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THE PRICING OF CREDIT RISKY SECURITIES UNDER STOCHASTIC INTEREST RATE MODEL WITH DEFAULT CORRELATION

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Abstract. In this paper, we study the pricing of credit risky securities under a three-firms contagion model. The interacting default intensities not only depend on the defaults of other firms in the system, but also depend on the default-free interest rate which follows jump diffusion stochastic differential equation, which extends the previous three-firms models (see R. A. Jarrow and F. Yu (2001), S. Y. Leung and Y. K. Kwok (2005), A. Wang and Z. Ye (2011)). By using the method of change of measure and the technology (H. S. Park (2008), R. Hao and Z. Ye (2011)) of dealing with jump diffusion processes, we obtain the analytic pricing formulas of defaultable zero-coupon bonds. Moreover, by the "total hazard construction", we give the analytic pricing formulas of credit default swap (CDS).

Keywords: credit risk; default correlation; defaultable bond; credit default swap; default intensity

MSC 2010: 60H30, 60G51

1. Introduction

Credit risk has long been a major problem plaguing financial institutions such as banks. Especially, after some financial crises such as the 1997 Asian financial crisis and the 2007 US subprime mortgage crisis, the contagion effect of credit risk has attracted huge attention of financial market regulators and institutions. Using credit derivatives to transfer, elude and hedge credit risk has become more and more important. To price credit derivatives fairly, the default contagion between the risky assets must be considered sufficiently. Therefore, we study the default contagion model based on reduced-form models in this paper.

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Typical reduced-form models are introduced in [1], [8], [15], [22], [7]. Jarrow and Lando [14] study the case in which the intensity for credit migration is constant. Litterman and Iben [21] give a Markov chain model of credit migration. In [8], [9] and [19], the default intensity is modeled as a random process. A common feature of reduced-form models is that default cannot be predicted and can occur at any time. Therefore, reduced-form models have been used to price a wide variety of instruments. The parameters of these models can be estimated ([5], [6]). Jarrow and Yu [17] set up an intensity-based model in which the parameters are estimated according to the prices of bonds and CDS. Leung and Kwok [20] give CDS valuation of a two-firms contagion model and three-firms contagion model by the method of change of measure. Wang and Ye [25] consider the effect that two parties default simultaneously on the third party. Bai, Hu and Ye [2] introduce a hyperbolic attenuation contagion model and obtain the analytic formula of CDS. A three-firms attenuation model with counterparty risk is introduced in [26], and the closed-form pricing expressions of defaultable bonds and CDS are obtained.

We mainly study the pricing of CDS. As one of the most important credit derivatives, CDS is a bilateral contract, which involves three parties: CDS protection buyer A, CDS protection seller B and reference asset C (see Fig. 1). Party A (CDS protection buyer) holds a corporate bond with some long maturity T_1 of party C (reference asset), and party C is subject to default. Party A faces the credit risk arising from the default of party C. To hedge this risk (or transfer this risk), party A enters a CDS contract with the maturity T ($T < T_1$), and makes premium payments, known as the swap premium, to party B (CDS protection seller). In exchange, party B promises to compensate A for its loss in the event of default or credit downgrade of the bond. In this paper, we consider the case that all three parties may default within the maturity T of CDS. Moreover, the default of three parties have the contagion effect. Therefore, to determine a fair swap rate of a CDS in the presence of counterparty risks, we give a three-firms default contagion model with interacting term. Also, we assume the default of three parties is related to the default-free interest rate, which extends the model in [25]. The structure of CDS with three-party default risk is as follows:

In Fig. 1, firm A (a corporate bond investing firm) holds a corporate bond (reference asset) issued by firm C (a corporate bond issuer)(refer to $1A$), and firm C is subject to default. At bond maturity, if firm C doesn't default, it will pay the bond principle and interest to firm A (see $1B$). Otherwise, it has no payments (refer to $1C$). On the other hand, to hedge the default risk of firm C, firm A and firm B (a monoline insurer) enter into a CDS contract. Firms A and B are also subject to default. If firms A and C have no default, firm A makes fixed premium payments, known as the swap premium to firm B (see $1D$). If either firm A or firm C defaults,

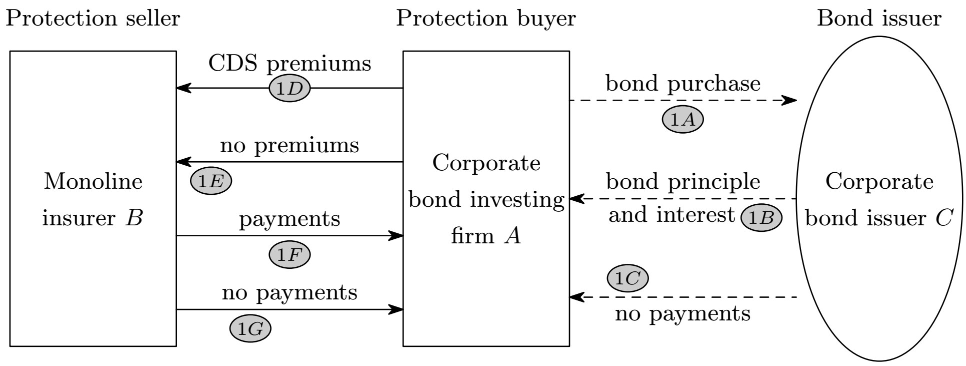

Figure 1. Structure of CDS with three-party default risk

there is no premium payments to firm B (refer to $1E$). In exchange, firm B promises to compensate A (if A doesn't default) for its loss in the event of default of the bond C as long as firm B doesn't default (refer to $1F$). If the protection seller B defaults prior to the default of either the reference asset C or the protection buyer A, the protection seller B can simply walk away from the contract and has no obligation to pay the compensation to the protection buyer A (see $1G$).

In this paper, we study a three-firms contagion model with interacting default risk and stochastic interest rate jump-diffusion risk. Under this model, the valuation of the defaultable zero-coupon bonds and CDS is obtained. The structure of this paper is as follows: in Section 2, we give the basic setup and the three-firms contagion model with interacting risk and a jump-diffusion stochastic interest rate process. In Section 3, we give the general bond pricing formulas, and closed-form pricing formulas of defaultable bonds are obtained. In Section 4, we give the joint conditional density function of default time for three firms and the analytical formula of CDS is provided. We conclude this paper in Section 5.

2. Basic setup and three-firms contagion model

2.1. Basic setup and construction of default time. We consider an uncertain economy with a time horizon of T^* described by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{T^*}, P)$ (in this paper we follow the symbols and notations of Jarrow and Yu [17]) satisfying the usual conditions of right-continuity and completeness with respect to P-null sets, where $\mathcal{F} = \mathcal{F}_{T^*}$ and P is an equivalent martingale measure, since we are only interested in the valuation of credit derivatives. We assume the existence and uniqueness of P, so that bond markets are complete and no arbitrage, as shown in discrete time [11] and in continuous time [12].

Let the $\mathbb{R}^d$-valued process X_t represent d dimensional economy-wide state variables. Point processes N^i $(i = 1, 2, 3)$ initialized at 0 represent the default processes of the firms in the economy so that the default of the ith firm occurs when N^i jumps from 0 to 1.

To be consistent with the information contained in the state variables and the default processes, let

$$\mathcal{F} = \mathcal{F}_t^X \vee \mathcal{F}_t^1 \vee \mathcal{F}_t^2 \vee \mathcal{F}_t^3, \tag{2.1}$$

where

$$\mathcal{F}_t^X = \sigma(X_s, 0 \leqslant s \leqslant t),$$

and

$$\mathcal{F}_t^i = \sigma(N_s^i, 0 \leqslant s \leqslant t) \tag{2.2}$$

are the filtrations generated by X_t and N_t^i, respectively.

Let

$$\begin{aligned}
\mathcal{G}_t^A &= \mathcal{F}_t^A \vee \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^B \vee \mathcal{F}_{T^*}^C = \mathcal{F}_t^A \vee \mathcal{G}_0^{-A},\\
\mathcal{G}_t^B &= \mathcal{F}_t^B \vee \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^A \vee \mathcal{F}_{T^*}^C = \mathcal{F}_t^B \vee \mathcal{G}_0^{-B},\\
\mathcal{G}_t^C &= \mathcal{F}_t^C \vee \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^A \vee \mathcal{F}_{T^*}^B = \mathcal{F}_t^C \vee \mathcal{G}_0^{-C},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{G}_0^{-A} &= \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^B \vee \mathcal{F}_{T^*}^C,\\
\mathcal{G}_0^{-B} &= \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^A \vee \mathcal{F}_{T^*}^C,\\
\mathcal{G}_0^{-C} &= \mathcal{F}_{T^*}^X \vee \mathcal{F}_{T^*}^A \vee \mathcal{F}_{T^*}^B.
\end{aligned}$$

We know that $\mathcal{G}_0^{-i}(i = A, B, C)$ contains complete information on the state variables and the default processes of all firms other than the ith, all the way up to time T^*.

According to the filtration $\mathcal{G}_t^i$, it's possible to select a nonnegative, $\mathcal{G}_0^i$-measurable process λ_t^i, satisfying $\int_0^t \lambda_s^i \,\mathrm{d}s < \infty, P$-a.s. for all $t \in [0, T^*]$, so that we can define an inhomogeneous Poisson process N^i, using the process λ_t^i as its intensity process.

Let τ^i denote the default time of firm i, namely, let τ^i be the first jump time of N^i. In a typical reduced-form model, which can be defined as

$$\tau^i = \inf\left\{t\colon \int_0^t \lambda_s^i \,\mathrm{d}s \geqslant E^i\right\}, \tag{2.3}$$

where $\{E^i\}_{i=1}^3$ is independent of X_t $(t \in [0, T^*])$.

According to the Doob-Meyer decomposition,

$$M_t^i = N_t - \int_0^{t\wedge\tau^i} \lambda_s^i \,\mathrm{d}s \tag{2.4}$$

is a $(P, \mathcal{F}_t)$-martingale.

Under the above characterization, the conditional survival probability of firm i is given by

$$P(\tau^i > t|\mathcal{G}_0^{-i}) = \exp\Big(-\int_0^t \lambda_s^i \,\mathrm{d}s\Big), \quad t \in [0, T^*]. \tag{2.5}$$

The unconditional survival probability of firm i is given by

$$P(\tau^i > t) = E\Big[\exp\Big(-\int_0^t \lambda_s^i \,\mathrm{d}s\Big)\Big], \quad t \in [0, T^*]. \tag{2.6}$$

2.2. Three-firms contagion model with stochastic interest rate process. In this subsection, we explore the three-firms contagion model with an interaction term and a stochastic interest rate. The default intensity of one firm is affected by the default risk of the other two firms and the jump-diffusion risk of default-free interest rate. In the three-firms contagion model, the inter-dependent structure between firm A, firm B and firm C is characterized by the correlated default intensities. The default intensities of A, B and C have the following forms

$$\lambda_t^A = a_0 + ar_t + a_1 \mathbb{1}_{\{\tau^B \leqslant t, \tau^C > t\}} + a_2 \mathbb{1}_{\{\tau^C \leqslant t, \tau^B > t\}} + a_3 \mathbb{1}_{\{\tau^B \leqslant t, \tau^C \leqslant t\}}, \tag{2.7}$$

$$\lambda_t^B = b_0 + br_t + b_1 \mathbb{1}_{\{\tau^A \leqslant t, \tau^C > t\}} + b_2 \mathbb{1}_{\{\tau^C \leqslant t, \tau^A > t\}} + b_3 \mathbb{1}_{\{\tau^A \leqslant t, \tau^C \leqslant t\}}, \tag{2.8}$$

$$\lambda_t^C = c_0 + cr_t + c_1 \mathbb{1}_{\{\tau^A \leqslant t, \tau^B > t\}} + c_2 \mathbb{1}_{\{\tau^B \leqslant t, \tau^A > t\}} + c_3 \mathbb{1}_{\{\tau^A \leqslant t, \tau^B \leqslant t\}}, \tag{2.9}$$

where $a_0 > 0$, $b_0 > 0$, $c_0 > 0$, $a > 0$, $b > 0$, $c > 0$, and satisfying $a_0 + a + a_1 + a_2 + a_3 > 0$, $b_0 + b + b_1 + b_2 + b_3 > 0$, $c_0 + c + c_1 + c_2 + c_3 > 0$.

The default-free interest rate satisfies the following affine jump diffusion stochastic differential equation

$$\mathrm{d}r_t = \alpha(K - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + q_t\,\mathrm{d}N_t, \tag{2.10}$$

where W_t is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^{T^*}, P)$. N_t is a Poisson process with intensity μ, which is independent of W_t. The parameters α and K are constant, and represent the reversion velocity and mean level, respectively. The parameter σ is the volatility, also a constant, q_t is a deterministic function.

In our model, we consider an interacting term, namely, we allow the effect of two parties' simultaneous default on the third party. Moreover, we assume the default-free firm follows the jump diffusion stochastic differential equation rather than a constant, which generalize the model [25].

From equation (2.10), r_t has the following explicit solution:

$$(2.11)\quad r_t = r_0 \mathrm{e}^{-\alpha t} + \alpha K \int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s + \sigma \int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}W_s + \int_0^t q_s \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}N_s.$$

Similar to the results in [17], we can also use time-t forward interest rate instead of time-0 forward interest rate r_t. Let $f(0,u) = r_0 \mathrm{e}^{-\alpha u}$ and for any $u \geqslant t$. Then equation (2.11) becomes

$$(2.12)\quad r_u = f(t,u) + \alpha K \int_t^u \mathrm{e}^{-\alpha(u-s)}\,\mathrm{d}s + \sigma \int_t^u \mathrm{e}^{-\alpha(u-s)}\,\mathrm{d}W_s + \int_t^u q_s \mathrm{e}^{-\alpha(u-s)}\,\mathrm{d}N_s,$$

where

$$(2.13)\quad \begin{aligned} f(t,u) &= f(0,u) \\ &\quad + \alpha K \int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}s + \sigma \int_0^t \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}W_s + \int_0^t q_s \mathrm{e}^{-\alpha(t-s)}\,\mathrm{d}N_s. \end{aligned}$$

Moreover, in accordance with the properties of Brownian motion W_s and Possion process N_t, we can obtain that the risk-free interest rate process r_t is a $\mathcal{F}_t^r$-Markov process, where $\mathcal{F}_t^r = \sigma(r_s, 0 \leqslant s \leqslant t)$.

Next, we employ the three-firms model specified by equations (2.7)–(2.9) with the stochastic interest rate process (2.10) to price defaultable bonds and CDS.

3. Bond pricing under three-firms contagion model

In this section, we assume that there are three firms A, B and C, and we consider that each firm holds defaultable bonds issued by the other two firms. Moreover, the three firms have default contagion, which is characterized by the correlated default intensities (2.7)–(2.9). Because of the symmetry of default intensities, we only consider one firm's bond when pricing the three firms' bonds.

3.1. The general pricing formulas.

Definition 3.1. A defaultable claim maturing at T is the quadruple (Y, A, W, τ), where Y is an $\mathcal{F}_T$-measurable random variable, $A = (A_t)_{t\in[0,T]}$ is an $\mathcal{F}$-adapted, continuous process of finite variation with $A_0 = 0$, $W = (W_t)_{t\in[0,T]}$ is an $\mathcal{F}$-predictable process, and τ is a random time.

Definition 3.2. The dividend process $D = (D_t)_{t\in\mathbb{R}^+}$ of the above defaultable claim maturing at T equals, for every $t \in \mathbb{R}^+$,

$$D_t = Y\mathbb{1}_{\{T<\tau\}}\mathbb{1}_{[T,\infty)}(t) + \int_{(0,t\wedge T]}(1-N_u)\,\mathrm{d}A_u + \int_{(0,t\wedge T]} W_u\,\mathrm{d}N_u, \tag{3.1}$$

where Y is the promised payoff, A represents the process of promised dividends and W is the recovery process.

Definition 3.3. The ex-dividend price process S of a defaultable claim (Y, A, W, τ) equals, for every $t \in [0, T]$,

$$S_t = E_t\Big[\int_{(t,T]} \frac{B_t}{B_u}\,\mathrm{d}D_u\Big], \tag{3.2}$$

where $B_t := B(t) = \exp\big(\int_0^t r_s\,\mathrm{d}s\big)$ is the money market account, r_t is a constant default-free spot rate, and E_t represents the conditional expectation on $\mathcal{F}_t$ under the equivalent martingale measure P.

For the defaultable zero-coupon bond which pays one dollar if not default, and pays δ times the price of a default-free bond at maturity, where δ is introduced by Jarrow and Turnbull [15] and Jarrow, Lando and Turnbull [14] as 'recovery of Treasury'.

Let $v^i(t,T)$ denote the time-t defaultable zero-coupon bond price, issued by firm i $(i = A, B, C)$, $\delta^i \in [0, 1]$ is the recovery rate of the firm i. By Definition 3.2 and Definition 3.3, $v^i(t,T)$ is given by

$$v^i(t,T) = E_t\Big[\frac{B_t}{B_T}(\delta^i\mathbb{1}_{\{\tau^i\leqslant T\}} + \mathbb{1}_{\{\tau^i>T\}})\Big]. \tag{3.3}$$

In this paper, we consider the valuation of defaultable zere-coupon bonds as expressed in (3.3). To obtain the analytic expression of (3.3), we mainly compute the conditional expectation in (3.3). To do this, deducing the joint conditional distribution is necessary.

3.2. The joint conditional distribution under two-firms model. To price bond valuations of three firms, we study a two-firms contagion model first and give its joint conditional distribution function.

We assume there are two firms A and B, and their default intensities λ_t^A and λ_t^B have the following forms

$$\lambda_t^A = a_0 + ar_t + a_1\mathbb{1}_{\{\tau^B\leqslant t\}}, \tag{3.4}$$

$$\lambda_t^B = b_0 + br_t + b_1\mathbb{1}_{\{\tau^A\leqslant t\}}, \tag{3.5}$$

where $a_0 > 0$, $b_0 > 0$, $a > 0$, $b > 0$, and satisfying $a_0 + a + a_1 + a_2 > 0$, $b_0 + b + b_1 + b_2 > 0$. The risk-free interest rate satisfies the jump diffusion stochastic differential equation (2.10).

We adopt the change of measure introduced by Collins-Dufresne et al. [4] to define a firm-specific probability measure P^i $(i = A, B)$ which puts zero probability on the paths where default occurs prior to maturity T. Specifically, the change of measure is defined by

$$Z_T := \frac{\mathrm{d}P^i}{\mathrm{d}P}\Big|_{\mathcal{F}_T} = \mathbb{1}_{\{\tau^i > T\}} \exp\biggl(\int_0^T \lambda_s^i \,\mathrm{d}s\biggr), \tag{3.6}$$

where P^i is a firm-specific (firm i $(i = A, B)$) probability measure which is absolutely continuous with respect to P on the stochastic interval $[0, \tau^i)$. To perform the calculations under the measure P^i, we enlarge the filtration to $\widetilde{\mathcal{F}}^i = (\widetilde{\mathcal{F}}_t^i)_{t\geqslant 0}$ as the completion of $\mathcal{F} = (\mathcal{F}_t)_{t\geqslant 0}$ by the null sets of the probability measure P^i.

Under the probability measures $P^i(i = A, B)$, the characteristics of the Brownian motion W_t and the Possion process Y_t have not changed and they are still independent. The macrovariable r_t is not influenced by the defaults of firms A, B.

Using the change of measure and Shreve [24], we can state the following lemma:

Lemma 3.1. *Let H be $\mathcal{F}_t$-measurable, s and t be real numbers, satisfying $0 \leqslant s \leqslant t \leqslant T$. Then,*

$$E^{P^i}[H \mid \widetilde{\mathcal{F}}_s^i \vee \mathcal{F}_{T^*}^r] = \frac{1}{Z_s^i} E^P[H \cdot Z_t^i \mid \mathcal{F}_s \vee \mathcal{F}_{T^*}^r], \tag{3.7}$$

where $\widetilde{\mathcal{F}}_s^i = \mathcal{F}_s \vee \mathcal{F}_s^i$.

With this lemma and the method in [10], we can obtain the joint conditional distribution function of two firms. The theorem is as follows:

Theorem 3.1. *Let intensity processes λ_t^i $(i = A, B)$ be given by (3.4)–(3.5). Then the conditional probability distribution of τ^A and τ^B is given by*

$$\begin{aligned} &P(\tau^A > t_1, \tau^B > t_2 \mid \mathcal{F}_t \vee \mathcal{F}_{T^*}^r) \\ &\quad = \exp(-b_0(t_2 - t) - bR_{t,t_2}) \biggl[\exp(-b_1(t_2 - t_1)) \\ &\quad - \exp(-(a_0(t_2 - t_1) - aR_{t_1,t_2})) + \exp(-a_0(t_2 - t) - aR_{t,t_2}) \\ &\quad + b_1 \int_{t_1}^{t_2} (\exp(-a_0(u - t_1) - b_1(t_2 - u)) - aR_{t_1,u}) \,\mathrm{d}u\biggr], \\ &\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } t < t_1 < t_2 < T, \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
(3.9)\qquad P(\tau^A > t_1, \tau^B > t_2 \mid \mathcal{F}_t \vee \mathcal{F}^r_{T^*}) &= \exp(-a_0(t_1 - t) - aR_{t,t_1}) \\
&\times \Big[\exp(-a_1(t_1 - t_2)) - \exp(-(b_0(t_1 - t_2) - bR_{t_2,t_1})) \\
&+ \exp(-b_0(t_1 - t) - bR_{t,t_1}) \\
&+ a_1 \int_{t_2}^{t_1} (\exp(-b_0(u - t_2) - a_1(t_1 - u)) - bR_{t_2,u})\,\mathrm{d}u \Big], \\
&\qquad \text{for } t < t_2 < t_1 < T.
\end{aligned}
$$

P r o o f. Under default intensities (3.4)–(3.5), for $t < t_1 < t_2 < T$,

$$
\begin{aligned}
(3.10)\qquad & P(\tau^A > t_1, \tau^B > t_2 \mid \mathcal{F}_t \vee \mathcal{F}^r_{T^*}) \\
&= E[\mathbb{1}_{\{\tau^A > t_1, \tau^B > t_2\}} \mid \mathcal{F}_t \vee \mathcal{F}^r_{T^*}] \\
&= E^B[\mathbb{1}_{\{\tau^A > t_1\}} \exp(-b_0(t_2 - t) - bR_{t,t_2} \\
&\quad - \mathbb{1}_{\{\tau^A \leqslant t_2\}} b_1(t_2 - \tau^A)) \mid \widetilde{\mathcal{F}}^A_t \vee \mathcal{F}^r_{T^*}] \\
&= E^B[[\mathbb{1}_{\{t_1 < \tau^A \leqslant t_2\}} \cdot \exp(-b_0(t_2 - t) - bR_{t,t_2} - b_1(t_2 - \tau^A)) \\
&\quad + \mathbb{1}_{\{\tau^A > t_2\}} \cdot \exp(-b_0(t_2 - t) - bR_{t,t_2})] \mid \widetilde{\mathcal{F}}^A_t \vee \mathcal{F}^r_{T^*}] \\
&\triangleq K_1 + K_2,
\end{aligned}
$$

where E^B denotes the expectation under probability measure P^B and $F^B(\cdot \mid \widetilde{\mathcal{F}}^A_t \vee \mathcal{F}^r_{T^*})$ the conditional probability distribution of τ^A.

Conditional on $\tau^A > t$, τ^A has the following distribution

$$
P(\tau^A > t_2 \mid \widetilde{\mathcal{F}}^A_t \vee \mathcal{F}^r_{T^*}) = \exp\Big(- \int_t^{t_2} \lambda^A_s \,\mathrm{d}s \Big).
$$

Moreover, under probability measure P^B,

$$
P^B(\tau^A > t_2 \mid \widetilde{\mathcal{F}}^A_t \vee \mathcal{F}^r_{T^*}) = \exp(-a_0(t_2 - t) - aR_{t,t_2}).
$$

Thus

$$
\begin{aligned}
K_1 &= \int_{t_1}^{t_2} \exp(-b_0(t_2 - t) - bR_{t,t_2} - b_1(t_2 - u))\,\mathrm{d}F^B_{\tau^A}(u) \\
&= \exp(-b_0(t_2 - t) - bR_{t,t_2}) \Big[\exp(-b_1(t_2 - t_1)) - \exp(-(a_0(t_2 - t_1) - aR_{t_1,t_2})) \\
&\quad + b_1 \int_{t_1}^{t_2} \exp(-a_0(u - t_1) - b_1(t_2 - u) - aR_{u,t_1})\,\mathrm{d}u \Big], \\
K_2 &= \exp(-b_0(t_2 - t) - bR_{t,t_2}) \cdot P^B(\tau^A > t_2 \mid \widetilde{\mathcal{F}}^A_t \vee \mathcal{F}^r_{T^*}) \\
&= \exp(-(a_0 + b_0)(t_2 - t) - (a + b)R_{t,t_2}).
\end{aligned}
$$

Substituting K_1 and K_2 in (3.10), we have

$$
\begin{aligned}
P(\tau^A > t_1, \tau^B > t_2 \mid \mathcal{F}_t \vee \mathcal{F}^r_{T^*}) &= \exp(-b_0(t_2 - t) - bR_{t,t_2}) \\
&\quad \times \Big[\exp(-b_1(t_2 - t_1)) - \exp(-(a_0(t_2 - t_1) - aR_{t_1,t_2})) \\
&\quad + \exp(-a_0(t_2 - t) - aR_{t,t_2}) \\
&\quad + b_1 \int_{t_1}^{t_2} (\exp(-a_0(u - t_1) - b_1(t_2 - u)) - aR_{t_1,u}) \, \mathrm{d}u \Big], \\
&\qquad \text{for } t < t_1 < t_2 < T.
\end{aligned}
$$

Analogously, we can obtain formula (3.9). This completes the proof. □

3.3. Bond pricing under three-firms model. We assume there are three firms A, B and C, and consider the case that each firm holds the other two firms' defaultable bonds with the same maturity date T ($< T^*$) and face value one dollar, so that when one party defaults, the other two parties' default probability will jump. The default intensities are described in (2.7)–(2.9).

Applying equation (3.3), we know that the defaultable bond price of firm i with the recovery rate δ^i is given by

$$
v^i(t,T) = E_t\left[\frac{B_t}{B_T}\left(\delta^i + \mathbb{1}_{\{\tau^i > t\}}(1 - \delta^i)\exp\left(-\int_t^T \lambda^i_s \, \mathrm{d}s\right)\right)\right], \quad t \leqslant T. \tag{3.11}
$$

For simplification, we assume the recovery rate δ^i of firm $i (i = A, B, C)$ is 0, then equation (3.11) becomes

$$
v^i(t,T) = \mathbb{1}_{\{\tau^i > t\}} E_t\left[\exp\left(-\int_t^T (r_s + \lambda^i_s) \, \mathrm{d}s\right)\right], \quad t \leqslant T. \tag{3.12}
$$

Because of the symmetry of default intensities, we need only compute the value of one of the three firms. In the remainder of this subsection, we will derive the closed-form pricing formula of firm C. From equations (3.12) and (2.9), the time-t value $v^C(t,T)$ of the defaultable bond C maturity at T satisfies

$$
\begin{aligned}
v^C(t,T) &= \mathbb{1}_{\{\tau^C > t\}} E_t\left[\exp\left(-\int_t^T (r_s + \lambda^C_s) \, \mathrm{d}s\right)\right] \\
&= \exp(-c_0(T - t)) E_t\Big[\exp(-(1 + c)R_{t,T}) \\
&\quad \times \exp\left(-\int_t^T (c_1 \mathbb{1}_{\{\tau^A \leqslant s, \tau^B > s\}} + c_2 \mathbb{1}_{\{\tau^B \leqslant s, \tau^A > s\}} + c_3 \mathbb{1}_{\{\tau^A \leqslant s, \tau^B \leqslant s\}}) \, \mathrm{d}s\right)\Big],
\end{aligned} \tag{3.13}
$$

where $R_{t,T} := \int_t^T r_s \, \mathrm{d}s$.

By Theorem 3.1, we have obtained the joint conditional distribution $P(\tau^A > t_1, \tau^B > t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r)$ under different circumstances. To obtain the analytic form of the price of bond C $v^C(t,T)$, computing the conditional expectation in (3.13) is necessary. It's critical to compute the conditional expectation $E_t[\mathrm{e}^{-mR_{t,T}}]$. According to the result of Hao and Ye [10], we give the following lemma:

Lemma 3.2. *Assume that r_t satisfies equation* (2.10) *or* (2.12) *and $E_t[\mathrm{e}^{-mR_{t,T}}]$ for all $m \in \mathbb{R}$ is the conditional expectation with respect to $\mathcal{F}_t$. Denote $L_1(m;t,T) := E_t[\mathrm{e}^{-mR_{t,T}}]$. Then*

$$\begin{aligned}(3.14)\qquad & L_1(m;t,T) \\ &= \exp\biggl[\int_t^T \Bigl(-mf(t,u) + \frac{1}{2}m^2\sigma^2 c_T^2(u) + \mu(\mathrm{e}^{-mq_u c_T(u)} - 1)\Bigr)\,\mathrm{d}u \\ &\quad - mK(T-t) + aKc_T(t)\biggr],\end{aligned}$$

where

$$c_T(u) = \frac{1-\mathrm{e}^{-\alpha(T-u)}}{\alpha}. \tag{3.15}$$

P r o o f. See [10] and [23]. □

Furthermore, we can obtain the conditional expectations $E_{t_0}[\mathrm{e}^{-m_1R_{t_0,t_1}-m_2R_{t_1,t_2}}]$ and $E_{t_0}[\mathrm{e}^{-m_1R_{t_0,t_1}-m_2R_{t_1,t_2}-m_3R_{t_2,t_3}}]$.

Lemma 3.3. *For any $0 \leqslant t_0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant T$, denote*

$$L_2(m_1,m_2;t_0,t_1,t_2) := E_{t_0}[\mathrm{e}^{-m_1R_{t_0,t_1}-m_2R_{t_1,t_2}}]$$

and

$$L_3(m_1,m_2,m_3;t_0,t_1,t_2,t_3) := E_{t_0}[\mathrm{e}^{-m_1R_{t_0,t_1}-m_2R_{t_1,t_2}-m_3R_{t_2,t_3}}],$$

where $m_1,m_2,m_3 \in \mathbb{R}$. Then, we have

$$\begin{aligned}(3.16)\quad L_2(m_1,m_2;t_0,t_1,t_2) &= \exp\biggl[-m_1\int_{t_0}^{t_1} f(t_0,u)\,\mathrm{d}u - F(m_2,t_0,t_1,t_2)\biggr] \\ &\times \exp\biggl[-\sum_{i=1}^{2} m_iK(t_i-t_{i-1}) + m_1Kc_{t_1}(t_0) + m_2(K-r_0)d(t_1,t_2,0)\biggr] \\ &\times \exp\biggl[\mu\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i} (\exp(-q_u(m_ic_{t_i}(u)) + m_{i+1}d(t_i,t_{i+1},u)\mathbb{1}_{\{i+1\leqslant 2\}}) - 1)\,\mathrm{d}u\biggr] \\ &\times \exp\biggl[\frac{1}{2}\sigma^2\sum_{i=1}^{2}\int_{t_{i-1}}^{t_i} (m_ic_{t_i}(u) + m_{i+1}d(t_i,t_{i+1},u)\mathbb{1}_{\{i+1\leqslant 2\}})^2\,\mathrm{d}u\biggr],\end{aligned}$$

and
(3.17)

$$L_3(m_1,m_2,m_3;t_0,t_1,t_2,t_3)=\exp\Big[-m_1\int_{t_0}^{t_1} f(t_0,u)\,\mathrm{d}u-\sum_{i=2}^{3}F(m_i,t_{i-2},t_{i-1},t_i)\Big]$$

$$\times\exp\Big[-\sum_{i=1}^{3}m_iK(t_i-t_{i-1})+m_1Kc_{t_1}(t_0)+\sum_{i=2}^{3}m_i(K-r_0)d(t_{i-1},t_i,0)\Big]$$

$$\times\exp\Big[\mu\sum_{i=1}^{3}\int_{t_{i-1}}^{t_i}(\mathrm{e}^{-q_u(m_ic_{t_i}(u)+m_{i+1}d(t_i,t_{i+1},u)_{\{i+1\leqslant 3\}})}-1)\,\mathrm{d}u\Big]$$

$$\times\exp\Big[\frac{1}{2}\sigma^2\sum_{i=1}^{3}\int_{t_{i-1}}^{t_i}(m_ic_{t_i}(u)+m_{i+1}d(t_i,t_{i+1},u)\mathbb{1}_{\{i+1\leqslant 3\}})^2\,\mathrm{d}u\Big],$$

where

$$d(t_i,t_{i+1},u)=\frac{1}{\alpha}(\mathrm{e}^{-\alpha(t_i-u)}-\mathrm{e}^{-\alpha(t_{i+1}-u)}),$$

$$F(m_i,t_{i-2},t_{i-1},t_i)=m_i\bigg(\int_0^{t_{i-2}}\sigma d(t_{i-1},t_i,u)\,\mathrm{d}W_t+\int_0^{t_{i-2}}q_ud(t_{i-1},t_i,u)\,\mathrm{d}N_t\bigg).$$

Proof. See [10]. □

Based on intensity processes λ_t^i $(i=A,B,C)$ given by (2.7)–(2.9) and the results given by Theorem 3.1, Lemma 3.2 and Lemma 3.3, we will give the calculation of the price of bond C $v^C(t,T)$ on different regions according to the other two bonds' defaults.

(1) Conditional on $\tau^A>t$, $\tau^B>t$, the default intensities λ_s^A and λ_s^B $(s\geqslant t)$ are given by

$$\lambda_s^A=a_0+ar_s+a_1\mathbb{1}_{\{\tau^B\leqslant s\}},$$
$$\lambda_s^B=b_0+br_s+b_1\mathbb{1}_{\{\tau^A\leqslant s\}}.$$

(3.18)
$$\begin{aligned}v^C(t,T)&=E_t\Big[\mathbb{1}_{\{\tau^C>t\}}\exp\Big(-\int_t^T(r_s+\lambda_s^C)\,\mathrm{d}s\Big)\Big]\\&=\mathrm{e}^{-c_0(T-t)}E_t\Big[\exp\Big(-\int_t^T((c+1)r_t+c_1\mathbb{1}_{\{\tau^A\leqslant s,\tau^B>s\}}\\&\quad+c_2\mathbb{1}_{\{\tau^B\leqslant s,\tau^A>s\}}+c_3\mathbb{1}_{\{\tau^A\leqslant s,\tau^B\leqslant s\}})\,\mathrm{d}s\Big)\Big]\\&=\mathrm{e}^{-c_0(T-t)}E_t[\mathrm{e}^{-(1+c)R_{t,T}}\cdot E_t[\exp(-(c_1\mathbb{1}_{\{\tau^A\leqslant s,\tau^B>s\}}\\&\quad+c_2\mathbb{1}_{\{\tau^B\leqslant s,\tau^A>s\}}+c_3\mathbb{1}_{\{\tau^A\leqslant s,\tau^B\leqslant s\}}))\mid\widetilde{\mathcal{F}}_t^C\vee\mathcal{F}_{T^*}^r]]\\&\triangleq\mathrm{e}^{-c_0(T-t)}E_t[\mathrm{e}^{-(1+c)R_{t,T}}\cdot V_1].\end{aligned}$$

Conditional on $\tau^A > t$, $\tau^B > t$, the region of integration is then appropriately divided into five pieces: $D_1\colon\ t \leqslant \tau^A \leqslant T,\ \tau^A \leqslant \tau^B \leqslant T$; $D_2\colon\ t \leqslant \tau^B \leqslant T,\ \tau^B \leqslant \tau^A \leqslant T$; $D_3\colon\ t \leqslant \tau^A \leqslant T,\ \tau^B \geqslant T$; $D_4\colon\ t \leqslant \tau^B \leqslant T,\ \tau^A \geqslant T$; $D_5\colon\ \tau^A \geqslant T,\ \tau^B \geqslant T$.

By the above division of the integration region, equation (3.18) becomes

$$v^C(t,T) \triangleq \mathrm{e}^{-c_0(T-t)} E_t[\mathrm{e}^{-(1+c)R_{t,T}} \cdot (J_1 + J_2 + J_3 + J_4 + J_5)], \tag{3.19}$$

where

$$\begin{aligned}
J_1 &= \iint_{D_1} \exp(-c_1(t_2 - t_1) - c_3(T - t_2))\,\mathrm{d}(1 - P(\tau^A > t_1, \tau^B > t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r)),\\
J_2 &= \iint_{D_2} \exp(-c_2(t_1 - t_2) - c_3(T - t_1))\,\mathrm{d}(1 - P(\tau^A > t_1, \tau^B > t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r)),\\
J_3 &= \iint_{D_3} \exp(-c_1(t_2 - t_1)) f_t(t_1, t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r)\,\mathrm{d}t_1\,\mathrm{d}t_2,\\
J_4 &= \iint_{D_4} \exp(-c_2(t_1 - t_2)) f_t(t_1, t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r)\,\mathrm{d}t_1\,\mathrm{d}t_2,\\
J_5 &= \iint_{D_5} f_t(t_1, t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r)\,\mathrm{d}t_1\,\mathrm{d}t_2,
\end{aligned}$$

and

(3.20)

$$\begin{aligned}
J_1 &= \iint_{D_1} \exp(-c_1(t_2 - t_1) - c_3(T - t_2))\,\mathrm{d}(1 - P(\tau^A > t_1, \tau^B > t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r))\\
&= c_1 \exp(-b_0(T-t) - bR_{t,T}) \int_t^T \exp(-(c_1 + a_0)(T - t_1) - aR_{t_1,T})\,\mathrm{d}t_1\\
&\quad - \exp(-b_0(T-t) - bR_{t,T})(1 - \exp(-(c_1 + a_0)(T - t) - c_3 R_{t,T}))\\
&\quad + \Big(a_0 - \frac{a}{b}(b_0 - c_3)\Big) \int_t^T \exp(-c_3(T - t_1) - b_0(t_1 - t) - bR_{t,t_1})\,\mathrm{d}t_1\\
&\quad + (c_3 - c_1) \int_t^T \exp(-(a_0 + b_0)(t_2 - t) - (a + b)R_{t,t_2})\,\mathrm{d}t_2\\
&\quad - \frac{a}{b}(\exp(b_0(T-t) - bR_{t,T}) - \exp(-c_3(T - t)))\\
&\quad - c_1(c_3 - c_1) \int_t^T \int_t^{t_2} \exp(-c_1(t_2 - t_1) - c_3(T - t_2))\\
&\quad \times \exp(-b_0(t_2 - t) - bR_{t,t_2}) \cdot \Big[\exp(-b_1(t_2 - t_1))\\
&\quad - \exp(-a_0(t_2 - t_1) - aR_{t_1,t_2}) \cdot \exp(-a_0(t_2 - t) - aR_{t,t_2})\\
&\quad + b_1 \int_{t_1}^{t_2} \mathrm{e}^{-a_0(u - t_1) - b_1(t_2 - u) - aR_{t_1,u}}\,\mathrm{d}u\Big]\,\mathrm{d}t_1\,\mathrm{d}t_2.
\end{aligned}$$

Similarly, we can obtain

$$
\begin{aligned}
(3.21)\qquad J_2 = {} & \iint_{D_2} \exp(-c_2(t_1-t_2) \\
& - c_3(T-t_1))\,\mathrm{d}(1-P(\tau^A>t_1,\tau^B>t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r)) \\
= {} & c_2\exp(-a_0(T-t)-aR_{t,T}) \\
& \times \int_t^T \exp(-(c_2+b_0)(T-t_2)-bR_{t_2,T})\,\mathrm{d}t_2 \\
& - \exp(-a_0(T-t)-aR_{t,T})(1-\exp(-(c_2+b_0)(T-t) \\
& - c_3R_{t,T})) + \Bigl(b_0-\frac{b}{a}(a_0-c_3)\Bigr) \\
& \times \int_t^T \exp(-c_3(T-t_2)-a_0(t_2-t)-aR_{t,t_2})\,\mathrm{d}t_2 \\
& + (c_3-c_2)\int_t^T \exp(-(a_0+b_0)(t_1-t)-(a+b)R_{t,t_1})\,\mathrm{d}t_1 \\
& - \frac{a}{b}(\exp(b_0(T-t)-bR_{t,T})-\exp(-c_3(T-t))) \\
& - c_2(c_3-c_2)\int_t^T\!\!\int_t^{t_1} \exp(-c_2(t_1-t_2)-c_3(T-t_1)) \\
& \times \exp(-a_0(t_1-t)-aR_{t,t_1})\cdot\Bigl[\exp(-a_1(t_1-t_2)) \\
& - \exp(-b_0(t_1-t_2)-bR_{t_2,t_1})\cdot\exp(-b_0(t_1-t)-bR_{t,t_1}) \\
& + a_1\int_{t_2}^{t_1} \mathrm{e}^{-b_0(u-t_2)-a_1(t_1-u)-bR_{t_2,u}}\,\mathrm{d}u\Bigr]\,\mathrm{d}t_2\,\mathrm{d}t_1.
\end{aligned}
$$

$$
\begin{aligned}
(3.22)\qquad J_3 = {} & \iint_{D_3} \exp(-c_1(t_2-t_1))f_t(t_1,t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r)\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
= {} & \exp(-b_0(T-t)-bR_{t,T}) \\
& \times \Bigl[c_1\int_t^T \exp(-(a_0+c_1)(T-t_1)-aR_{t_1,T})\,\mathrm{d}t_1 \\
& - (1-\exp(-(a_0+c_1)(T-t)-aR_{t,T}))\Bigr] \\
& - c_1\int_T^\infty\!\!\int_t^T \exp(-c_1(t_2-t_1)) \\
& \times \exp(-b_0(t_2-t)-bR_{t,t_2}) \\
& \times \Bigl[\exp(-b_1(t_2-t_1))-\exp(-a_0(t_2-t_1)-aR_{t_1,t_2}) \\
& + b_1\int_{t_1}^{t_2} \exp(-a_0(u-t_1)-b_1(t_2-u))\cdot\exp(-aR_{t_1,u})\,\mathrm{d}u \\
& + \exp(-a_0(t_2-t)-aR_{t,t_2})\Bigr]\,\mathrm{d}t_1\,\mathrm{d}t_2.
\end{aligned}
$$

Similarly,

(3.23)

$$
\begin{aligned}
J_4 = & \iint_{D_4} \exp(-c_2(t_1 - t_2)) f_t(t_1, t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r) \, \mathrm{d}t_1 \, \mathrm{d}t_2 \\
= & \exp(-a_0(T-t) - aR_{t,T}) \Big[c_2 \int_t^T \exp(-(b_0 + c_2)(T - t_2) - bR_{t_2,T}) \, \mathrm{d}t_2 \\
& - (1 - \exp(-(b_0 + c_2)(T-t) - bR_{t,T})) \Big] - c_2 \int_T^\infty \int_t^T \exp(-c_2(t_1 - t_2)) \\
& \times \exp(-a_0(t_1 - t) - aR_{t,t_1}) \Big[\exp(-a_1(t_1 - t_2)) - \exp(-b_0(t_1 - t_2) - bR_{t_2,t_1}) \\
& + a_1 \int_{t_2}^{t_1} \exp(-b_0(u - t_2) - a_1(t_1 - u)) \cdot \exp(-bR_{t_2,u}) \, \mathrm{d}u \\
& + \exp(-b_0(t_1 - t) - bR_{t,t_1}) \Big] \, \mathrm{d}t_2 \, \mathrm{d}t_1.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(3.24) \qquad J_5 = & \iint_{D_5} f_t(t_1, t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r) \, \mathrm{d}t_1 \, \mathrm{d}t_2 = P(\tau^A > t, \tau^B > t \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r) \\
= & \exp(-(a_0 + b_0)(T-t) - (a+b)R_{t,T}).
\end{aligned}
$$

Hence, conditional on $\tau^A > t$, $\tau^B > t$, we can obtain the time-t bond price of firm C

$$
\begin{aligned}
(3.25) \qquad v^C(t,T) = & \mathrm{e}^{-c_0(T-t)} E_t[\mathrm{e}^{-(1+c)R_{t,T}} (J_1 + J_2 + J_3 + J_4 + J_5)] \\
\triangleq & V_1,
\end{aligned}
$$

where J_1, J_2, J_3, J_4, J_5 are given by equations (3.20)–(3.24), and $L_1(\cdot;\cdot,\cdot)$, $L_2(\cdot,\cdot;\cdot,\cdot,\cdot)$, $L_3(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot,\cdot)$ are defined as in Lemma 3.2 and 3.3.

(2) Conditional on $\tau^A \leqslant t$, $\tau^B > t$, the default intensities λ_s^A and λ_s^B ($s \geqslant t$) are given by

$$
\begin{aligned}
\lambda_s^A &= a_0 + ar_s + a_1 \mathbb{1}_{\{\tau^B \leqslant s\}}, \\
\lambda_s^B &= b_0 + br_s + b_1.
\end{aligned}
$$

The conditional survival function of τ^B is given by

$$
P(\tau^B > t_2 \mid \widetilde{\mathcal{F}}_t^C \vee \mathcal{F}_{T^*}^r) = \exp(-(b_0 + b_1)(t_2 - t) - bR_{t,t_2}).
$$

Then,

$$E_t\left[\mathbb{1}_{\{\tau^C>t\}}\exp\left(-\int_t^T(r_s+\lambda_s^C)\,\mathrm{d}s\right)\right]=\mathrm{e}^{-c_0(T-t)}E_t \tag{3.26}$$
$$\times\left[\mathrm{e}^{-(1+c)R_{t,T}}\exp\left(-\int_t^T(c_1\mathbb{1}_{\{\tau^A\leqslant s,\tau^B>s\}}+c_3\mathbb{1}_{\{\tau^A\leqslant s,\tau^B\leqslant s\}})\,\mathrm{d}s\right)\right]$$
$$=\mathrm{e}^{-c_0(T-t)}E_t[\mathrm{e}^{-(1+c)R_{t,T}}\cdot E_t[\exp(-(c_1\mathbb{1}_{\{\tau^A\leqslant s,\tau^B>s\}}$$
$$+c_3\mathbb{1}_{\{\tau^A\leqslant s,\tau^B\leqslant s\}}))\mid\widetilde{\mathcal{F}}_t^C\vee\mathcal{F}_{T^*}^r]]$$
$$\triangleq\mathrm{e}^{-c_0(T-t)}E_t[\exp(-(1+c)R_{t,T})\cdot J_6],$$

where

$$J_6=E_t[\exp(-(c_1\mathbb{1}_{\{\tau^A\leqslant s,\tau^B>s\}}+c_3\mathbb{1}_{\{\tau^A\leqslant s,\tau^B\leqslant s\}}))\mid\widetilde{\mathcal{F}}_t^C\vee\mathcal{F}_{T^*}^r] \tag{3.27}$$
$$=E_t[\mathbb{1}_{\{t<\tau^B\leqslant T\}}\exp(-c_1(t_2-t)-c_3(T-t_2))$$
$$+\mathbb{1}_{\{\tau^B>T\}}\exp(-c_1(T-t))\mid\widetilde{\mathcal{F}}_t^C\vee\mathcal{F}_{T^*}^r]$$
$$=\int_t^T\exp(-c_1(t_2-t)-c_3(T-t_2))\,\mathrm{d}(1-P(\tau^B>t_2\mid\widetilde{\mathcal{F}}_t^C\vee\mathcal{F}_{T^*}^r))$$
$$+\int_T^\infty\mathrm{e}^{-c_1(T-t)}\,\mathrm{d}(1-P(\tau^B>t_2\mid\widetilde{\mathcal{F}}_t^C\vee\mathcal{F}_{T^*}^r))$$
$$=(c_3-c_1)\int_t^T\exp(-(b_0+b_1+c_1)(t_2-t)-c_3(T-t_2)-bR_{t,t_2})\,\mathrm{d}t_2$$
$$-\exp(-(b_0+b_1+c_1)(T-t)-bR_{t,T}).$$

Substituting (3.27) to (3.26), we can obtain

$$v^C(t,T)=\mathrm{e}^{-c_0(T-t)} \tag{3.28}$$
$$\times E_t\left[(c_3-c_1)\int_t^T\exp(-(b_0+b_1+c_1)(t_2-t)-c_3(T-t_2))\right.$$
$$\times\exp(-(1+b+c)R_{t,t_2}-(1+c)R_{t_2,T})\,\mathrm{d}t_2$$
$$\left.-\exp(-(b_0+b_1+c_1)(T-t)-(1+b+c)R_{t,T})\right]\triangleq V_2.$$

(3) Conditional on $\tau^A>t$, $\tau^B\leqslant t$, the default intensities of λ_s^A and λ_s^B ($s\geqslant t$) become

$$\lambda_t^A=a_0+ar_s+a_1,$$
$$\lambda_t^B=b_0+br_s+b_1\mathbb{1}_{\{\tau^A\leqslant s\}}.$$

Hence, as in the case of (2), we can derive the time-t bond price of firm C conditional on $\tau^A > t$, $\tau^B \leqslant t$,

$$(3.29)\qquad \begin{aligned} v^C(t,T) = {} & \mathrm{e}^{-c_0(T-t)} \\ & \times E_t\biggl[(c_3-c_2)\int_t^T \exp(-(a_0+a_1+c_2)(t_1-t)-c_3(T-t_1)) \\ & \times \exp(-(1+a+c)R_{t,t_1}-(1+c)R_{t1,T})\,\mathrm{d}t_2 \\ & - \exp(-(a_0+a_1+c_2)(T-t)-(1+a+c)R_{t,T})\biggr] \triangleq V_3. \end{aligned}$$

(4) Conditional on $\tau^A \leqslant t$, $\tau^B \leqslant t$, the time-t bond price of firm C is given by

$$(3.30)\qquad \begin{aligned} v^C(t,T) &= E_t\biggl[\mathbb{1}_{\{\tau^C>t\}}\exp\Bigl(-\int_t^T (r_s+\lambda_s^C)\,\mathrm{d}s\Bigr)\biggr] \\ &= \mathrm{e}^{-(c_0+c_3)(T-t)}E_t[\mathrm{e}^{-(1+c)R_{t,T}}] \triangleq V_4. \end{aligned}$$

By now, we can give the complete expression of the price of bond C by the following theorem:

Theorem 3.2. *Let default intensity processes λ_t^i $(i=A,B,C)$ be given by* (2.7)–(2.9). *Then conditional on $\tau^C > t$, the price $v^C(t,T)$ of the defaultable bond C is given by*

$$(3.31)\qquad \begin{aligned} v^C(t,T) = {} & \mathbb{1}_{\{\tau^A>t,\tau^B>t\}}V_1 + \mathbb{1}_{\{\tau^A\leqslant t,\tau^B>t\}}V_2 \\ & + \mathbb{1}_{\{\tau^A>t,\tau^B\leqslant t\}}V_3 + \mathbb{1}_{\{\tau^A\leqslant t,\tau^B\leqslant t\}}V_4, \end{aligned}$$

where V_1, V_2, V_3 and V_4 are given by equations (3.25), (3.28), (3.29) *and* (3.30), *respectively.*

Remark 3.1. In the pricing of defaultable bonds, the three firms contagion model can be extended to the case of n firms. For the case that the n firms are homogenous, the derivation of joint density function can be found in [28].

4. CDS valuation under three-firms model with jump-diffusion stochastic interest rate

Based on the reduced form approach with correlated market and credit risks, the closed form valuation formula for the swap rate of a CDS is obtained in [16]. Jarrow and Yu [17] consider the impact of counterparty risk on the pricing of a CDS, and they assume an inter-dependent default structure that avoids "looping default" by involving Primary-Secondary framework and simplifies the payoff structure. Hull and White [13] apply the credit index model for valuing CDS with counterparty risk. Kim and Kim [18] conclude that if the default correlation between the counterparty and reference bond is ignored, the pricing error in a CDS can be quite substantial. A generalized affine model to price CDS under default correlations and counterparty risk is developed in [3]. Yu [27] uses the "total hazard" approach to construct the default process and obtains an analytic expression of the joint distribution of default times in his two-firms and three-firms contagion models. Leung and Kwok [20] use the "change of measure" approach introduced by Collins-Dufresne et al. [4] to price the CDS in two-firms model and three-firms contagion model and obtain the closed form formulas. Bai, Hu and Ye [2] put forward a hyperbolic attenuation default contagion model, and obtain the analytic expression of CDS. An interacting term is considered in [25].

Based on the above results and methods, we give the pricing of CDS on the model (2.7)–(2.9) with the jump diffusion interest rate risk (2.10).

As discussed in Section 1, we need to consider the credit risk from all three parties and the jump diffusion risk when pricing the swap rate of CDS during this period (from 0 to T, where T is the maturity of CDS). In the contract, we assume that each party is obligated to pay until its own default regardless of whether the other party has defaulted or not. Let the swap rate needed to fully insure one dollar of reference asset from time 0 to T be denoted by s. For simplification, we assume the relevant recovery rates are zero.

To price CDS, it's necessary to compute the joint density function $f(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*})$ of $\tau = (\tau^A, \tau^B, \tau^C)$. By the method of "total hazard construction" [25], $f(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*})$ is given by the following lemma:

Lemma 4.1. *Assume that λ^i_t $(i = A, B, C)$ are given by the model* (2.7)–(2.9) *and r_t is given by* (2.10). *Then the joint conditional density function of $\tau = (\tau^A, \tau^B, \tau^C)$ is given by*

$$f(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*}) = \begin{cases} f_1(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*}), & t_1 \leqslant t_2 \leqslant t_3 \leqslant T, \\ f_2(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*}), & t_1 \leqslant t_3 \leqslant t_2 \leqslant T, \\ f_3(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*}), & t_2 \leqslant t_1 \leqslant t_3 \leqslant T, \\ f_4(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*}), & t_2 \leqslant t_3 \leqslant t_1 \leqslant T, \\ f_5(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*}), & t_3 \leqslant t_1 \leqslant t_2 \leqslant T, \\ f_6(t_1, t_2, t_3 \mid \mathcal{F}^r_{T^*}), & t_3 \leqslant t_2 \leqslant t_1 \leqslant T \end{cases} \tag{4.1}$$

where

$$
\begin{aligned}
f_1(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*}) &= (a_0 + ar_{t_1})(b_0 + b_1 + br_{t_2})(c_0 + c_3 + cr_{t_3})\\
&\quad \times \exp(-(aR_{0,t_1} + bR_{0,t_2} + cR_{0,t_3}))\\
&\quad \times \exp(-(a_0 - b_1 - c_1)t_1 - (b_0 + b_1 + c_1 - c_3)t_2 - (c_0 + c_3)t_3),\\
f_2(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*}) &= (a_0 + ar_{t_1})(c_0 + c_1 + cr_{t_3})(b_0 + b_3 + br_{t_2})\\
&\quad \times \exp(-(aR_{0,t_1} + bR_{0,t_2} + cR_{0,t_3}))\\
&\quad \times \exp(-(a_0 - b_1 - c_1)t_1 - (b_0 + b_3)t_2 - (c_0 + c_1 + b_1 - b_3)t_3),\\
f_3(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*}) &= (b_0 + br_{t_2})(a_0 + a_1 + ar_{t_1})(c_0 + c_3 + cr_{t_3})\\
&\quad \times \exp(-(aR_{0,t_1} + bR_{0,t_2} + cR_{0,t_3}))\\
&\quad \times \exp(-(a_0 + a_1 - c_1)t_1 - (b_0 - a_1 + c_1 - c_3)t_2 - (c_0 + c_3)t_3),\\
f_4(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*}) &= (b_0 + br_{t_2})(c_0 + c_2 + cr_{t_3})(a_0 + a_3 + ar_{t_1})\\
&\quad \times \exp(-(aR_{0,t_1} + bR_{0,t_2} + cR_{0,t_3}))\\
&\quad \times \exp(-(a_0 + a_3)t_1 - (b_0 - c_2 - a_1)t_2 - (c_0 + c_2 + a_1 - a_3)t_3),\\
f_5(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*}) &= (c_0 + cr_{t_3})(a_0 + a_2 + ar_{t_1})(b_0 + b_3 + br_{t_2})\\
&\quad \times \exp(-(aR_{0,t_1} + bR_{0,t_2} + cR_{0,t_3}))\\
&\quad \times \exp(-(a_0 + a_2 + b_2 - b_3)t_1 - (b_0 + b_3)t_2 - (c_0 - a_2 - b_2)t_3)),\\
f_6(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*}) &= (c_0 + cr_{t_3})(a_0 + a_3 + ar_{t_1})(b_0 + b_2 + br_{t_2})\\
&\quad \times \exp(-(aR_{0,t_1} + bR_{0,t_2} + cR_{0,t_3}))\\
&\quad \times \exp(-(a_0 + a_3)t_1 - (b_0 + b_2 + a_2 - a_3)t_2 - (c_0 - a_2 - b_2)t_3).
\end{aligned}
$$

P r o o f. The proof is similar to the construction in [25], we omit it here. □

To price the valuation of CDS, we need only discuss the cash flows of the payment leg and the contingent leg.

For the payment leg, party A pays the swap premium to party B until one of the three parties defaults. Thus, the market value of the payment leg at time 0 is

$$
E\biggl[s\int_0^T \exp\biggl(-\int_0^s r_u\,\mathrm{d}u\biggr)\mathbb{1}_{\{\tau^A\wedge\tau^B\wedge\tau^C>s\}}\,\mathrm{d}s\biggr]. \tag{4.2}
$$

For the contingent leg, if party C defaults before or at time T and parties A and B haven't defaulted before C's default, then B will pay for the loss of A at time τ^C

immediately. Thus, the market value of the contingent leg at time 0 is

$$E\Big[\exp\Big(-\int_0^{\tau^C} r_u\,\mathrm{d}u\Big)\mathbb{1}_{\{\tau^A>\tau^C,\tau^B>\tau^C,\tau^C\leqslant T\}}\Big]. \tag{4.3}$$

Theorem 4.1. *Let default intensities* λ_t^i $(i=A,B,C)$ *and the risk-free interest rate* r_t *be given by* (2.7)–(2.9) *and* (2.10). *Then the* CDS *swap rate* s *under the contagion model is given by the ratio of expressions* (4.2) *and* (4.3).

P r o o f. According to the arbitrage-free pricing principle, using (4.2) and (4.3), we have the expression of the swap rate s

$$s=\frac{E\big[\exp(-\int_0^{\tau^C} r_u\,\mathrm{d}u)\mathbb{1}_{\{\tau^A>\tau^C,\tau^B>\tau^C,\tau^C\leqslant T\}}\big]}{E\big[\int_0^T\exp(-\int_0^s r_u\,\mathrm{d}u)\mathbb{1}_{\{\tau^A\wedge\tau^B\wedge\tau^C>s\}}\,\mathrm{d}s\big]} \tag{4.4}$$

First, we calculate the denominator in expression (4.4): by Fubini's theorem,

$$\begin{aligned}
&E\Big[\int_0^T\exp\Big(-\int_0^t r_u\,\mathrm{d}u\Big)\mathbb{1}_{\{\tau^A\wedge\tau^B\wedge\tau^C>s\}}\,\mathrm{d}t\Big] \qquad (4.5)\\
&=E\Big[\int_0^T E\Big[\exp\Big(-\int_0^t r_u\,\mathrm{d}u\Big)\mathbb{1}_{\{\tau^A\wedge\tau^B\wedge\tau^C>t\}}\mid\mathcal{F}^r_{T^*}\Big]\,\mathrm{d}t\Big]\\
&=E\Big[\int_0^T \exp(-R_{0,t})E^C\Big[\exp\Big(-\int_0^t \lambda_u^C\,\mathrm{d}u\Big)\mathbb{1}_{\{\tau^A\wedge\tau^B>t\}}\mid\mathcal{F}^r_{T^*}\Big]\,\mathrm{d}s\Big]\\
&=E\Big[\int_0^T \exp(-R_{0,t})E^C\Big[\exp\Big(-\int_0^t (c_0+cr_u)\,\mathrm{d}u\Big)\mathbb{1}_{\{\tau^A\wedge\tau^B>t\}}\mid\mathcal{F}^r_{T^*}\Big]\,\mathrm{d}t\Big]\\
&=E\Big[\int_0^T \exp(-c_0t-(1+c)R_{0,t})E^C[\mathbb{1}_{\{\tau^A\wedge\tau^B>t\}}\mid\mathcal{F}^r_{T^*}]\,\mathrm{d}t\Big]\\
&=E\Big[\int_0^T \exp(-(a_0+b_0+c_0)t-(1+a+b+c)R_{0,t})\,\mathrm{d}t\Big]\\
&=\int_0^T\exp(-(a_0+b_0+c_0)t)\cdot L_1(1+a+b+c;0,t)\,\mathrm{d}t,
\end{aligned}$$

where $L_1(\cdot;\cdot,\cdot)$ is given by (3.14).

Next, we compute the numerator in expression (4.4),

$$\begin{aligned}
&E\Big[\exp\Big(-\int_0^{\tau^C} r_u\,\mathrm{d}u\Big)\mathbb{1}_{\{\tau^A>\tau^C,\tau^B>\tau^C,\tau^C\leqslant T\}}\Big] \qquad (4.6)\\
&\quad=E\Big[E\Big[\exp\Big(-\int_0^{\tau^C} r_u\,\mathrm{d}u\Big)\mathbb{1}_{\{\tau^A>\tau^C,\tau^B>\tau^C,\tau^C\leqslant T\}}\mid\mathcal{F}^r_{T^*}\Big]\Big]
\end{aligned}$$

$$
\begin{aligned}
&= E\Big[\int_0^T\int_{t_3}^\infty\int_{t_3}^\infty \exp\Big(-\int_0^{t_3} r_u\,\mathrm{d}u\Big) f(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*})\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\Big] \\
&= E\Big[\int_0^T\int_{t_3}^\infty\int_{t_3}^{t_2} \exp\Big(-\int_0^{t_3} r_u\,\mathrm{d}u\Big) f_5(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*})\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\Big] \\
&\quad + E\Big[\int_0^T\int_{t_3}^\infty\int_{t_2}^\infty \exp\Big(-\int_0^{t_3} r_u\,\mathrm{d}u\Big) f_6(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*})\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\Big] \\
&\triangleq J_7 + J_8,
\end{aligned}
$$

where

$$
\begin{aligned}
(4.7)\quad J_7 &= E\Big[\int_0^T\int_{t_3}^\infty\int_{t_3}^{t_2} \exp(-R_{0,t_3}) f_5(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*})\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\Big] \\
&= E\Big[\int_0^T\int_{t_3}^\infty\int_{t_3}^{t_2} (c_0 + c r_{t_3})(b_0 + b_3 + b r_{t_2}) \\
&\quad \times \exp(-(b_0+b_3)t_2 - (c_0 - a_2 - b_2)t_3) \\
&\quad \times [(a_0 + a_2 + b_2 - b_3 + a r_{t_1}) - (b_2 - b_3)] L_1(1+c; 0, t_3) \\
&\quad \times \exp(-(a_0 + a_2 + b_2 - b_3)t_1 - a R_{0,t_1})\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\Big] \\
&= \frac{a+2b}{a+b}\Big(c_0 - \frac{c(a_0+b_0+c_0)}{1+a+b+c}\Big) \\
&\quad \times \int_0^T \exp(-(a_0+b_0+c_0)t_3) L_1(1+a+b+c; 0, t_3)\,\mathrm{d}t_3 \\
&\quad - \frac{a(b_0+b_2) - b(a_0+a_2)}{a+b}\Big[\int_0^T \frac{c}{1+c}\Big[\exp(-(a_0+a_2+b_0+b_2)t_2) \\
&\quad \times L_1(a+b; 0, t_2) - \exp(-(a_0+b_0+c_0)t_2)\cdot L_1(1+a+b+c; 0, t_2) \\
&\quad - (c_0 - a_2 - b_2)\int_0^{t_2} \exp(-(a_0+a_2+b_0+b_2)t_2 - (c_0-a_2-b_2)t_3) \\
&\quad \times L_2(1+a+b+c, a+b; 0, t_3, t_2)\,\mathrm{d}t_3\Big]\,\mathrm{d}t_2 \\
&\quad + \int_T^\infty \frac{c}{1+c}\Big[\exp(-(a_0+a_2+b_0+b_2)t_2)\cdot L_1(a+b; 0, t_2) \\
&\quad - \exp(-(a_0+a_2+b_0+b_2)t_2 - (c_0-a_2-b_2)T) \\
&\quad \times L_2(1+a+b+c, a+b; 0, T, t_2) \\
&\quad - (c_0 - a_2 - b_2)\int_0^T \exp(-(a_0+a_2+b_0+b_2)t_2 - (c_0-a_2-b_2)t_3) \\
&\quad \times L_2(1+a+b+c, a+b; 0, t_3, t_2)\,\mathrm{d}t_3\Big]\,\mathrm{d}t_2\Big],
\end{aligned}
$$

and

$$\begin{aligned}
J_8 &= E\Big[\int_0^T\int_{t_3}^\infty\int_{t_2}^\infty \exp\Big(-\int_0^{t_3} r_u\,\mathrm{d}u\Big) f_6(t_1,t_2,t_3 \mid \mathcal{F}^r_{T^*})\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\Big] \\
&= E\Big[\int_0^T\int_{t_3}^\infty\int_{t_2}^\infty (c_0+cr_{t_3})(a_0+a_3+ar_{t_1})(b_0+b_2+br_{t_2}) \\
&\quad \times \exp(-(a_0+a_3)t_1-(b_0+b_2+a_2-a_3)t_2-(c_0-a_2-b_2)t_3) \\
&\quad \times \exp(-(aR_{0,t_1}+bR_{0,t_2}+(1+c)R_{0,t_3}))\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3 \\
&= E\Big[-\int_0^T\int_{t_3}^\infty (c_0+cr_{t_3})(b_0+b_2+br_{t_2}) \\
&\quad \times \exp(-(a_0+a_2+b_0+b_2)t_2-(a+b)R_{0,t_2}) \\
&\quad \times \exp(-(c_0-a_2-b_2)t_3-(1+c)R_{0,t_3})\,\mathrm{d}t_2\,\mathrm{d}t_3\Big].
\end{aligned}$$

Further, we can obtain that

(4.8)

$$\begin{aligned}
J_8 = &-c_0(b_0+b_2)\int_0^T\int_{t_3}^\infty \exp(-(a_0+a_2+b_0+b_2)t_2-(c_0-a_2-b_2)t_3) \\
&\times L_2(1+a+b+c,a+b;0,t_3,t_2)\,\mathrm{d}t_2\,\mathrm{d}t_3 \\
&-\frac{c(b_0+b_2)}{1+c}\Big[(c_0-a_2-b_2)\int_0^T\int_{t_3}^\infty \exp(-(c_0-a_2-b_2)t_3 \\
&-(a_0+a_2+b_0+b_2)t_2)\cdot L_2(1+a+b+c,a+b;0,t_3,t_2)\,\mathrm{d}t_2\,\mathrm{d}t_3 \\
&+\int_0^T \exp(-(a_0+b_0+c_0)t_2)\cdot L_1(1+a+b+c;0,t_2)\,\mathrm{d}t_2 \\
&-\int_0^T \exp(-(a_0+a_2+b_0+b_2)t_2)\cdot L_1(a+b;0,t_2)\,\mathrm{d}t_2 \\
&+\int_T^\infty \exp(-(c_0-a_2-b_2)T-(a_0+a_2+b_0+b_2)t_2) \\
&\times L_2(1+a+b+c,a+b;0,T,t_2)\,\mathrm{d}t_2 \\
&-\int_T^\infty \exp(-(a_0+a_2+b_0+b_2)t_2)\cdot L_1(a+b;0,t_2)\,\mathrm{d}t_2\Big] \\
&+\frac{b(a_0+b_0)}{a+b}\int_0^T\int_{t_3}^\infty \exp(-(a_0+a_2+b_0+b_2)t_2)\cdot L_1(a+b;0,t_2)\,\mathrm{d}t_2\,\mathrm{d}t_3 \\
&-\frac{b}{a+b}\int_0^T \exp(-(a_0+a_2+b_0+b_2)t_3)\cdot L_1(a+b;0,t_3)\,\mathrm{d}t_3,
\end{aligned}$$

where $L_1(\cdot;\cdot,\cdot)$ and $L_2(\cdot,\cdot;\cdot,\cdot,\cdot)$ are given by (3.14) and (3.16), respectively.

Combining equations (4.4)–(4.8), we see that s has the following expression

$$s = \frac{J_7 + J_8}{\int_0^T \exp(-(a_0 + b_0 + c_0)t) \cdot L_1(1 + a + b + c; 0, t)\,\mathrm{d}t}, \tag{4.9}$$

where J_7 and J_8 are given by (4.7) and (4.8), respectively. The proof is complete. □

Remark 4.1. From equation (4.9), we can see that the default of the three parties and the interest risk all have impact on the swap rate s. The contagion effect of the reference asset and the protection buyer (or seller) on the protection seller (or buyer) has effect on the swap rate s. The contagion effect of the protection buyer and seller on the reference asset has effect on the swap rate s. This shows that when pricing CDS in "loop-default" models, without loss of generality, we can assume that the reference asset is the primary firm and the protection buyer and the seller are secondary firms.

5. Conclusion

In this paper, we mainly study a three-firms contagion model with an interaction term and the stochastic interest rate jump diffusion risk, and obtain the analytical expressions of defaultable bonds and CDS. From these expressions, we claim that the default risk of three parties and the default-free interest rate risk have effect on the valuation of defaultable bonds and CDS. Also, the contagion effect of the two parties' simultaneous default on the third party is not ignorable. Therefore, the contagion model in our paper is more realistic. Since there are only three parties involved in CDS, studying n-firms ($n > 3$) contagion models is meaningless for CDS valuation. However, some other credit derivatives such as basket swaps and CDO contain more parties. Then the default correlation for n-firms ($n > 3$) should be considered. But this is a difficult problem. Some special case has been discussed in [28].

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