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FLOWS ON THE JOIN OF TWO GRAPHS

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Abstract. The join of two graphs G and H is a graph formed from disjoint copies of G and H by connecting each vertex of G to each vertex of H. We determine the flow number of the resulting graph. More precisely, we prove that the join of two graphs admits a nowhere-zero 3-flow except for a few classes of graphs: a single vertex joined with a graph containing an isolated vertex or an odd circuit tree component, a single edge joined with a graph containing only isolated edges, a single edge plus an isolated vertex joined with a graph containing only isolated vertices, and two isolated vertices joined with exactly one isolated vertex plus some number of isolated edges.

Keywords: nowhere-zero flow; graph join

MSC 2010: 05C21

1. Introduction

The study of nowhere-zero flows was initiated by Tutte in 1949 in his famous paper [18]. Since then, this notion has become an important part of the graph theory and has received considerable attention. One of the main reasons are three conjectures that Tutte proposed about nowhere-zero flows [7].

Conjecture 1.1 (5-flow conjecture, 1954). *Every bridgeless graph has a nowhere-zero 5-flow.*

Conjecture 1.2 (4-flow conjecture, 1966). *Every bridgeless graph that does not contain the Petersen graph as a minor has a nowhere-zero 4-flow.*

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Conjecture 1.3 (3-flow conjecture 1972)**.** *Every graph without a* 3*-edge-cut has a nowhere-zero* 3*-flow.*

A great deal of interest has been directed towards Tutte's 5-flow conjecture and has led to a significant progress. Among other, it has been shown that the smallest counterexample to the 5-flow conjecture is a 6-cyclically edge-connected snark of the girth at least 11, see [9]. Robertson, Seymour, and Thomas [11] proved that the cubic 4-flow conjecture can be reduced to almost planar graphs, which in turn could be treated by a suitable modification of the 4-colour theorem. The 3-flow conjecture has received somewhat less attention. Perhaps the lack of progress in this area has motivated several authors to examine flow numbers of graphs resulting from various graph operations, most notably graph products [5], [6], [10], [12], [15], [20]. Very recently Thomassen proved the 3-flow conjecture for 8-edge-connected graphs [17]. However, for graphs with smaller edge-connectivity this problem remains widely open. In this paper we therefore examine another operation on graphs—the join.

The join of two graphs G and H is a graph formed from disjoint copies of G and H by connecting every vertex of G to every vertex of H. We determine the flow number of the resulting graph. We show that, except for a few cases, the flow number of the join of two graphs is 3. The hardest part of our analysis arises when one of graphs is "small". The argument splits into a number of cases, but our approach allows us to deal with large number of cases efficiently.

2. Preliminaries

All graphs in this paper are finite; they may contain loops and multiple edges. Let G be a graph and A an abelian group. An *A-flow* is an orientation of edges of G and an assignment of values from A to the oriented edges of G such that the total sum of values on incoming edges equals the total sum of values on outgoing edges (the so called Kirchhoff's law). An A-flow which does not use 0 is called *nowhere-zero*. Note that a graph G admits a nowhere-zero A-flow if and only if G is bridgeless. Moreover, if $A = \mathbb{Z}$, a nowhere-zero *k-flow* is a nowhere-zero $\mathbb{Z}$-flow that uses only values from $\{\pm 1, \dots, \pm(k-1)\}$. The existence of a nowhere-zero k-flow and the existence of a nowhere-zero A-flow can be easily linked. Indeed, G has a nowhere-zero A-flow if and only if it has also a nowhere-zero $|A|$-flow. We will use this to switch freely between nowhere-zero group flows and nowhere-zero integer flows. The *flow number* of a bridgeless graph G is the smallest k such that G has a nowhere-zero k-flow.

We say that graphs G and H are *homeomorphic* if there is an isomorphism from some subdivision of G to some subdivision of H. Note that homeomorphic graphs have the same flow number.

While defining a flow on a graph, it is impractical to define both the orientation and value separately. That is why when we say that the flow value on the edge uv is x, we mean that the edge is oriented from u to v and the flow value on the edge is x. To *add a value* x to an oriented (closed) path $P = v_1v_2 \dots v_k$ ($v_k = v_1$) means to add the value x to each edge v_iv_{i+1} that is oriented from v_i to v_{i+1} and to subtract the value x if v_iv_{i+1} is oriented from v_{i+1} to v_i. Again, if an orientation of some edge v_iv_{i+1} of P is not given yet, we orient it from v_i to v_{i+1} and assign x to v_iv_{i+1}.

For any $k > 2$, the problem of deciding whether a graph G has a nowhere-zero k-flow is in $\mathcal{NP}$. For $k = 3$, this problem is in fact $\mathcal{NP}$-complete, even when G is planar [1, page 562]. However, for $k = 2$ and in some classes of graphs things are quite easy. For instance a graph G has a nowhere-zero 2-flow if and only if all vertices of G have even valency [1], [3]. A cubic graph has the flow number equal to 3 if and only if it is bipartite [1], [3], [7]. Note also that if G is a graph which can be decomposed to several edge-disjoint subgraphs and if each of these subgraphs has a nowhere-zero k-flow, then G also has a nowhere-zero k-flow.

Several authors examined the flow number of different products of graphs. Imrich and Škrekovski proved that the Cartesian product of two nontrivial graphs has a nowhere-zero 4-flow [6]. This result was later improved in two directions. Shu and Zhang [15] characterized the case when the Cartesian product has a nowhere-zero 3-flow and Rollová and Škoviera stated that every Cartesian bundle of two graphs without isolated vertices has a nowhere-zero 4-flow [12]. The flow number of the direct product of graphs was determined by Zhang, Zheng and Mamut [20]. Imrich, Peterin, Špacaran and Zhang [5] considered the flow number of the strong product of two graphs.

We will consider another graph operation: the join of two graphs. Let G and H be two disjoint graphs. The *join* of graphs G and H is the graph $G^* = G + H$ with the vertex set $V = V(G) \cup V(H)$ where two vertices u and v are adjacent if

- ▷ $u, v \in V(G)$ and $uv \in E(G)$ or
- ▷ $u, v \in V(H)$ and $uv \in E(H)$ or
- ▷ $u \in V(G)$ and $v \in V(H)$.

The graphs G and H will be called *summands*. The vertices of G^* originating from the vertices of $V(G)$ will be called *G-vertices* of G^* and the vertices of G^* originating from the vertices of $V(H)$ will be called *H-vertices* of G^*. An edge connecting two vertices from $V(G)$ ($V(H)$) will be called a *G-edge* (an *H-edge*) of G^*. An edge connecting the vertices originating from different graphs will be called a *cross edge*. Clearly, the cross edges form a complete bipartite graph $K_{|V(G)|,|V(H)|}$. Several different graphs and classes of graphs can be created as the join of graph—complete n-partite graphs, wheels, stars, cone graphs, fan graphs (see Figure 1).

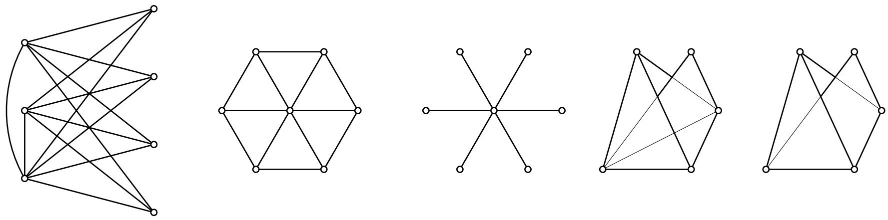

Figure 1. Graphs that can be created as the join of two graphs.

Note that the fact whether the join of two graphs has a nowhere-zero 2-flow can be easily checked.

Theorem 2.1. *A graph $G + H$ has a nowhere-zero 2-flow if and only if for each $v \in G$ the valency of v has the same parity as $|V(H)|$ and for each $w \in H$ the valency of w has the same parity as $|V(G)|$.*

This criterion is easy to verify. Therefore in the rest of the paper we will not try to distinguish among flow numbers 2 and 3.

The problem of determining whether the join of two graphs has a nowhere-zero 3-flow is partially answered by [4]. Indeed, if neither G nor H contain an isolated vertex, the resulting graph is triangularly connected and the results of that paper apply. However, in this case it may not be completely obvious which choices for G or H create a graph that does not have a nowhere-zero 3-flow. Some parts of the proof could be slightly simplified by using the result from [4], but the improvement is not significant, therefore we decided to write the proofs independently of [4].

The main result of this paper is the following:

Theorem 2.2. *Let G and H be two arbitrary graphs without loops, where $|V(G)| \leqslant |V(H)|$. If $G = K_1$ and H contains an isolated vertex then $G + H$ has no nowhere-zero flow. Otherwise $G + H$ has a nowhere-zero 3-flow, except for the following cases where $G + H$ has flow number 4:*

- ▷ *$G = K_1$ and H has no isolated vertex and one of the components of H is an odd-circuit-tree—a graph in which all blocks are odd cycles,*
- ▷ *$G = K_1 \cup K_2$ and $H = n \cdot K_1$, where $n \geqslant 2$,*
- ▷ *$G = K_2$ and $H = n \cdot K_2$, where $n \geqslant 1$,*
- ▷ *$G = 2 \cdot K_1$ and $H = K_1 \cup (n \cdot K_2)$, where $n \geqslant 1$,*

where $m \cdot L$ stands for the graph consisting of m disjoint copies of L and $K \cup L$ stands for the graph consisting of disjoint copies of K and L.

Cases when both summands are "big" enough can be solved in many ways. Perhaps the most straightforward is to use a known result about the group connectivity of

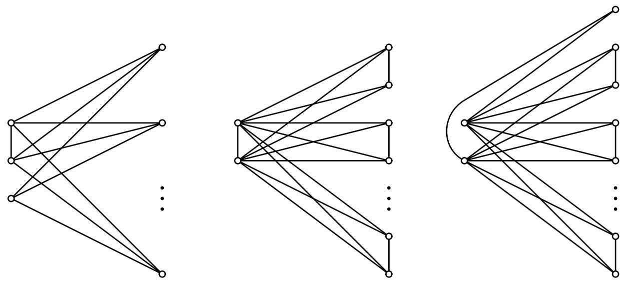

Figure 2. Examples of joins that have the flow number 4.

graphs. Let A be an abelian group and G a bridgeless graph. We fix an arbitrary orientation. The graph G is said to be A-*connected* if for every zero-sum function $f\colon V(G) \to A$, there exists an assignment $\xi\colon E(G) \to (A - \{0\})$ such that for every $v \in G$, the sum of values on edges incoming to v minus the sum of values on edges outgoing from v is $f(v)$. Note that if a graph is A-connected, it also has a nowhere-zero A-flow. Moreover, if a graph G is A-connected and we add an edge to G obtaining a graph G', the graph G' is also A-connected [8]. The join of two graphs is in fact a complete bipartite graph with additional edges. A complete bipartite graph is $\mathbb{Z}_3$-connected if and only if each partition has the size at least 4, see [2].

Lemma 2.1. *Let G and H be two graphs such that $4 \leqslant |V(G)| \leqslant |V(H)|$. Then $G + H$ has a nowhere-zero 3-flow.*

Therefore, we only need to consider the cases when one of the summands is "small". The rest of the proof of Theorem 2.2 will be divided into four chapters. In the next chapter we present lemmas used in Chapters 4, 5 and 6. Chapters 4, 5 and 6 prove Theorem 2.2 for the situation where $|V(G)|$ is 1, 2 and 3 respectively.

3. Tools

Our general approach is to try to decompose the edge-set of a graph in such a way that every partition has a certain nowhere-zero flow. Of course, bipartite graphs will play here an important role.

Lemma 3.1 ([14])**.** *Let $K_{m,n}$ be a complete bipartite graph with partition sizes m and n, where $m, n \geqslant 2$. Then $K_{m,n}$ has a nowhere-zero 3-flow.*

Similarly, graphs which are almost eulerian have a nowhere-zero 3-flow.

Lemma 3.2. *Each bridgeless graph with exactly two vertices of an odd valency has a nowhere-zero* 3*-flow.*

P r o o f. Let G be a graph satisfying the conditions of the lemma and let v_1 and v_2 be two vertices of an odd valency. Consider the component H of G that contains v_1 (and therefore it contains also v_2). Since G is bridgeless, H is 2-connected. Therefore by Menger's theorem ([3]) there are two edge-disjoint paths P and Q between v_1 and v_2. Delete the edges of P in G to obtain G'. Since there is a path Q between v_1 and v_2, both v_1 and v_2 are in the same component of G'; we denote it by H'. Note that all vertices of H' have even valency, therefore H' has an eulerian trail T. We send the flow value 1 along T. Moreover, we send the value 1 along the closed trial created from P and from the subtrial T' of T in the section from v_2 to v_1. This produces the value 2 on T'. The graph $G - E(H' \cup P)$ has a nowhere-zero 2-flow. Therefore, the whole graph G has a nowhere-zero 3-flow. $\square$

Note that the condition that the graph is bridgeless is necessary and we will need to check it everytime when applying Lemma 3.2.

An *amalgamation* of two edges $e_1 = u_1v_1$ of a graph G_1 and $e_2 = u_2v_2$ of a graph G_2 is a graph created by identifying u_1 with u_2 and v_1 with v_2 and then deleting one of the two edges corresponding to e_1 and e_2. The other edge will be called the *amalgamated edge.* The following observations can be made regarding the existence of a nowhere-zero $\mathbb{Z}_3$-flow.

Theorem 3.1. *Let G_1 and G_2 be two non-trivial graphs. Let G be the amalgamation of two arbitrary edges $u_1v_1 \in G_1$ and $u_2v_2 \in G_2$. Then*

(1) *if both G_1 and G_2 have a nowhere-zero $\mathbb{Z}_3$-flow then also G has a nowhere-zero $\mathbb{Z}_3$-flow;*

(2) *if both G_1 and G_2 have no nowhere-zero $\mathbb{Z}_3$-flow then also G has no nowhere-zero $\mathbb{Z}_3$-flow.*

P r o o f. Let both G_1 and G_2 have a nowhere-zero $\mathbb{Z}_3$-flow. Because of an automorphism of $\mathbb{Z}_3$ which maps 1 to 2, we can choose nowhere-zero flows φ_1 and φ_2 on G_1 and G_2, respectively, such that $\varphi_1(u_1v_1) = 1$ and $\varphi_2(u_2v_2) = 1$. The sum of φ_1 and φ_2 is a nowhere-zero 3-flow of G.

On the other hand, suppose that neither G_1 nor G_2 have a nowhere-zero flow and suppose on the contrary that G has a nowhere-zero $\mathbb{Z}_3$-flow φ. Let φ_1 and φ_2 be two $\mathbb{Z}_3$-flows on G_1 and G_2, respectively, which we obtain from φ as follows. Let $\varphi_i(e) = \varphi(e)$ for each edge e of G_i different from u_iv_i $(i = 1, 2)$. From Kirchhoff's law $\varphi_i(u_iv_i)$ in G_i is uniquely determined. Since there is no nowhere-zero flow on G_i and for each edge $e \neq u_iv_i$ we have $\varphi(e) \neq 0$, it follows that $\varphi_i(u_iv_i) = 0$. The flow $\varphi_1 + \varphi_2$ on G equals the flow φ for all edges $e \in G$, $e \neq uv$, where uv is the amalgamated edge. But from Kirchhoff's law also $\varphi(uv) = (\varphi_1 + \varphi_2)(uv) = 0$, which is a contradiction. $\square$

Other useful graphs will be graphs derived from K_4. Let K_n be the complete graph of order n. The graphs which can be obtained from K_n by adding a (parallel) edge and by removing an edge will be denoted by K_n^+ and K_n^-, respectively. The flow number of K_4^+ and K_4^- can be easily determined.

Lemma 3.3. *The flow number of K_4 is 4. The flow number of both K_4^+ and K_4^- is 3.*

Corollary 3.1. *The join of K_2 and $n \cdot K_2$ has flow number 4 for all $n \geqslant 1$. The join of $2 \cdot K_1$ and $K_1 \cup n \cdot K_2$ has flow number 4 for all $n \geqslant 1$.*

Proof. The first graph is an amalgamation of n copies of K_4, so by Lemma 3.3 and Theorem 3.1, it does not admit a nowhere-zero 3-flow. Since this graph is a union of graphs with a nowhere-zero 4-flow, namely K_4 and $n-1$ copies of K_4^-, it has a nowhere-zero 4-flow. So the flow number of the first graph is 4. The second graph is homeomorphic to the first. $\square$

There is one other useful graph that has the flow number 4, but cannot be obtained easily from K_4.

Lemma 3.4. *The flow number of the graph $(K_1 \cup K_2) + (n \cdot K_1)$ is 4 for all $n \geqslant 2$.*

Proof. Suppose that the graph $G = (K_1 \cup K_2) + (n \cdot K_1)$ has a nowhere-zero $\mathbb{Z}_3$-flow φ. Let us choose the orientation of G in such a way that $\varphi(e) = 1$ for every edge e of G. For each $(n \cdot K_1)$-vertex all its three edges must be oriented in the same way. Let a be the number of $(n \cdot K_1)$-vertices with all edges outgoing and let b be the number of $(n \cdot K_1)$-vertices with all edges incoming. Let v be the $(K_1 \cup K_2)$-vertex isolated in $K_1 \cup K_2$. From Kirchhoff's law in v it follows that $b - a \equiv 0 \pmod 3$. However, the other two vertices in $K_1 \cup K_2$ give conditions $b - a \equiv 1 \pmod 3$ and $b - a \equiv -1 \pmod 3$, which is a contradiction.

Now we show that G has a nowhere-zero 4-flow. The graph $(K_1 \cup K_2) + (2 \cdot K_1)$ is homeomorphic to K_4, so it has a nowhere-zero 4-flow. The graph $(K_1 \cup K_2) + (3 \cdot K_1)$ has $K_{3,3}$ as a spanning subgraph. Since $K_{3,3}$ is $\mathbb{Z}_4$-connected [2], $(K_1 \cup K_2) + (3 \cdot K_1)$ has a nowhere-zero 4-flow. For $n > 3$, we can always decompose the graph to $(K_1 \cup K_2) + (2 \cdot K_1)$ and the complete bipartite graph $K_{3,n-2}$. $\square$

In this paper we will use the fact that several graphs have a nowhere-zero 3-flow (for proofs see Figure 3).

Lemma 3.5. *The join of the path on three vertices P_2 with two isolated vertices $2 \cdot K_1$ has a nowhere-zero 3-flow.*

Lemma 3.6. *The join of the star S_3 with the complete graph K_2 has a nowhere-zero 3-flow.*

Lemma 3.7. *The join of the graph $K_1 \cup K_2$ with the graph $K_1 \cup K_2^+$ has a nowhere-zero 3-flow.*

Lemma 3.8. *The graph $(3 \cdot K_1) + (K_1 \cup (2 \cdot K_2))$ has a nowhere-zero 3-flow.*

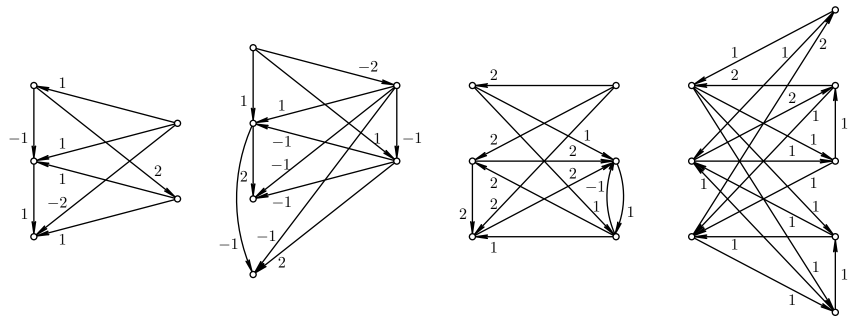

Figure 3. Nowhere-zero 3-flows on $P_2 + (2 \cdot K_1)$, $S_3 + K_2$, $(K_1 \cup K_2) + (K_1 \cup K_2^+)$ and $(3 \cdot K_1) + (K_1 \cup (2 \cdot K_2))$

4. Join with a vertex

If one of the summands G or H has only one vertex, its edges must be loops. Since we can ignore loops in flow problems, we only need to consider the case when one of the summands is an isolated vertex. Without loss of generality let $V(G) = \{u\}$. If H contains an isolated vertex, the join $G + H$ contains a bridge and thus $G + H$ admits no nowhere-zero flow. From now on we assume that H contains no isolated vertex.

Let v be a vertex of H of valency at least 3 and let $e = wv$ and $f = vy$ be two different non-loop edges incident with v. Then we can create the graph $H_{v,e,f}$ by deleting the edges e and f and adding a new edge wy. We name such an operation a *splitting* of the graph H in v. It is not difficult to see that if $\{u\} + H_{v,e,f}$ has a nowhere-zero k-flow for some v, e and f, then also $\{u\} + H$ has a nowhere-zero k-flow. It is obvious that every graph can be split into a collection of paths and circuits. Let us examine these three cases.

Lemma 4.1. *The join $K_1 + P_n$ of a vertex and a path on $n+1$ vertices has a nowhere-zero 3-flow for all $n \geqslant 1$.*

P r o o f. The graph is an amalgamation of several triangles, thus the result follows according to Theorem 3.1. $\square$

Lemma 4.2. *The join $K_1 + C_{2n}$ of a vertex and an even cycle has a nowhere-zero 3-flow for all $n \geqslant 1$.*

P r o o f. The case when $n = 1$ is trivial. Let $n \geqslant 2$ and $C_{2n} = x_1x_2 \ldots x_{2n}$. We send a value 1 along the circuits $ux_{2i+1}x_{2i+2}$ for $i \in \{0, 1, \ldots, n-1\}$. We also send a value 1 along circuits $ux_{2i+1}x_{2i}$ for $i \in \{1, 2, \ldots, n-1\}$ and ux_1x_{2n}. This is a nowhere-zero 3-flow on $K_1 + C_{2n}$. $\square$

Lemma 4.3. *The join $K_1 + C_{2n+1}$ of a vertex with an odd cycle has flow number 4 for all $n \geqslant 1$.*

P r o o f. First we show that $G^* = K_1 + C_{2n+1}$ has no nowhere-zero 3-flow. Consider, for a contradiction, a nowhere-zero $\mathbb{Z}_3$-flow on G^* with an orientation chosen in such a way that all edges of G^* have value 1. Since each C_{2n+1}-vertex is 3-valent, its incident edges are all oriented either towards it or away from it. Therefore the orientation of the edges must alternate along C_{2n+1}, which is not possible, because the circuit has an odd length.

A nowhere-zero $\mathbb{Z}_4$-flow can be constructed from a nowhere-zero $\mathbb{Z}_4$-flow on $\{u\} + P_{2n}$ by sending a value l along the triangle ux_0x_{2n} where x_0 and x_{2n} are two end-vertices of P_{2n}. Note that we can always choose a suitable value l because $\mathbb{Z}_4$ has 3 non-zero elements and at most two of them are already used on the edges of the triangle ux_0x_{2n}. $\square$

Therefore the join of a graph H that does not contain an isolated vertex with a single vertex always has a nowhere-zero 4-flow. Moreover, it has a nowhere-zero 3-flow only if H can be split into a collection of paths and even cycles. It is known that this can be done if and only if none of the components of H is an odd-circuit-tree [15], which is a graph where each block is a circuit of an odd length. On the other hand, let H contain a component T that is an odd-circuit-tree. Then $\{u\} + T$ is an amalgamation of several graphs created as the join of a vertex with an odd cycle. If H has only one component we are done. If H has more than one component, we contract $\{u\} + (H - T)$ into one vertex. We get a graph isomorphic to $\{u\} + T$. Since the contraction of a subgraph cannot increase the flow number we have that the flow number of $\{u\} + H$ is at least 4.

We summarize our results in the next lemma:

Lemma 4.4. *The join of a graph $G = \{u\}$ with a graph H without any isolated vertices has flow number 4 if and only if H has a component which is an odd-circuit-tree. Otherwise $G + H$ has a nowhere-zero 3-flow.*

5. Join with a graph on two vertices

In this section, we will consider the case when the smaller summand, say G, has exactly two vertices. Let g_e and h_e be, respectively, the number of vertices of an even valency in G and H. Let g_o and h_o be, respectively, the number of vertices of an odd valency in G and in H. Of course, both g_o and h_o are even numbers.

In this chapter we will apply the vertex splitting operation even on vertices of valency 2, which creates isolated vertices (note that even if we use this kind of splitting and the resulting graph admits a nowhere-zero k-flow, the original graph also admits a nowhere-zero k-flow). Since the existence of loops makes no difference in flow problems, we will ignore the resulting loops while analyzing a flow.

Let g_e=2. Let us use the vertex splitting operation on both G and H as many times as possible (even if it creates loops) and name the result G' and H' respectively. Since the vertex splitting operation preserves the parity of vertex-valency, g_e, g_o, h_e and h_o stay the same. Thus the graph G' contains only two isolated vertices (ignoring loops) and the graph H' is a collection of isolated vertices and isolated edges.

If the number of isolated vertices in H' is 0 (i.e., $h_e = 0$), then the graph $G' + H'$ can be decomposed into several copies of K_4^- (one for each isolated edge of H'). Due to Lemma 3.3 each of them has a nowhere-zero 3-flow and so does $G' + H'$ and then also $G + H$. Suppose now that $h_e \geqslant 2$. Then the graph $G' + H'$ is a union of K_{2,h_e} and $h_o/2$ copies of K_4^-. Both the graphs have a nowhere-zero 3-flow by Lemma 3.1 and Lemma 3.3, respectively, and so does $G + H$.

If $h_e = 1$, then $h_o \geqslant 2$, so H' contains at least one copy of K_2. By Corollary 3.1 the graph $G' + H'$ has flow number 4. If $G + H = G' + H'$, then $G + H$ also has flow number 4. Suppose now that $G + H \neq G' + H'$. If $H \neq H'$, then there was a series of splittings used on H and let H'' be a graph one splitting before we get H'. In this situation we will analyze the graph $G' + H''$. If $H = H'$, let G'' be the graph one splitting before we get G'. In this situation we will analyze the graph $G'' + H$. There are five distinct joins to analyze:

(1) the join of G' and $H'' = P_2 \cup (n \cdot K_2)$,
(2) the join of G' and $H'' = P_2^+ \cup (n \cdot K_2)$,
(3) the join of G' and $H'' = K_1 \cup S_3 \cup (n \cdot K_2)$,
(4) the join of G' and $H'' = P_3^{+c} \cup K_1 \cup (n \cdot K_2)$,
(5) the join of $G'' = C_2$ and H,

where P_2^+ denotes a path of length two with one multiple edge and P_3^{+c} denotes a path of length 3 with a multiple center edge.

In the first case $G' + H''$ can be decomposed into several copies of K_4^- and $(2 \cdot K_1) + P_2$, which are the graphs that have according to Lemma 3.3 and Lemma 3.5 a nowhere-zero 3-flow. In the second case H'' can be split in such a way that we get $P_2 \cup (n \cdot K_2)$. In the third case $G' + H''$ can be decomposed into a subgraph $[(2 \cdot K_1) + (K_1 \cup S_3)]$ and several copies of K_4^-. The first subgraph is homeomorphic to $K_2 + S_3$, which has a nowhere-zero 3-flow due to Lemma 3.6, and the second subgraph has a nowhere-zero 3-flow due to Lemma 3.3. In the fourth case splitting of P_3^{+c} in such a way that we obtain S_3 transforms this case to the previous one. In the fifth case $G'' + H$ can be decomposed into $C_2 + K_1$ and several copies of K_4^-. Note that again they both have a nowhere-zero 3-flow. Thus each case leads to a nowhere-zero 3-flow of $G + H$. We conclude that if one of summands in $G + H$, say G, has two vertices of even degree in G and the other, say H, has at least two vertices, then $G + H$ has no nowhere-zero 3-flow if and only if $G = 2 \cdot K_1$ and $H = K_1 \cup (n \cdot K_2)$.

Let g_o=2. Let us consider $(G - e) + (H \cup \{v\})$, where e is an arbitrary non-loop edge of G and v is a new vertex added to H. The graph $(G - e) + (H \cup \{v\})$ is homeomorphic to $G + H$, so they have the same flow number. Note that both the vertices of $G - e$ have even valency and thus by the previous case the graph $(G - e) + (H \cup \{v\})$ has no nowhere-zero 3-flow if and only if $G - e = 2 \cdot K_1$ and $H + \{v\} = K_1 \cup (n \cdot K_2)$. Therefore the graph $G + H$ has no nowhere-zero 3-flow if and only if $G = K_2$ and $H = n \cdot K_2$. In that case, it has the flow number 4.

6. Join with a graph on three vertices

In this section, we will consider that the smaller summand, say G, has exactly three vertices. Again, let g_e and h_e be, respectively, the number of vertices of an even valency in G and in H and let g_o and h_o be, respectively, the number of vertices of an odd valency in G and in H. Of course, g_o and h_o are even numbers.

Similarly to the previous chapter, we will apply a splitting even if it creates an isolated vertex or a loop. Again, we will ignore loops while analyzing a nowhere-zero flow.

Again we use the vertex splitting operation on both G and H as many times as possible and thus we obtain G' and H'.

Let $g_e = 3$. Then $G' = 3 \cdot K_1$. If $h_e = 0$ then $G' + H'$ can be decomposed into several copies of $(3 \cdot K_1) + K_2$, which according to the previous section has a nowhere-zero 3-flow and hence so does $G + H$. If $h_e \geqslant 2$ then $G' + H'$ can be partitioned to $(3 \cdot K_1) + (h_e \cdot K_1)$ and several $(3 \cdot K_1) + K_2$. The first graph is the complete bipartite graph with partitions of sizes at least 2 and the second has a nowhere-zero 3-flow by the previous section and hence so does $G + H$.

Therefore, the only problem can arise when $h_e = 1$. Let $h_o > 2$, we claim that $G' + H'$ has the flow number 3. Let $h_o = 4$. Then $G' + H'$ has a nowhere-zero 3-flow due to Lemma 3.8. If $h_o > 4$, $G' + H'$ can be decomposed into the graph $(3 \cdot K_1) + (K_1 \cup (2 \cdot K_2))$ and several copies of $(3 \cdot K_1) + K_2$. Each has a nowhere-zero 3-flow, the first due to Lemma 3.8, and hence so does $G + H$.

If $h_o = 2$, then according to Lemma 3.4 $G' + H'$ has the flow number 4. If $G + H = G' + H'$, then the flow number of $G + H$ is also 4. Suppose now that $G' = 3 \cdot K_1$ and $H' = K_1 \cup K_2$ and $G \neq G'$ or $H \neq H'$. Let us analyze the last splitting operation. Suppose first it was done on G'' yielding G'. There is only one possibility, $G'' = K_1 \cup C_2$. According to Lemma 3.7 $G'' + H'$ has a nowhere-zero 3-flow. On the other hand, suppose that the last splitting operation was done on some H'' yielding H'. Then $H'' = P_2$ or $H'' = P_2^+$ or $H'' = K_1 \cup K_2^{++}$, where K_2^{++} is a triple edge. The graph $(3 \cdot K_1) + P_2$ can be decomposed to $K_{2,3}$ and $K_1 + P_2$, which both have a nowhere-zero 3-flow (Lemma 3.1 and Lemma 4.1). If $H'' = P_2^+$, a splitting operation on H'' can by applied in a different way to obtain P_2. If $H'' = K_1 \cup K_2^{++}$, then $G' + H''$ can be decomposed to $K_{3,3}$ and K_2^{++}. Note that they both have a nowhere-zero 3-flow and hence so does $G + H$. Therefore $G + H$ has no nowhere-zero 3-flow if and only if $G = 3 \cdot K_1$ and $H = K_1 \cup K_2$.

Let $g_e = 1$ and $g_o = 2$. Suppose first that $h_o \geqslant 2$. If $h_e > 1$, then let us delete the complete bipartite graph from $G' + H'$ between all G'-vertices and all isolated H'-vertices, which has a nowhere-zero 3-flow according to Lemma 3.1. The remaining graph has exactly two vertices of odd valency, both in G'. Since $h_o \geqslant 2$, we have at least two edge-disjoint paths between these two vertices. According to Lemma 3.2, such a graph has a nowhere-zero 3-flow and hence so does $G + H$.

If $h_e = 1$, then $h_o \geqslant 2$ (each summand has at least 3 vertices). We can decompose $G' + H'$ into $(K_1 \cup K_2) + (K_1 \cup K_2)$ and several copies of $(3 \cdot K_1) + K_2$. The first graph is bridgeless with exactly two vertices of odd valency, and $(3 \cdot K_1) + K_2$ is homeomorphic to K_4^+. They both have a nowhere-zero 3-flow (according to Lemma 3.2 and Lemma 3.3) and hence so does $G + H$.

If $h_e = 0$, then this graph has a nowhere-zero 3-flow according to Lemma 3.2.

Let $h_o = 0$. Then $G' + H'$ has the flow number 4 according to Lemma 3.4. If $G + H = G' + H'$, then the flow number of $G + H$ is also 4. Suppose now that $G \neq G'$ or $H \neq H'$. Let us analyze the last splitting operation. Suppose first it was done on G'' yielding $G' = K_1 \cup K_2$. Then either $G'' = P_2$ or $G'' = P_2^+$ or $G'' = K_1 \cup K_2^{++}$, where K_2^{++} is a triple edge. The graph $P_2 + (n \cdot K_1)$ can be decomposed to $K_1 + P_2$ and $K_{3,n-1}$. Since $n \geqslant 3$, both the graphs have a nowhere-zero 3-flow according to Lemma 4.1 and Lemma 3.1, respectively. If $G'' = P_2^+$, then a splitting operation on G'' can be done differently to obtain P_2. If $G'' = K_1 \cup K_2^{++}$, then $G'' + H'$ can be decomposed into K_2^{++} and $K_{3,n}$, which both admit a nowhere-zero 3-flow. Thus it has a nowhere-zero 3-flow and so does $G + H$. If the last splitting operation was applied to H'' yielding $H' = n \cdot K_1$, then $H'' = K_2^{+} \cup (n-2) \cdot K_1$. The graph $G' + H''$ can be decomposed to $G' + C_2$ and $K_{3,n-2}$, which both have a nowhere-zero 3-flow, if $n > 3$. If $n = 3$, the graph $G' + H''$ has a nowhere-zero 3-flow due to Lemma 3.7. Therefore $G + H$ has no nowhere-zero 3-flow if and only if $G = K_1 \cup K_2$ and $H = n \cdot K_1, n \geqslant 3$. This concludes our analysis.

7. Summary and concluding remarks

As we have seen, when studying flow properties of joins, the cases when one graph is "small" and the other is "big" are the most interesting. In fact, for some related notions even characterizing joins with a single vertex may be an interesting question. For instance, when considering the group connectivity it is not hard to see that $K_1 + P_n$ is not $\mathbb{Z}_3$-connected. On the other hand, $K_1 + C_{2n}$ is $\mathbb{Z}_3$-connected for all $n \geqslant 1$.

Question 7.1. *Let G and H be two graphs, $V(G) \leqslant 3$. When is $G + H$ $\mathbb{Z}_3$-connected, $\mathbb{Z}_4$-connected and $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-connected?*

References

[1] *J. A. Bondy, U. S. R. Murty*: Graph Theory. Graduate Texts in Mathematics 244, Springer, Berlin, 2008.

[2] *J. J. Chen, E. Eschen, H.-J. Lai*: Group connectivity of certain graphs. Ars Comb. *89* (2008), 141–158.

[3] *R. Diestel*: Graph Theory. Third ed. Graduate Texts in Mathematics 173, Springer, Berlin, 2005.

[4] *G. Fan, H. Lai, R. Xu, C.-Q. Zhang, Ch. Zhou*: Nowhere-zero 3-flows in triangularly connected graphs. J. Comb. Theory, Ser. B *98* (2008), 1325–1336.

[5] *W. Imrich, I. Peterin, S. Špacarpan, C.-Q. Zhang*: NZ-flows in strong products of graphs. J. Graph Theory *64* (2010), 267–276.

[6] *W. Imrich, R. Škrekovski*: A theorem on integer flows on Cartesian products of graphs. J. Graph Theory *43* (2003), 93–98.

[7] *F. Jaeger*: Nowhere-zero flow problems. Selected Topics in Graph Theory 3 (L. W. Beineke, R. J. Wilson, eds.). Academic Press, San Diego, CA, 1988, pp. 71–95.

[8] *F. Jaeger, N. Linial, C. Payan, M. Tarsi*: Group connectivity of graphs—a nonhomogeneous analogue of nowhere-zero flow properties. J. Comb. Theory, Ser. B *56* (1992), 165–182.

[9] *M. Kochol*: Smallest counterexample to the 5-flow conjecture has girth at least eleven. J. Comb. Theory, Ser. B *100* (2010), 381–389.

[10] *M. Nánásiová, M. Škoviera*: Nowhere-zero 3-flows in Cayley graphs and Sylow 2-subgroups. J. Algebr. Comb. *30* (2009), 103–111.

[11] *N. Robertson, P. Seymour, R. Thomas*: Tutte's edge-colouring conjecture. J. Comb. Theory, Ser. B *70* (1997), 166–183.
[12] *E. Rollová, M. Škoviera*: Nowhere-zero flows in Cartesian bundles of graphs. Eur. J. Comb. *33* (2012), 867–871.
[13] *P. D. Seymour*: Nowhere-zero 6-flows. J. Comb. Theory, Ser. B *30* (1981), 130–135.
[14] *H. Shahmohamad*: On minimum flow number of graphs. Bull. Inst. Comb. Appl. *35* (2002), 26–36.
[15] *J. Shu, C.-Q. Zhang*: Nowhere-zero 3-flows in products of graphs. J. Graph Theory *50* (2005), 79–89.
[16] *E. Steffen*: Circular flow numbers of regular multigraphs. J. Graph Theory *36* (2001), 24–34.
[17] *C. Thomassen*: The weak 3-flow conjecture and the weak circular flow conjecture. J. Comb. Theory, Ser. B *102* (2012), 521–529.
[18] *W. T. Tutte*: On the imbedding of linear graphs in surfaces. Proc. Lond. Math. Soc., II. Ser. *51* (1949), 474–483.
[19] *R. Xu, C.-Q. Zhang*: Nowhere-zero 3-flows in squares of graphs. Electron. J. Comb. *10* (2003), Research paper R5, 8 pages; printed version J. Comb. *10* (2003).
[20] *Z. Zhang, Y. Zheng, A. Mamut*: Nowhere-zero flows in tensor product of graphs. J. Graph Theory *54* (2007), 284–292.

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