorname{tr}_g h,
\end{aligned}$$

where we used

$$\begin{aligned}
h_{ik,ki} &= \frac{1}{2}\Delta R - \alpha [2\varphi_j \varphi_{jii} + (\Delta \varphi)^2 + |\nabla^2 \varphi|^2 - R_{ij} \varphi_i \varphi_j], \\
\frac{\partial}{\partial t} (|\nabla \varphi|^2) &= \left(\frac{\partial}{\partial t} g^{ij}\right) \varphi_i \varphi_j + g^{ij} \frac{\partial}{\partial t} (\varphi_i \varphi_j) \\
&= \Delta (|\nabla \varphi|^2) - 2|\nabla^2 \varphi|^2 - 2\alpha |\nabla \varphi|^4 - 2(ar + b)|\nabla \varphi|^2.
\end{aligned}$$

We complete the proof of Lemma 2.2. \square

3. L^2 -estimates and some applications

For any a, b , we will prove that the nonnegativity of $\operatorname{tr}_g h(t)$ is preserved by the evolution equation (1.3).

Theorem 3.1. *Let $g(t)$ be a solution to the evolution equation (1.3) on a compact manifold M^n . If $\operatorname{tr}_g h \geq 0$ holds at $t = 0$, then also for all $t > 0$ as long as the solution exists.*

PROOF: Let

$$\beta = e^{2 \int_0^t [ar(s)+b] ds} (\operatorname{tr}_g h).$$

Then

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\beta &= e^{2 \int_0^t [ar(s)+b] ds} [2(ar + b)(\operatorname{tr}_g h) + \frac{\partial}{\partial t}(\operatorname{tr}_g h) - \Delta(\operatorname{tr}_g h)] \\
&= e^{2 \int_0^t [ar(s)+b] ds} [2|h_{ij}|^2 + 2\alpha(\Delta \varphi)^2] \\
&\geq 0,
\end{aligned}$$

where the second equality used Lemma 2.2. From the maximum principle of parabolic equations, we obtain that $\operatorname{tr}_g h(t)$ remains nonnegative if $\operatorname{tr}_g h(0)$ is nonnegative under the initial metric $g(0)$. \square

Next, we begin with an elementary lemma.

Lemma 3.1. *Let $g(t)$ be a solution to the evolution equation (1.3) on a compact manifold M^n . Then for any $f \in C^2(\mathbb{R}_+ \times M^n)$, it holds that*

$$(3.1) \quad \begin{aligned} \frac{d}{dt} \int_M f dV_g &= \int_M \left(\frac{\partial}{\partial t} - \Delta \right) f dV_g \\ &\quad - \int_M (\operatorname{tr}_g h) f dV_g + n(ar + b) \int_M f dV_g \end{aligned}$$

and

$$(3.2) \quad \begin{aligned} \frac{d}{dt} \int_M |\nabla f|^2 dV_g &= -2 \int_M |\nabla^2 f|^2 dV_g - \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g \\ &\quad - 2\alpha \int_M \langle \nabla \varphi, \nabla f \rangle^2 dV_g \\ &\quad - 2 \int_M (\Delta f) \left(\frac{\partial}{\partial t} - \Delta \right) f dV_g + (n-2)(ar + b) \int_M |\nabla f|^2 dV_g. \end{aligned}$$

PROOF: By (4) of Lemma 2.1, we have

$$\frac{\partial}{\partial t} dV_g = -(\operatorname{tr}_g \tilde{h}) dV_g = [-\operatorname{tr}_g h + n(ar + b)] dV_g.$$

Therefore,

$$\begin{aligned} \frac{d}{dt} \int_M f dV_g &= \int_M \left\{ \frac{\partial}{\partial t} f + [-\operatorname{tr}_g h + n(ar + b)] f \right\} dV_g \\ &= \int_M \frac{\partial}{\partial t} f dV_g - \int_M (\operatorname{tr}_g h) f dV_g + n(ar + b) \int_M f dV_g \\ &= \int_M \left(\frac{\partial}{\partial t} - \Delta \right) f dV_g - \int_M (\operatorname{tr}_g h) f dV_g + n(ar + b) \int_M f dV_g, \end{aligned}$$

which completes the proof of (3.1).

According to the Bochner formula, we have

$$(3.3) \quad \Delta |\nabla f|^2 = 2|\nabla^2 f|^2 + 2R^{ij} f_i f_j + 2\langle \nabla \Delta f, \nabla f \rangle.$$

By virtue of (1) of Lemma 2.1, we get

$$\frac{\partial}{\partial t} |\nabla f|^2 = 2\tilde{h}^{ij} f_i f_j + 2\langle \nabla \left(\frac{\partial}{\partial t} f \right), \nabla f \rangle,$$

so that

$$\begin{aligned}
 & \left(\frac{\partial}{\partial t} - \Delta\right)|\nabla f|^2 \\
 (3.4) \quad & = -2|\nabla^2 f|^2 + 2\langle \nabla\left(\frac{\partial}{\partial t} - \Delta\right)f, \nabla f \rangle + 2(\tilde{h}^{ij} - R^{ij})f_i f_j \\
 & = -2|\nabla^2 f|^2 + 2\langle \nabla\left(\frac{\partial}{\partial t} - \Delta\right)f, \nabla f \rangle - 2\alpha\langle \nabla\varphi, \nabla f \rangle^2 - 2(ar + b)|\nabla f|^2
 \end{aligned}$$

and

$$\begin{aligned}
 \frac{d}{dt} \int_M |\nabla f|^2 dV_g & = \int_M \left\{ \frac{\partial}{\partial t} |\nabla f|^2 + [-\text{tr}_g h + n(ar + b)]|\nabla f|^2 \right\} dV_g \\
 & = \int_M \left\{ \left(\frac{\partial}{\partial t} - \Delta\right)|\nabla f|^2 + [-\text{tr}_g h + n(ar + b)]|\nabla f|^2 \right\} dV_g \\
 & = -2 \int_M |\nabla^2 f|^2 dV_g - \int_M (\text{tr}_g h)|\nabla f|^2 dV_g - 2\alpha \int_M \langle \nabla\varphi, \nabla f \rangle^2 dV_g \\
 & \quad - 2 \int_M (\Delta f)\left(\frac{\partial}{\partial t} - \Delta\right)f dV_g + (n-2)(ar + b) \int_M |\nabla f|^2 dV_g.
 \end{aligned}$$

We complete the proof of Lemma 3.1. \square

Theorem 3.2. *Let $g(t)$ be a solution to the evolution equation (1.3) on a compact manifold M^n . Then for any $f \in C^2(\mathbb{R}_+ \times M^n)$,*

$$\begin{aligned}
 (3.5) \quad \frac{d}{dt} \int_M |\nabla f|^2 dV_g & \leq - \int_M (\text{tr}_g h)|\nabla f|^2 dV_g + (n-2)(ar + b) \int_M |\nabla f|^2 dV_g \\
 & \quad + \frac{n}{8} \int_M \left[\left(\frac{\partial}{\partial t} - \Delta\right)f\right]^2 dV_g.
 \end{aligned}$$

PROOF: Using the Cauchy inequality, we have

$$|\nabla^2 f|^2 \geq \frac{1}{n}(\Delta f)^2,$$

so that by (3.2)

$$\begin{aligned}
 \frac{d}{dt} \int_M |\nabla f|^2 dV_g & \leq -\frac{2}{n} \int_M (\Delta f)^2 dV_g - \int_M (\text{tr}_g h)|\nabla f|^2 dV_g \\
 & \quad - 2 \int_M (\Delta f)\left(\frac{\partial}{\partial t} - \Delta\right)f dV_g + (n-2)(ar + b) \int_M |\nabla f|^2 dV_g \\
 & \leq - \int_M (\text{tr}_g h)|\nabla f|^2 dV_g + (n-2)(ar + b) \int_M |\nabla f|^2 dV_g \\
 & \quad + \frac{n}{8} \int_M \left[\left(\frac{\partial}{\partial t} - \Delta\right)f\right]^2 dV_g,
 \end{aligned}$$

which completes the proof of Theorem 3.2. \square

Together with the evolution for $\text{tr}_g h$ in Lemma 2.2, one can establish the following

Theorem 3.3. *Let $g(t)$ be a solution to the evolution equation (1.3) on a compact manifold M^n . Then*

$$(3.6) \quad \begin{aligned} \frac{d}{dt} \int_M |\nabla(\text{tr}_g h)|^2 dV_g &\leq - \int_M (\text{tr}_g h) |\nabla(\text{tr}_g h)|^2 dV_g \\ &+ (n-2)(ar+b) \int_M |\nabla(\text{tr}_g h)|^2 dV_g \\ &+ \frac{n}{8} \int_M [2|h_{ij}|^2 + 2\alpha(\Delta\varphi)^2 - 2(ar+b)\text{tr}_g h]^2 dV_g. \end{aligned}$$

Next, we give the evolution equation of eigenvalues with respect to the Laplacian.

Theorem 3.4. *Let $g(t)$ be a solution to the evolution equation (1.2) on a compact manifold M^n .*

(1) *If $\lambda(t)$ denotes the non-negative eigenvalue of the Laplacian on M^n , then*

$$\frac{d}{dt} \lambda = 2 \int_M h^{ij} f_i f_j dV_g - \int_M (\text{tr}_g h) |\nabla f|^2 dV_g + \lambda \int_M (\text{tr}_g h) f^2 dV_g.$$

In particular, if $\text{tr}_g h \geq 0$ holds at $t = 0$, then

$$\frac{d}{dt} \lambda \leq 2 \int_M h^{ij} f_i f_j dV_g + \lambda \int_M (\text{tr}_g h) f^2 dV_g.$$

(2) *If $\lambda(t)$ denotes the non-negative eigenvalue of $\Delta - \frac{1}{2}\text{tr}_g h$ on M^n , then*

$$\frac{d}{dt} \lambda = 2 \int_M h^{ij} f_i f_j dV_g + \int_M f^2 [|h_{ij}|^2 + \alpha(\Delta\varphi)^2] dV_g.$$

In particular, if $h_{ij}(t) \geq 0$, then the eigenvalue of the operator $-\Delta + \frac{1}{2}\text{tr}_g h$ are nondecreasing.

(3) *If $\lambda(t)$ denotes the non-negative eigenvalue of $\Delta + \frac{1}{4}\text{tr}_g h$ on M^n , then*

$$\frac{d}{dt} \lambda \leq \int_M \left[\frac{n-1}{2n} (\text{tr}_g h)^2 + \frac{5}{2} \lambda (\text{tr}_g h) \right] f^2 dV_g + 2\lambda^2.$$

Remark 3.1. In particular, taking $\alpha = 2$, then (2) in Theorem 3.4 becomes Theorem 1.10 in [4].

Remark 3.2. We remark that the second and third inequalities of p. 160 in [9] is slight incorrect. The right version of the third inequality of p. 160 in [9] should

be

$$\begin{aligned}
\frac{d}{dt}\lambda &= -2 \int_M |\nabla^2 f|^2 dV_g + (2\varphi - 1) \int_M R |\nabla f|^2 dV_g \\
&\quad + \int_M (2\varphi S - \frac{2}{3}\varphi AR + \varphi' R) f^2 dV_g \\
&\quad - \varphi \int_M R(R - 4\varphi R - \frac{2}{3}A + 2\lambda) f^2 dV_g \\
&\quad + \lambda \int_M R(R - 4\varphi R - \frac{2}{3}A + 2\lambda) f^2 dV_g.
\end{aligned}$$

As a result, for $n = 3$, the estimate $\frac{d}{dt}\lambda \leq \frac{19}{8}\lambda^2$ of Theorem 6.4 in [9] cannot be obtained.

PROOF: Let us consider an eigenfunction f associated with the eigenvalue λ , that is,

$$-(\Delta + V)f = \lambda f, \quad \int_M f^2 dV_g = 1$$

where V is a potential function to be chosen later. Then, we obtain from (3.2) that

$$\begin{aligned}
\frac{d}{dt} \int_M |\nabla f|^2 dV_g &= -2 \int_M |\nabla^2 f|^2 dV_g - \int_M (\text{tr}_g h) |\nabla f|^2 dV_g \\
&\quad - 2\alpha \int_M \langle \nabla \varphi, \nabla f \rangle^2 dV_g \\
(3.7) \quad &\quad + 2 \int_M (V + \lambda)^2 f^2 dV_g + \int_M (V + \lambda) \frac{\partial f^2}{\partial t} dV_g \\
&\quad + (n - 2)(ar + b) \int_M |\nabla f|^2 dV_g.
\end{aligned}$$

By virtue of the divergence theorem,

$$\int_M |\nabla f|^2 dV_g = \int_M f(-\Delta)f dV_g = \int_M (V + \lambda)f^2 dV_g,$$

so that

$$\begin{aligned}
(3.8) \quad \frac{d}{dt} \int_M |\nabla f|^2 dV_g &= \int_M f^2 \frac{\partial}{\partial t} (V + \lambda) dV_g + \int_M (V + \lambda) \frac{\partial f^2}{\partial t} dV_g \\
&\quad + \int_M [-\text{tr}_g h + n(ar + b)] (V + \lambda) f^2 dV_g.
\end{aligned}$$

Thus, (3.8) combines with (3.7) to give

$$\begin{aligned}
 \frac{d}{dt}\lambda &= -2 \int_M |\nabla^2 f|^2 dV_g - \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g - 2\alpha \int_M \langle \nabla \varphi, \nabla f \rangle^2 dV_g \\
 (3.9) \quad &+ 2 \int_M (V + \lambda)^2 f^2 dV_g - \int_M f^2 \frac{\partial V}{\partial t} dV_g \\
 &+ \int_M [(\operatorname{tr}_g h) - 2(ar + b)](V + \lambda) f^2 dV_g.
 \end{aligned}$$

Choose $V = c \operatorname{tr}_g h$, where c is a constant to be determined. Then

$$\frac{\partial V}{\partial t} = c \frac{\partial}{\partial t} \operatorname{tr}_g h$$

so that

$$\int_M f^2 \frac{\partial V}{\partial t} dV_g = c \int_M f^2 \frac{\partial}{\partial t} \operatorname{tr}_g h dV_g.$$

Using integration by parts again and by use of Lemma 2.2, we have

$$\begin{aligned}
 \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g &= \int_M (\operatorname{tr}_g h)(c \operatorname{tr}_g h + \lambda) f^2 dV_g + \frac{1}{2} \int_M f^2 \Delta(\operatorname{tr}_g h) dV_g \\
 &= \int_M (\operatorname{tr}_g h)(c \operatorname{tr}_g h + \lambda) f^2 dV_g + \frac{1}{2} \int_M f^2 \frac{\partial}{\partial t} \operatorname{tr}_g h dV_g \\
 &\quad - \int_M f^2 [|h_{ij}|^2 + \alpha(\Delta \varphi)^2 - (ar + b) \operatorname{tr}_g h] dV_g.
 \end{aligned}$$

In other words,

$$\begin{aligned}
 - \int_M f^2 \frac{\partial V}{\partial t} dV_g &= -c \int_M f^2 \frac{\partial}{\partial t} \operatorname{tr}_g h dV_g \\
 &= -2c \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g + 2c \int_M (\operatorname{tr}_g h)(c \operatorname{tr}_g h + \lambda) f^2 dV_g \\
 &\quad - 2c \int_M f^2 [|h_{ij}|^2 + \alpha(\Delta \varphi)^2 - (ar + b) \operatorname{tr}_g h] dV_g.
 \end{aligned}$$

It follows that

$$\begin{aligned}
\frac{d}{dt}\lambda &= -2 \int_M |\nabla^2 f|^2 dV_g - \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g - 2\alpha \int_M \langle \nabla \varphi, \nabla f \rangle^2 dV_g \\
&\quad + 2 \int_M (V + \lambda)^2 f^2 dV_g - \int_M f^2 \frac{\partial V}{\partial t} dV_g \\
&\quad + \int_M [(\operatorname{tr}_g h) - 2(ar + b)](V + \lambda) f^2 dV_g \\
&= -2 \int_M |\nabla^2 f|^2 dV_g - (1 + 2c) \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g - 2\alpha \int_M \langle \nabla \varphi, \nabla f \rangle^2 dV_g \\
&\quad + \int_M [(1 + 4c)(\operatorname{tr}_g h) + 2\lambda - 2(ar + b)](c \operatorname{tr}_g h + \lambda) f^2 dV_g \\
&\quad - 2c \int_M f^2 [|h_{ij}|^2 + \alpha(\Delta \varphi)^2 - (ar + b)\operatorname{tr}_g h] dV_g.
\end{aligned}$$

Integrating (3.3) yields

$$\begin{aligned}
(3.10) \quad & -2 \int_M |\nabla^2 f|^2 dV_g - 2\alpha \int_M \langle \nabla \varphi, \nabla f \rangle^2 dV_g \\
&= \int_M (2R^{ij} f_i f_j + 2\langle \nabla \Delta f, \nabla f \rangle) dV_g \\
&\quad - 2\alpha \int_M \langle \nabla \varphi, \nabla f \rangle^2 dV_g \\
&= 2 \int_M h^{ij} f_i f_j dV_g - 2 \int_M (c \operatorname{tr}_g h + \lambda)^2 f^2 dV_g.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
(3.11) \quad & \frac{d}{dt}\lambda = 2 \int_M h^{ij} f_i f_j dV_g - (1 + 2c) \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g \\
&\quad + \int_M [(1 + 2c)(\operatorname{tr}_g h) - 2(ar + b)](c \operatorname{tr}_g h + \lambda) f^2 dV_g \\
&\quad - 2c \int_M f^2 [|h_{ij}|^2 + \alpha(\Delta \varphi)^2 - (ar + b)\operatorname{tr}_g h] dV_g.
\end{aligned}$$

Now we are in the position to complete the proof of Theorem 3.4. Notice that under the evolution equation (1.2), we have $a = b = 0$.

If we choose $c = 0$, then (3.11) reduces to

$$\frac{d}{dt}\lambda = 2 \int_M h^{ij} f_i f_j dV_g - \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g + \lambda \int_M (\operatorname{tr}_g h) f^2 dV_g$$

and (1) of Theorem 3.4 follows.

If we choose $c = -\frac{1}{2}$, then (3.11) reduces to

$$(3.12) \quad \frac{d}{dt}\lambda = 2 \int_M h^{ij} f_i f_j dV_g + \int_M f^2 [|h_{ij}|^2 + \alpha(\Delta\varphi)^2] dV_g.$$

Hence, we obtain (2) of Theorem 3.4.

If we choose $c = \frac{1}{4}$, then (3.11) reduces to

$$(3.13) \quad \begin{aligned} \frac{d}{dt}\lambda &= 2 \int_M h^{ij} f_i f_j dV_g - \frac{3}{2} \int_M (\operatorname{tr}_g h) |\nabla f|^2 dV_g \\ &\quad + \frac{3}{2} \int_M (\operatorname{tr}_g h) \left(\frac{1}{4} \operatorname{tr}_g h + \lambda\right) f^2 dV_g \\ &\quad - \frac{1}{2} \int_M f^2 [|h_{ij}|^2 + \alpha(\Delta\varphi)^2] dV_g. \end{aligned}$$

It follows from (3.10) that

$$2 \int_M h^{ij} f_i f_j dV_g \leq 2 \int_M \left(\frac{1}{4} \operatorname{tr}_g h + \lambda\right) f^2 dV_g.$$

Thus, we deduce from (3.13) the following

$$(3.14) \quad \begin{aligned} \frac{d}{dt}\lambda &\leq 2 \int_M \left(\frac{1}{4} \operatorname{tr}_g h + \lambda\right) f^2 dV_g \\ &\quad + \frac{3}{2} \int_M (\operatorname{tr}_g h) \left(\frac{1}{4} \operatorname{tr}_g h + \lambda\right) f^2 dV_g - \frac{1}{2} \int_M f^2 |h_{ij}|^2 dV_g. \end{aligned}$$

Applying the Cauchy inequality

$$|h_{ij}|^2 \geq \frac{1}{n} (\operatorname{tr}_g h)^2$$

into (3.14) gives

$$\frac{d}{dt}\lambda \leq \int_M \left[\frac{n-1}{2n} (\operatorname{tr}_g h)^2 + \frac{5}{2} \lambda (\operatorname{tr}_g h) \right] f^2 dV_g + 2\lambda^2.$$

We complete the proof of Theorem 3.4. □

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