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Mohamed Ali Toumi

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ON EXTENSIONS OF ORTHOSYMMETRIC LATTICE BIMORPHISMS

Mohamed Ali Toumi, Bizerte

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Abstract. In the paper we prove that every orthosymmetric lattice bilinear map on the cartesian product of a vector lattice with itself can be extended to an orthosymmetric lattice bilinear map on the cartesian product of the Dedekind completion with itself. The main tool used in our proof is the technique associated with extension to a vector subspace generated by adjoining one element. As an application, we prove that if $(A, *)$ is a commutative d-algebra and $A^{\mathfrak{d}}$ its Dedekind completion, then, $A^{\mathfrak{d}}$ can be equipped with a d-algebra multiplication that extends the multiplication of A.

Moreover, we indicate an error made in the main result of the paper: M. A. Toumi, Extensions of orthosymmetric lattice bimorphisms, Proc. Amer. Math. Soc. 134 (2006), 1615–1621.

Keywords: d-algebra; f-algebra; lattice homomorphism; lattice bimorphism

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1. Introduction

In [13] the author proved the following result: let A be an Archimedean vector lattice, let $A^{\mathfrak{d}}$ be its Dedekind completion and let B be a Dedekind complete vector lattice. If $\Psi_0\colon A \times A \to B$ is an orthosymmetric lattice bilinear map, then every lattice extension $\Psi\colon A^{\mathfrak{d}} \times A^{\mathfrak{d}} \to B$ of Ψ_0 is again orthosymmetric. Unfortunately, the result and its proof given in [13] are not correct. The aim of the present paper is to indicate the error in the proof of the above mentioned result and to give a correct formulation concerning extension of orthosymmetric lattice bilinear maps. With help of a suitable example we show that ([13], Theorem 1) and its proof cannot be improved. Apart from that, we are going to investigate the problem of extension of multiplications for d-algebras, which is a question posed by Huijsmans in [8] (last paragraph of section 7): can the multiplication of d-algebra be extended to its

Dedekind completion? Under some assumptions (being a slight modification of the hypotheses imposed in [13]) we prove a result on the extension of orthosymmetric lattice bilinear maps. The main tool used in our proof is the technique associated with extension to a vector subspace generated by adjoining one element.

For terminology, notations and concepts that are not explained in this paper we refer to the standard monographs [1], [11] and [14].

We notice that in [4], the author furnished examples showing that the main result in [13] is false.

2. Definitions and notations

We shall assume throughout this paper that all vector lattices (or Riesz spaces) under consideration are Archimedean.

A vector subspace E of a vector lattice A is said to be *majorizing (dominating)* A if for each $x \in A$ there exists $a \in E$ such that $|x| \leqslant a$. A linear mapping T defined on a vector lattice A with values in the vector lattice B is called *positive* if $T(A^+) \subset B^+$ (notation $T \in \mathcal{L}^+(A,B)$ or $T \in \mathcal{L}^+(A)$ if $A = B$). The linear mapping $T \in \mathcal{L}^+(A,B)$ is called a *lattice* (or *Riesz*) *homomorphism* (notation $T \in \operatorname{Hom}(A,B)$ or $T \in \operatorname{Hom}(A)$ if $A = B$) whenever $a \wedge b = 0$ implies $T(a) \wedge T(b) = 0$ see [10].

An (real) algebra A which is simultaneously a vector lattice is called *lattice ordered algebra* (*ℓ-algebra*). In an ℓ-algebra A we denote the collection of all nilpotent elements of A by $N(A)$. The ℓ-algebra A is referred to as *semiprime* if $N(A) = \{0\}$. The ℓ-algebra A is called an *f-algebra* if A verifies the property that $a \wedge b = 0$ and $c \geqslant 0$ imply $ac \wedge b = ca \wedge b = 0$. Any f-algebra is automatically commutative and has positive squares. Every unital f-algebra (i.e., an f-algebra with a unit element) is semiprime. For more information about this field, we refer the reader to [1].

Also we need the following definitions. An ℓ-algebra A is called a *d-algebra* whenever it follows from $a \wedge b = 0$ and $c \geqslant 0$ that $ac \wedge bc = ca \wedge cb = 0$ (equivalently, whenever $|ab| = |a||b|$). In other words, multiplications by positive elements in the d-algebra A are lattice homomorphisms. Contrary to the f-algebras, d-algebras need not be commutative nor have positive squares. For the elementary theory of d-algebras we refer to [2], [6], [9]. An ℓ-algebra A is called an *almost f-algebra* whenever it follows from $a \wedge b = 0$ that $ab = 0$.

A vector lattice A is called *universally complete* if A is a Dedekind complete vector lattice and every positive orthogonal system in A has a supremum in A. Every vector lattice A has a universal completion A^u, this means that there exists a unique (up to a Riesz isomorphism) universally complete vector lattice A^u such that A can be identified with an order dense Riesz subspace of A^u. The vector lattice A^u is equipped with an f-algebra multiplication, under which A^u is an f-algebra with unit element, see [1, Section 8, Exercise 13] for an interesting approach to the existence of the universal completion using orthomorphisms.

We end this section with some definitions of bilinear maps on vector lattices. Let A, B and C be vector lattices. A bilinear map Ψ from $A \times B$ into C is said to be *positive* whenever $a \in A^+$ and $b \in B^+$ imply $\Psi(a,b) \in C^+$ (equivalently $|\Psi(a,b)| \leqslant \Psi(|a|,|b|)$ for all $a \in A$ and $b \in B$). The bilinear map Ψ is called *lattice* (or *Riesz*) *bilinear map* whenever the partial operators

$$\begin{aligned}\Psi(a,\cdot)\colon B &\longrightarrow C\\ c &\longmapsto \Psi(a,c)\end{aligned} \qquad \text{and} \qquad \begin{aligned}\Psi(\cdot,b)\colon A &\longrightarrow C\\ c &\longmapsto \Psi(c,b)\end{aligned}$$

are lattice homomorphisms for every $a \in A^+$ and $b \in B^+$(equivalently $|\Psi(a,b)| = \Psi(|a|,|b|)$ for all $a \in A$ and $b \in B$). A bilinear map Ψ from $A \times A$ into C is said to be *orthosymmetric* if $a \wedge b = 0$ implies $\Psi(a,b) = 0$.

3. The main results

Grobler and Labuschagne [8], proved that if A and B are vector lattices and if C is a Dedekind complete vector lattice, then every lattice bilinear map $\Psi_0\colon A \times B \to C$ can be extended to a lattice bilinear map Ψ to $A^{\delta} \times B^{\delta}$ into C, where A^{δ} (or B^{δ}) is the Dedekind completion of A (of B, respectively). The question arises whether Ψ is still orthosymmetric when Ψ_0 is, in addition, orthosymmetric. The answer is affirmative.

To reach this aim we need the following result.

Proposition 3.1 ([13], Proposition 1)**.** *Let A be a vector lattice, let A^{δ} be its Dedekind completion, let B be a Dedekind complete vector lattice and let $\Psi_0\colon A \times A \to B$ be an orthosymmetric lattice bilinear map. If $\Psi\colon A^{\delta} \times A^{\delta} \to B$ is a lattice bilinear map extension of Ψ_0 to $A^{\delta} \times A^{\delta}$ into B, then for all $x, y \in A^{\delta}$ such that $x \wedge y = 0$, for all $x_i \in A$ ($y_i \in A$, respectively) such that $0 \leqslant x_i \leqslant x$ ($0 \leqslant y_i \leqslant y$, respectively), we have*

$$\text{(AF)} \qquad \Psi(x,y_i) = \Psi(y_i,x) = \Psi(x_i,y) = \Psi(y,x_i) = \Psi(x_i,y_i) = 0.$$

R e m a r k 3.1. It is natural to ask if any positive extension of an orthosymmetric lattice bilinear map is again orthosymmetric. The answer is negative and this is illustrated by the following example:

E x a m p l e 3.1. Let A be the vector lattice of all real stationary sequences. Its Dedekind completion $A^{\mathfrak{d}}$ is the vector lattice of all real bounded sequences ($A^{\mathfrak{d}} \equiv \ell^{\infty}(\mathbb{N}) \equiv C(\beta\mathbb{N})$). We consider $A \times A$ with the coordinatewise vector space operations and partial ordering. Let $\Psi_0\colon A \times A \to \mathbb{R}$, defined by $\Psi_0(f,g) = (\lim\limits_{n\to\infty} f(n))(\lim\limits_{n\to\infty} g(n))$ for all $(f,g) \in A\times A$. For any $u, v \in \beta\mathbb{N} \setminus \mathbb{N}$, such that $u \neq v$, let $\Psi\colon A^{\mathfrak{d}} \times A^{\mathfrak{d}} \to \mathbb{R}$ defined by $\Psi(f,g) = ((\beta f)(u))((\beta g)(v))$. It is easily seen that Ψ_0 is an orthosymmetric lattice bilinear map, whereas Ψ is a lattice extension of Ψ_0 which is not orthosymmetric.

The foregoing example shows that ([13], Theorem 1) and its proof are regrettably wrong.

In [12, Remark 19.5] De Pagter proved that for any uniformly complete vector lattice A with a strong order unit e, there exists a unique multiplication in A such that A is an f-algebra with a unit element e. Next, we give the well known result which is dealing with the existence of unital f-algebra multiplications on universally complete vector lattices.

Lemma 3.1 ([1, Section 8, Exercise 13]). *Let A be a universally complete vector lattice and let e be a weak order unit of A. Then there exists a unique multiplication in A such that A is an f-algebra with a unit element e.*

R e m a r k 3.2. We remark that any universally complete vector lattice can be seen as a universally complete unital f-algebra. So in the sequel, we denote its f-algebra multiplication by juxtaposition.

A simple combination of Lemma 3.1 and Proposition 3.1 gives:

Proposition 3.2. *Let A be a vector lattice, let $A^{\mathfrak{d}}$ be its Dedekind completion, let $A^{\mathfrak{u}}$ be its universal completion and let B be a Dedekind complete vector lattice. If $\Psi_0\colon A \times A \to B$ is a positive orthosymmetric bilinear map, then every positive bilinear map extension Ψ of Ψ_0 to $A \times A^{\mathfrak{d}}$ in B satisfies the following property:*

$$\Psi(f,g) \leqslant \Psi(f',g)$$

for all $(f,f',g) \in A \times A \times A^{\mathfrak{d}}$ such that $fg \leqslant f'g$.

P r o o f. Let $(f,f',g) \in A \times A \times A^{\mathfrak{d}}$ such that $fg \leqslant f'g$. Then,

$$((f-f')g)^{+} = (f-f')^{+}g^{+} + (f-f')^{-}g^{-} = 0.$$

Hence, $(f-f')^+ \wedge g^+ = (f-f')^- \wedge g^- = 0$. Since Ψ satisfies the property (AF), it follows that $\Psi((f-f')^+, g^+) = \Psi((f-f')^-, g^-) = 0$. Consequently,

$$\Psi((f-f'), g) = -\Psi((f-f')^+, g^-) - \Psi((f-f')^-, g^+) \leqslant 0,$$

which means that $\Psi(f,g) \leqslant \Psi(f',g)$ and the proof is complete. □

For the rest of the paper, we shall fix the following notation and assumptions. Let A be a vector lattice, let $A^{\mathfrak{d}}$ be its Dedekind completion, let $A^{\mathfrak{u}}$ be its universal completion, let B be a Dedekind complete vector lattice and let $\Psi_0 \colon A \times A^{\mathfrak{d}} \to B$ be a positive bilinear map satisfying:

$$\Psi_0(f,g) = 0$$

for all $(f,g) \in A \times A^{\mathfrak{d}}$ such that $f \wedge g = 0$. Since $A^{\mathfrak{u}}$ can be equipped with unital f-algebra multiplication (denoted by juxtaposition) in such a manner that $A^{\mathfrak{u}}$ becomes an f-algebra, we define for all $(f,g) \in A^{\mathfrak{d}} \times (A^{\mathfrak{d}})_+$,

$$H(f,g) = \Big\{(f', g_1, \ldots, g_n) \in A \times ((A^{\mathfrak{d}})_+)^n \colon \quad f'g_i \geqslant fg_i, \ \forall\, 1 \leqslant i \leqslant n, \\ \sum_{i=1}^{n} g_i = g, \ \forall n \in \mathbb{N}^* \Big\}$$

and

$$\Psi_1(f,g) = \inf\Big\{\sum_{i=1}^{n} \Psi_0(f', g_i) \colon \ (f', g_1, \ldots, g_n) \in H(f,g)\Big\},$$

where the infimum is taken over all finite positive decompositions of g (i.e., $g_i \in (A^{\mathfrak{d}})_+$, $\sum\limits_{i=1}^{n} g_i = g$) and over all $f' \in A$ such that $f'g_i \geqslant fg_i$. One can easily see, that $\Psi_1(\lambda f, g) = \Psi_1(f, \lambda g) = \lambda \Psi_1(f,g)$ for all $(f,g) \in A^{\mathfrak{d}} \times A^{\mathfrak{d}}$ and for all $\lambda \in \mathbb{R}_+$. The following lemma captures the basic features of Ψ_1.

Lemma 3.2. *The mapping Ψ_1 satisfies the following properties:*

(1) $\Psi_1(f,g) = \Psi_0(f,g)$, *for all* $(f,g) \in A \times (A^{\mathfrak{d}})_+$,
(2) $\Psi_1(f,g) = 0$, *for all* $(f,g) \in A^{\mathfrak{d}} \times A^{\mathfrak{d}}$ *such that* $f \wedge g = 0$,
(3) $\Psi_1(f_0 + f, g) = \Psi_0(f_0, g) + \Psi_1(f,g)$, *for all* $(f_0, f, g) \in A \times A^{\mathfrak{d}} \times (A^{\mathfrak{d}})_+$,
(4) $\Psi_1(f, g+g') = \Psi_1(f,g) + \Psi_1(f,g')$, *for all* $(f,g,g') \in A^{\mathfrak{d}} \times (A^{\mathfrak{d}})_+ \times (A^{\mathfrak{d}})_+$.

P r o o f. (1) Let $(f,g) \in A \times (A^{\mathfrak{d}})_+$ and let $f' \in A$, $g_i \in (A^{\mathfrak{d}})_+$, $f'g_i \geqslant fg_i$, $\sum\limits_{i=1}^{n} g_i = g$. By Proposition 3.2, since $f'g_i \geqslant fg_i$ then $\Psi_0(f', g_i) \geqslant \Psi_0(f, g_i)$ and since

$f \in A$, it follows that $\Psi_0(f,g) = \sum_{i=1}^{n} \Psi_0(f,g_i)$ and then $\sum_{i=1}^{n} \Psi_0(f',g_i) \geqslant \Psi_0(f,g)$. Therefore $\Psi_1(f,g) \geqslant \Psi_0(f,g)$. Moreover, since $fg \leqslant fg$, it follows that $\Psi_1(f,g) \leqslant \Psi_0(f,g)$. Hence $\Psi_1(f,g) = \Psi_0(f,g)$.

(2) Let $(f,g) \in A \times A^{\mathfrak{d}}$ such that $f \wedge g = 0$. Then $fg = 0$. Let $f' \in A, g_i \in (A^{\mathfrak{d}})_+$ such that $f'g_i \geqslant fg_i$ and $\sum_{i=1}^{n} g_i = g$. It follows that $\sum_{i=1}^{n} fg_i = fg = 0$. Then, $fg_i = 0$ for all $1 \leqslant i \leqslant n$. Consequently, $\Psi_0(f',g_i) \geqslant 0$ and then $\Psi_1(f,g) \geqslant 0$. Moreover, in the case where $f \wedge g = 0$, it is evident that $\Psi_1(f,g) \leqslant 0$. As a conclusion, $\Psi_1(f,g) = 0$ for all $(f,g) \in A \times A^{\mathfrak{d}}$ such that $f \wedge g = 0$.

(3) Let $(f_0, f, g) \in A \times A^{\mathfrak{d}} \times (A^{\mathfrak{d}})_+$. Then,

$$
\begin{aligned}
\Psi_1(f_0 + f, g) &= \inf\left\{\sum_{i=1}^{n} \Psi_0(f',g_i)\colon\ (f',g_1,\ldots,g_n) \in H(f+f_0,g)\right\} \\
&= \inf\left\{\sum_{i=1}^{n} \Psi_0(f',g_i)\colon\ (f',g_1,\ldots,g_n) \in H(f+f_0,g)\right\} - \Psi_0(f_0,g) + \Psi_0(f_0,g) \\
&= \inf\left\{\sum_{i=1}^{n} \Psi_0(f',g_i) - \Psi_0(f_0,g)\colon\ (f',g_1,\ldots,g_n) \in H(f+f_0,g)\right\} + \Psi_0(f_0,g) \\
&= \inf\left\{\sum_{i=1}^{n} \Psi_0(f',g_i) - \sum_{i=1}^{n} \Psi_0(f_0,g_i)\colon\ (f',g_1,\ldots,g_n) \in H(f+f_0,g)\right\} + \Psi_0(f_0,g) \\
&= \inf\left\{\sum_{i=1}^{n} \Psi_0(f'-f_0,g_i)\colon\ (f',g_1,\ldots,g_n) \in H(f+f_0,g)\right\} + \Psi_0(f_0,g) \\
&= \Psi_1(f,g) + \Psi_0(f_0,g).
\end{aligned}
$$

(4) Let $(f, g, g') \in A^{\mathfrak{d}} \times (A^{\mathfrak{d}})_+ \times (A^{\mathfrak{d}})_+$. Then

$$
\begin{aligned}
\Psi_1(f,g) + \Psi_1(f,g') = \inf\Bigg\{&\sum_{i=1}^{n} \Psi_0(f',g_i) + \sum_{j=1}^{m} \Psi_0(f'',g_j)\colon\ (f',g_1,\ldots,g_n) \in H(f,g), \\
&(f'',g_1,\ldots,g_m) \in H(f,g')\Bigg\} \geqslant \Psi_1(f,g+g').
\end{aligned}
$$

Since

$$
\Psi_1(f,g+g') = \inf\left\{\sum_{i=1}^{n} \Psi_0(f',g_i)\colon\ (f',g_1,\ldots,g_n) \in H(f,g+g')\right\}
$$

by the Riesz decomposition theorem [7, V, Theorem 1], there exist $u_1,\ldots,u_n,v_1,\ldots,v_n \in (A^{\mathfrak{d}})_+$ such that

$$
\begin{gathered}
u_i + v_i = g_i \qquad (i = 1,\ldots,n) \\
g = u_1 + \ldots + u_n, \qquad g' = v_1 + \ldots + v_n.
\end{gathered}
$$

By our calculation, since $f'(u_i+v_i) \geqslant f(u_i+v_i)$, it follows that $(f-f')(u_i+v_i) \leqslant 0$. Hence,

$$((f-f')(u_i+v_i))^+ = (f-f')^+(u_i+v_i) = 0.$$

Therefore,

$$(f-f')^+u_i = (f-f')^+v_i = 0.$$

Consequently, $f'u_i \geqslant fu_i$ and $f'v_i \geqslant fv_i$. Hence,

$$\Psi_1(f, g+g') = \inf \Bigg\{ \sum_{i=1}^n \Psi_0(f', u_i) + \sum_{i=1}^n \Psi_0(f', v_i)\colon\ (f', u_1, \ldots, u_n) \in H(f, g), \\ (f', v_1, \ldots, v_n) \in H(f, g') \Bigg\} \geqslant \Psi_1(f, g) + \Psi_1(f, g').$$

Therefore,

$$\Psi_1(f, g+g') = \Psi_1(f, g) + \Psi_1(f, g'),$$

which gives the desired result. □

Let $x_0 \in A^{\mathfrak{d}} \setminus A$ and let's denote by A_{x_0} the space $\langle A + \mathbb{R}x_0 \rangle$.

All the preparations have been made for the first central result in the paper:

Theorem 3.1. *Let A be a vector lattice, let $A^{\mathfrak{d}}$ be its Dedekind completion, let $0 \leqslant x_0 \in A^{\mathfrak{d}} \setminus A$ and let B be a Dedekind complete vector lattice. If $\Psi_0\colon A \times A^{\mathfrak{d}} \to B$ is an orthosymmetric lattice bilinear map, then Ψ_0 has a positive orthosymmetric extension Ψ' to $A_{x_0} \times A^{\mathfrak{d}}$ in B.*

Proof. Denote also by Ψ_0 the restriction of Ψ_0 to $A \times (A^{\mathfrak{d}})_+$. Let us take $\Psi\colon A_{x_0} \times (A^{\mathfrak{d}})_+ \to B$, defined by $\Psi'(k, g) = t\Psi_1(x_0, g) + \Psi_0(a, g)$, where $k = a + tx_0 \in A_{x_0}$ $(a \in A, t \in \mathbb{R})$, for all $g \in (A^{\mathfrak{d}})_+$. By Lemma 3.2,

$$\Psi'(k, g_1+g_2) = \Psi'(k, g_1) + \Psi'(k, g_2)$$

for all $g_1, g_2 \in (A^{\mathfrak{d}})_+$. Then Ψ has a unique positive extension to $A_{x_0} \times A^{\mathfrak{d}}$ defined by

$$\Psi'(k, g) = t\Psi_1(x_0, g^+) - t\Psi_1(x_0, g^-) + \Psi_0(a, g)$$

for all $k = a + tx_0 \in A_{x_0}$ $(a \in A,\ t \in \mathbb{R})$ and for all $g \in A^{\mathfrak{d}}$.

We may prove now that Ψ is orthosymmetric. Let $a + tx_0 \in A_{x_0}$ and $g \in (A^{\mathfrak{d}})_+$ such that $(a+tx_0) \wedge g = 0$. If $t = 0$, then $a \wedge g = 0$. It follows, by using Proposition 3.1, that $\Psi(a, g) = 0$. If $t > 0$, then $tx_0 \geqslant -a$. Let $f' \in A, g_i \in (A^{\mathfrak{d}})_+$ such that $f'g_i \geqslant$

tx_0g_i and $\sum\limits_{i=1}^{n} g_i = g$. Hence, $tx_0g_i \geqslant -ag_i$. Since $\Psi(-a, g_i) = \Psi_0(-a, g_i) = -\Psi_0(a, g_i)$ and by using Proposition 3.2, it follows that $\Psi(f', g_i) \geqslant -\Psi_0(a, g_i)$. Therefore,

$$\Psi'(k, g) = \Psi_1(k, g) = t\Psi_1(x_0, g) + \Psi_0(a, g) \geqslant 0.$$

Moreover, by using the fact that $(a+tx_0)g = 0$ and by using again Proposition 3.2, we have $\Psi'(k, g) = \Psi_1(k, g) \leqslant 0$. Therefore, $\Psi'(k, g) = 0$. Now if $t < 0$, then $-tx_0 \leqslant a$. Hence, $-tx_0g \leqslant ag$. Let $f' \in A, g_i \in (A^{\mathfrak{d}})_+$ such that $f'g_i \geqslant -tx_0g_i$ and $\sum\limits_{i=1}^{n} g_i = g$. It follows that

$$-t\Psi_1(x_0, g) = \Psi_1(-tx_0, g) \leqslant \Psi_1(a, g) = \Psi_0(a, g).$$

Consequently,

$$t\Psi_1(x_0, g) + \Psi_0(a, g) \geqslant 0.$$

Now since $(a + tx_0)g = 0$ and by using again Proposition 3.2, we have $\Psi'(k, g) \leqslant 0$. Therefore, $\Psi'(k, g) = 0$, and the proof is complete. $\square$

The extension Ψ' from Theorem 3.1 satisfies the following:

Proposition 3.3. *Let A be a vector lattice, let $A^{\mathfrak{d}}$ be its Dedekind completion, let $A^{\mathfrak{u}}$ be its universal completion and let B be a Dedekind complete vector lattice. If $\Psi_0\colon A \times A^{\mathfrak{d}} \to B$ is a positive orthosymmetric bilinear map, then the positive orthosymmetric extension Ψ' to $A_{x_0} \times A^{\mathfrak{d}}$ in B satisfies the following property:*

$$\Psi'(f, g) \leqslant \Psi'(f', g)$$

for all $(f, f', g) \in A_{x_0} \times A_{x_0} \times (A^{\mathfrak{d}})_+$ such that $fg \leqslant f'g$.

Proof. It is sufficient to prove that if $(f, g) \in A_{x_0} \times (A^{\mathfrak{d}})_+$ such that $fg \geqslant 0$, then $\Psi(f, g) \geqslant 0$. To this end, let $(f, g) \in A_{x_0} \times (A^{\mathfrak{d}})_+$ such that $fg \geqslant 0$. Hence, $f = a + tx_0 \in A_{x_0}$ $(a \in A,\ t \in \mathbb{R})$. If $t = 0$, then $f = a$ and $ag \geqslant 0$. It follows, by using Proposition 3.2, that $\Psi(a, g) = \Psi(f, g) \geqslant 0$. If $t > 0$, then $tx_0g \geqslant -ag$. Let $f' \in A, g_i \in (A^{\mathfrak{d}})_+$ such that $f'g_i \geqslant tx_0g_i$ and $\sum\limits_{i=1}^{n} g_i = g$. Hence, $tx_0g_i \geqslant -ag_i$. Since $\Psi(-a, g_i) = \Psi_0(-a, g_i) = -\Psi_0(a, g_i)$ and by using Proposition 3.2, it follows that $\Psi(f', g_i) \geqslant -\Psi_0(a, g_i)$. Therefore,

$$\Psi'(f, g) = \Psi_1(k, g) = t\Psi_1(x_0, g) + \Psi_0(a, g) \geqslant 0.$$

Now if $t < 0$, then $-tx_0 \leqslant a$. Hence, $-tx_0 g \leqslant ag$. Let $f' \in A, g_i \in (A^{\mathfrak{d}})_+$ such that $f' g_i \geqslant tx_0 g_i$ and $\sum\limits_{i=1}^{n} g_i = g$. It follows that

$$-t\Psi_1(x_0, g) = \Psi_1(-tx_0, g) \leqslant \Psi_1(a, g) = \Psi_0(a, g).$$

Consequently,

$$\Psi'(f, g) = t\Psi_1(x_0, g) + \Psi_0(a, g) \geqslant 0$$

and we are done. $\square$

Using the same argument as for Ψ, we define for all $(f, g) \in A^{\mathfrak{d}} \times (A^{\mathfrak{d}})_+$

$$\Psi_1'(f, g) = \inf \left\{ \sum_{i=1}^{n} \Psi'(f', g_i) \colon\ f' \in A_{x_0}, g_i \in (A^{\mathfrak{d}})_+, f' g_i \geqslant f g_i, \sum_{i=1}^{n} g_i = g \right\},$$

where the infimum is taken over all finite positive decompositions of g (i.e., $g_i \in (A^{\mathfrak{d}})_+$, $\sum\limits_{i=1}^{n} g_i = g$) and over all $f_i \in A$ such that $f_i g_i \geqslant f g_i$. As easily seen, that $\Psi_1'(\lambda f, g) = \Psi_1'(f, \lambda g) = \lambda \Psi_1'(f, g)$ for all $(f, g) \in A^{\mathfrak{d}} \times A^{\mathfrak{d}}$ and for all $\lambda \in \mathbb{R}_+$. The following lemma capture the basic features of Ψ_1'.

Lemma 3.3. *The mapping Ψ_1' satisfies the following properties:*

(1) $\Psi_1'(f, g) = \Psi'(f, g)$, *for all* $(f, g) \in A_{x_0} \times (A^{\mathfrak{d}})_+$,
(2) $\Psi_1'(f, g) = 0$, *for all* $(f, g) \in A^{\mathfrak{d}} \times A^{\mathfrak{d}}$ *such that* $f \wedge g = 0$,
(3) $\Psi_1'(f_0 + f, g) = \Psi'(f_0, g) + \Psi_1'(f, g)$, *for all* $(f_0, f, g) \in A_{x_0} \times A^{\mathfrak{d}} \times (A^{\mathfrak{d}})_+$,
(4) $\Psi_1'(f, g + g') = \Psi_1'(f, g) + \Psi_1'(f, g')$, *for all* $(f, g, g') \in A^{\mathfrak{d}} \times (A^{\mathfrak{d}})_+ \times (A^{\mathfrak{d}})_+$.

Proof. Using the same argument as in Lemma 3.2, we can deduce properties (1), (3) and (4).

(2) Let $(f, g) \in A \times A^{\mathfrak{d}}$ such that $f \wedge g = 0$. Then $fg = 0$. Let $f' \in A_{x_0}, g_i \in (A^{\mathfrak{d}})_+$ such that $f' g_i \geqslant f g_i$ and $\sum\limits_{i=1}^{n} g_i = g$. It follows that $\sum\limits_{i=1}^{n} f g_i = fg = 0$. Hence, $\sum\limits_{i=1}^{n} g_i f = 0$. Then, $g_i f = 0$ for all $1 \leqslant i \leqslant n$. Hence, by using Proposition 3.3, $\Psi'(f', g_i) \geqslant 0$ and then $\Psi_1'(f, g) \geqslant 0$. Moreover, in the case where $f \wedge g = 0$, it is evident that $\Psi_1'(f, g) \leqslant 0$. A a conclusion, $\Psi_1'(f, g) = 0$ for all $(f, g) \in A \times (A^{\mathfrak{d}})_+$ such that $f \wedge g = 0$. $\square$

By a standard application of the Kuratowski-Zorn lemma, we get from Theorem 3.1 and Lemma 3.3:

Corollary 3.1. *Let A be a vector lattice, let $A^{\mathfrak{d}}$ be its Dedekind completion and let B be a Dedekind complete vector lattice. If $\Psi_0\colon A\times A \to B$ is an orthosymmetric lattice bilinear map, then Ψ_0 can be extended as an orthosymmetric lattice bilinear map Ψ to $A^{\mathfrak{d}} \times A^{\mathfrak{d}}$ in B.*

P r o o f. By Proposition 3.1, Ψ_0 can be extended as a lattice bilinear map, denoted also by Ψ_0, to $A\times A^{\mathfrak{d}}$ in B which is orthosymmetric. By a standard application of the Kuratowski-Zorn lemma, we deduce from Theorem 3.1, Proposition 3.3 and Lemma 3.3 that Ψ_0 can be extended as an orthosymmetric positive bilinear map Ψ to $A^{\mathfrak{d}} \times A^{\mathfrak{d}}$ in B. Then (by [5]), Ψ is symmetric. It remains to prove that Ψ is a lattice bilinear map. Let $f, f' \in A^{\mathfrak{d}}$ such that $f \wedge f' = 0$ and let $g \in (A^{\mathfrak{d}})_+$. Then, by the symmetry of Ψ,

$$\Psi(f \wedge f', g) = \Psi(g, f \wedge f') = 0.$$

Moreover, let $g_0 \in A$ such that $g \leqslant g_0$. Consequently,

$$0 \leqslant \Psi(g, f) \wedge \Psi(g, f') \leqslant \Psi(g_0, f) \wedge \Psi(g_0, f') = \Psi(g_0, f \wedge f') = 0.$$

It follows that $\Psi(g, f) \wedge \Psi(g, f') = \Psi(g, f \wedge f') = 0$. Since Ψ is symmetric, we have

$$\Psi(f \wedge f', g) = \Psi(g, f \wedge f') = \Psi(g, f) \wedge \Psi(g, f') = \Psi(f, g) \wedge \Psi(f', g) = 0$$

and we are done. $\square$

The Corollary 3.1 will be applied to d-algebras.

For the rest of the paper we shall fix the following notations and assumptions. Let $(A, *)$ be a d-algebra. Let $\Psi_0\colon A \times A \to A$; $(a, b) \mapsto a * b$ be the lattice bilinear map associated with the d-algebra product of A and let $\Psi\colon A^{\mathfrak{d}} \times A^{\mathfrak{d}} \to A^{\mathfrak{d}}$ be a lattice bilinear map extension of Ψ_0. Hence we construct a new multiplication denoted also by $*$. Next, we will give a necessary and sufficient condition for the associativity of the extended product. In order to hit this mark, we give the following definition:

Definition 3.1. A lattice bilinear map $\Omega\colon A \times A \to A$ is said to be two power orthosymmetric if $\Omega_{a,b}\colon A \times A \to A; (x, y) \mapsto \Omega(\Omega(a, x), \Omega(y, b))$ is orthosymmetric, for all $a, b \in A_+$.

R e m a r k 3.3. Let A be a vector lattice and let B be a cofinal vector sublattice of A. It is not hard to see that the property "$\Omega\colon A \times A \to A$ is two power orthosymmetric" is equivalent to the following property: $\Omega_{a,b}\colon A \times A \to A; (x, y) \mapsto (\Omega(a, x), \Omega(y, b))$ is orthosymmetric, for all $a, b \in B_+$.

Theorem 3.2. *If Ψ is two power orthosymmetric, then the new multiplication $*$ is associative, that is $(a*b)*c = a*(b*c)$, for all $0 \leqslant a, b, c \in A^{\mathfrak{d}}$.*

Proof. Let $0 \leqslant a, c \in A^{\mathfrak{d}}$. Let $T_{a,c}\colon A \to A^{\mathfrak{d}};\ x \mapsto a*x*c$. Then $T_{a,c}$ is a lattice homomorphism, so it can be extended as a lattice homomorphism in two different ways, which are

$$T_1\colon A^{\mathfrak{d}} \to A^{\mathfrak{d}};\ x \mapsto (a*x)*c$$

and

$$T_2\colon A^{\mathfrak{d}} \to A^{\mathfrak{d}};\ x \mapsto a*(x*c).$$

We claim that $T_1 + T_2$ is a lattice homomorphism. Indeed, if $x_1 \wedge x_2 = 0$ in $A^{\mathfrak{d}}$, we have in the unital f-algebra a universal completion A^u of A, where the f-algebra multiplication is denoted by juxtaposition,

$$\begin{aligned}((T_1 + T_2)(x_1))&((T_1 + T_2)(x_2)) \\ &= [((a*x_1)*c) + (a*(x_1*c))][((a*x_2)*c) + (a*(x_2*c))] \\ &= ((a*x_1)*c)(a*(x_2*c)) + (a*(x_1*c))((a*x_2)*c).\end{aligned}$$

Let $e = (a*x_1)+c+a+(x_2*c)+(x_1*c)+(a*x_2)$, let $f = e*e$ and let $\Phi\colon A_e^{\mathfrak{d}} \times A_e^{\mathfrak{d}} \to A_f^{\mathfrak{d}}$, defined by $(x, y) \mapsto (x*y)$. It is well known that there exists a unique multiplication (denoted by $\bullet$) on $A_f^{\mathfrak{d}}$ such that $A_f^{\mathfrak{d}}$ becomes an f-algebra with a unit element f (see [12, Remark 19.5]). Moreover, by [3, Theorem 1], we have

$$x*y = (x*e) \bullet (e*y),$$

for all $x, y \in A_e^{\mathfrak{d}}$. Then,

$$0 \leqslant ((a*x_1)*c) \bullet (a*(x_2*c)) \leqslant ((a*x_1)*e) \bullet (e*(x_2*c)) = (a*x_1)*(x_2*c) = 0.$$

Hence,

$$((a*x_1)*c) \wedge (a*(x_2*c)) = 0.$$

Then,

$$\begin{aligned}(a*(x_1*c)) \bullet ((a*x_2)*c) &= ((a*x_2)*c) \bullet (a*(x_1*c)) \\ &\leqslant ((a*x_2)*e) \bullet (e*(x_1*c)) = (a*x_2)*(x_1*c) = 0.\end{aligned}$$

Then,

$$(a*(x_1*c)) \wedge ((a*x_2)*c) = 0.$$

Consequently,

$$((a * x_1) * c)(a * (x_2 * c)) = (a * (x_1 * c))((a * x_2) * c) = 0.$$

Therefore,

$$((T_1 + T_2)(x_1))((T_1 + T_2)(x_2)) = 0.$$

Since any unital f-algebra is semiprime, $((T_1 + T_2)(x_1)) \wedge ((T_1 + T_2)(x_2)) = 0$ and then, $T_1 + T_2$ is a lattice homomorphism. Let us take $S = T_1/2 + T_2/2$. Consequently, S is a lattice extension to $A^{\mathfrak{d}}$ of $T_{a,c}$, which implies that S is an extreme extension of $T_{a,c}$. Hence $T_1 = S = T_2$. Finally, we have $(a*b)*c = a*(b*c)$, for all $0 \leqslant a, b, c \in A^{\mathfrak{d}}$ and we are done. $\square$

Thus we deduce the following corollaries.

Corollary 3.2. *Let $(A, *)$ be a d-algebra and $A^{\mathfrak{d}}$ its Dedekind completion. Then, $A^{\mathfrak{d}}$ can be equipped with a d-algebra multiplication that extends the multiplication of A if and only if there exists a two power orthosymmetric map $\Psi\colon A^{\mathfrak{d}} \times A^{\mathfrak{d}} \to A^{\mathfrak{d}}$ a lattice bilinear map extension of Ψ_0, the lattice bilinear map associated with the d-algebra A.*

Corollary 3.3. *Let $(A, *)$ be a commutative d-algebra and $A^{\mathfrak{d}}$ its Dedekind completion. Then, $A^{\mathfrak{d}}$ can be equipped with a d-algebra multiplication that extends the multiplication of A.*

P r o o f. Let $\Psi_0\colon A \times A \to A;\ (a, b) \mapsto a*b$ be the lattice bilinear map associated with the commutative d-algebra product of A. Then, Ψ_0 is an orthosymmetric lattice bilinear map, then by using Corollary 1, Ψ_0 can be extended as an orthosymmetric lattice bilinear map Ψ to $A^{\mathfrak{d}} \times A^{\mathfrak{d}}$ in $A^{\mathfrak{d}}$. Let $a, b \in A_+$ and let $x_1 \wedge x_2 = 0$ in $A^{\mathfrak{d}}$. Since Ψ is symmetric, we have

$$\Psi(\Psi(a, x), \Psi(y, b)) = \Psi(\Psi(x, a), \Psi(y, b)).$$

Moreover, since Ψ is a lattice bilinear map, $\Psi(x, a) \wedge \Psi(y, b) = 0$. By the fact that Ψ is orthosymmetric,

$$\Psi(\Psi(a, x), \Psi(y, b)) = 0,$$

which gives the desired result. $\square$

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Author's address: *Mohamed Ali Toumi*, Départment des Mathématiques, Faculté des Sciences de Bizerte, 7021 Zarzouna, Bizerte, Tunisia, e-mail: `MohamedAli.Toumi@fsb.rnu.tn`.