Weilin Zou; Fengquan Li; Boqiang Lv

On a nonlocal problem for a confined plasma in a Tokamak

Applications of Mathematics, Vol. 58 (2013), No. 6, 609–642

Persistent URL: `http://dml.cz/dmlcz/143501`

ON A NONLOCAL PROBLEM FOR A CONFINED PLASMA IN A TOKAMAK

Weilin Zou, Nanchang, Fengquan Li, Dalian, Boqiang Lv, Nanchang

(Received January 9, 2012)

Abstract. The paper deals with a nonlocal problem related to the equilibrium of a confined plasma in a Tokamak machine. This problem involves terms $u'_*(|u > u(x)|)$ and $|u > u(x)|$, which are neither local, nor continuous, nor monotone. By using the Galerkin approximate method and establishing some properties of the decreasing rearrangement, we prove the existence of solutions to such problem.

Keywords: nonlinear elliptic equation; relative rearrangement; Tokamak; decreasing rearrangement; plasma physics

MSC 2010: 35D99, 35M10, 35Q35

1. Introduction and statement of the main results

In this paper, we are mainly interested in the resolution of a class of nonlocal problems governing the equilibrium of a plasma in a Tokamak (a toroidal machine). For a detailed presentation of this model, we refer the reader to [3], [23], [33] and the references therein. The configuration is assumed to be axi-symmetric (for example the cylindrical machine), thus the problem can be reduced to a two-dimensional one in the meridian section of the torus. From the Maxwell equations and the magneto-hydrodynamic theory of equilibrium in the plasma, one can deduce that the flux function u satisfies the problem

$$\text{(P)}\qquad \begin{cases} -\Delta u = f(x,u) & \text{in } \{u > 0\},\\ -\Delta u = 0 & \text{in } \{u \leqslant 0\},\\ u = \gamma \text{ (a negative constant to be determined)} & \text{on } \partial\Omega,\\ -\int_{\partial\Omega} \partial u/\partial n = I \text{ (a given positive constant)}, \end{cases}$$

This work is supported by Natural Science Foundation of Jiangxi (No. 20132BAB211006), NSFC (No. 10401009) and NCET (No. 060275) of China.

where Ω (representing the cross section of the Tokamak) is a bounded, open and connected subset of $\mathbb{R}^2$ with regular boundary $\partial\Omega$ with the outward unit normal n. The set $\Omega_p := \{u > 0\}$ is the region occupied by the plasma, the set $\Omega_v := \{u \leqslant 0\}$ is the vacuum region.

The term $f(x,u)$ represents the derivative $\mathrm{d}p/\mathrm{d}u$ where p is the pressure term. The exact expression of f cannot be obtained from the MHD system and some constitutive law must be assumed. A simple model is proposed that f depends on u in a local way (the typical example is $f(u) = \lambda u_+$). This typical model was considered by several authors to study the equilibrium states of a confined plasma in Tokamak devices (see [2], [33], [35]) and in Stellarator devices (see [7], [8], [9], [25]).

Assuming that the fluid is adiabatic, a more sophisticated model is considered by H. Grad in [14] (see also [23], [34] and [17]): f depending on u in a nonlocal way, i.e. f depends on x, u, $|u > u(x)|$, $u'_*(|u > u(x)|)$ or even the term $u''_*(|u > u(x)|)$, where u_* denotes the decreasing rearrangement of u (see Section 2 below) and $u'_*(s) = \mathrm{d}u_*/\mathrm{d}s$.

When f depends on the nonlocal term $|u > u(x)|$, this model has also been studied in literature (see [13], [18], [19], [21], [22], [26], [29]). Rakotoson has studied a Grad-Shafranov problem in [28] in the case of $f(x,u) = k(|u > u(x)|)u'_*(|u > u(x)|)$. Ferone et al. [10] (see also [11]) have considered the case of $f(x,u) = G(x,u,|u > u(x)|, k(|u > u(x)|)u'_*(|u > u(x)|))$, satisfying the growth condition $\mu_0 s_+ + \delta' \leqslant G(x,s,t,r) \leqslant \kappa_1|s| + \kappa_2$, where μ_0, δ', κ_1, κ_2 are positive constants and the function k is defined by

$$k(s) = \min\{s^{1/2}, (|\Omega| - s)^{1/2}\}. \tag{1.1}$$

Concerning the case $f(x,u) = g(u, |u > u(x)|, u'_*(|u > u(x)|))$ in a Stellarator model, the existence of solutions has been studied recently in [37].

In this paper, we consider problem (P) with the general pressure law

$$f(x,u) = \lambda\varphi(u_+)|k(|u > u(x)|)u'_*(|u > u(x)|)|^q + g(x), \tag{1.2}$$

where $\varphi\colon [0,\infty] \to [0,\infty]$ is a continuous function, q is a positive constant with $1 \leqslant q < 2$ and $0 \leqslant g \in L^r(\Omega)$ with $r > 1$. As in [6] (see also [3], [33], and [28]), $\lambda > 0$ is the parameter which represents the ratio between the particle pressure and the magnetic pressure.

Clearly, under this pressure law, problem (P) is equivalent to the problem

$$(\mathscr{P})\quad \begin{cases} -\Delta u = \lambda\varphi(u_+)|k(|u > u(x)|)u'_*(|u > u(x)|)|^q\chi_{\{u>0\}} + g(x)\chi_{\{u>0\}} & \text{in } \Omega,\\ u = \gamma \text{ (a negative constant to be determined)} & \text{on } \partial\Omega,\\ -\int_{\partial\Omega} \partial u/\partial n = I \text{ (a given positive constant).} \end{cases}$$

Our main goal in this paper is to study the existence of solutions to problem $(\mathscr{P})$. The main difficulties lie in the boundary conditions, and the facts that the operator is in general not coercive and the nonlinearity g is only known to be continuous on $\mathcal{V} = \{v \in H^1(\Omega)\colon |\{x\colon \nabla v(x) = 0\}| = 0\}$. To overcome these difficulties, we will introduce a truncated problems $(\mathscr{P}_h)$ which will be approximated by a family of problems $(\mathscr{P}_{h\varepsilon})$, and solve the problem $(\mathscr{P}_{h\varepsilon})$ by means of the Galerkin method and a topological degree theory. Finally, thanks to the L^∞ estimates on u_{h+} (see Theorem 4.3), we prove that $u = u_h$ is a solution of problem $(\mathscr{P})$ if h is large enough. Thus, as in [21] (see also [29]), our main results are stated as follows.

Theorem 1.1. *Let $\alpha = \min\{2/q, r\}$. Suppose that $\int_\Omega g(x)\,\mathrm{d}x > I$ and*

$$\frac{1}{4\pi}\int_0^{|\Omega|}\frac{1}{\theta}\int_0^{\theta}\Big(g_*(s) + \frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s\,\mathrm{d}\theta < \sup_{s>0} W(s), \tag{1.3}$$

where W is defined as

$$W(s) = \int_0^s \exp\Big(-\frac{q}{8\pi}\int_\theta^s \varphi^{2/q}(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}\theta. \tag{1.4}$$

Assume that (1.2) *holds and φ is a monotone increasing function, then problem $(\mathscr{P})$ admits a solution $u \in H^1(\Omega)\cap W^{2,\alpha}(\Omega)$ with $u_+ \in L^\infty(\Omega)$ in the following sense:*

(1) $-\Delta u = \hat{f}\chi_{\{u>0\}} + g(x)\chi_{\{u>0\}}$, *where $\hat{f} \in L^\alpha(\Omega)$ is such that*

$$\hat{f}(x) \in \lambda\varphi(u_+)|k(\beta(u(x)))u_*'(\beta(u(x)))|^q,$$

where $\beta(u(x)) = [|u > u(x)|, |u \geqslant u(x)|]$.

(2) $u_{|\partial\Omega} = \gamma < 0$.

(3) $-\int_{\partial\Omega}\partial u/\partial n = I$.

Remark 1.1. We point out that $\int_\Omega g(x)\,\mathrm{d}x > I$ is a sufficient condition for the existence of a free boundary $u_{|\partial\Omega} < 0$ and $u_+ \not\equiv 0$. Indeed, we have

$$-\int_{\partial\Omega}\frac{\partial u}{\partial n} = \int_{\{u>0\}}\hat{f} + g(x)\,\mathrm{d}x = I > 0,$$

thus $u_+ \not\equiv 0$. Moreover, arguing as in (4.40), we obtain $u_{|\partial\Omega} < 0$. The existence of a free boundary to problem $(\mathscr{P})$ (or (P)) is physically expected, since the plasma can not touch the vacuum vessel $\partial\Omega$ in this case.

R e m a r k 1.2. We observe that the above function u does not satisfy the standard notion of solutions to problem $(\mathscr{P})$, since the term $\lambda\varphi(u_+)|k(|u > u(x)|)u'_*(|u > u(x)|)|^q$ does not appear in the equation of problem $(\mathscr{P})$, but is replaced by $\hat{f}$. This is due to the fact that the nonlinearity is only known to be continuous on $\mathcal{V}$. Under some additional assumptions on g, we prove that $|\{x\colon\ u(x) > 0 \text{ and } \nabla u(x) = 0\}| = 0$ and thus u is a solution of problem $(\mathscr{P})$ in the standard sense. More precisely, we have

Theorem 1.2. *Assume that $g(x) > 0$ a.e. in Ω. If the assumptions of Theorem* 1.1 *hold, then there exists a solution $u \in H^1(\Omega) \cap W^{2,\alpha}(\Omega)$ with $u_+ \in L^\infty(\Omega)$ to problem $(\mathscr{P})$ in the standard sense.*

R e m a r k 1.3. Condition (1.3) is important to obtain the L^∞ estimates on the function u_{h+} (see Theorem 4.3) and then to get the existence of the solution u. If $\varphi \in L^{2/q}[0,\infty)$, then we can also obtain L^∞ estimates on u_{h+} and prove the existence of solutions to problem $(\mathscr{P})$ without condition (1.3). Moreover, the assumption that φ is monotone increasing is removed. This result is stated as follows.

Theorem 1.3. *Let q be a positive constant with $1 \leqslant q < 2$ and $0 \leqslant g \in L^r(\Omega)$. Assume that $\int_\Omega g(x)\,\mathrm{d}x > I$ and $\varphi\colon\ [0,\infty] \to [0,\infty]$ is a continuous function with $\varphi \in L^{2/q}[0,\infty)$. Then problem $(\mathscr{P})$ admits a solution $u \in H^1(\Omega) \cap W^{2,\alpha}(\Omega)$ with $u_+ \in L^\infty(\Omega)$ in sense of Theorem* 1.1. *Moreover, if $g(x) > 0$ a.e. in Ω, then u is a solution to problem $(\mathscr{P})$ in the standard sense.*

R e m a r k 1.4. If $\varphi \equiv c_0$ (a positive constant), then $\varphi \notin L^{2/q}[0,\infty)$. However, we can still get the existence of a solution u to problem $(\mathscr{P})$ without condition (1.3). This result is stated as follows.

Theorem 1.4. *Assume that $\int_\Omega g(x)\,\mathrm{d}x > I$ and $\varphi(\cdot) \equiv c_0$, then problem $(\mathscr{P})$ admits a solution $u \in H^1(\Omega) \cap W^{2,\alpha}(\Omega)$ in the sense of Theorem* 1.1. *Moreover, if $g(x) > 0$ a.e. in Ω, then u is a solution to problem $(\mathscr{P})$ in the standard sense.*

R e m a r k 1.5. The operator Δ can be extended to a more general operator of the form $\operatorname{div}(a(x)\nabla u)$ with $a \in C^{1,\beta}(\overline{\Omega})$. Furthermore, it is possible to adopt our results to the more general problem

$$\begin{cases} -\operatorname{div}(a(x)\nabla u) = f(x,u,|u > u(x)|, u'_*(|u > u(x)|))\chi_{\{u>0\}} & \text{in } \Omega,\\ u = \gamma \text{ (a negative constant to be determined)} & \text{on } \partial\Omega,\\ -\int_{\partial\Omega} \partial u/\partial n = I \text{ (a given positive constant),} \end{cases}$$

under some appropriate assumptions on f.

This paper is organized as follows: In Section 2, we first recall the notions of the monotone and relative rearrangements of a function, as well as some of their properties. In Section 3, we prove some results used in the Galerkin approximation. In Section 4, we first introduce a family of truncation problems $(\mathscr{P}_h)$ and then prove the existence of solutions to $(\mathscr{P}_h)$. Finally, we complete the proofs of the main results.

2. Properties of the decreasing and relative rearrangement

Let Ω be a connected and bounded open measurable subset of $\mathbb{R}^2$ (here we consider the two dimensional case, but the definitions and some of the results hold for any dimension $N \geqslant 2$), we denote by $|E|$ the Lebesgue measure of a measurable set E.

Given a measurable function $u\colon \Omega \to \mathbb{R}$, we will say that u has a flat region at the level t if $|u = t| = |\{x \in \Omega\colon u(x) = t\}| > 0$. We recall that there exists an at most countable family D of flat regions $P_u(t_i) = \{u = t_i\}$ (see [7], [22]). The union of all the flat regions of u is denoted by $P(u) = \bigcup\limits_{i\in D} P_u(t_i)$.

We define the distribution function $\mu_u(t)$ of u as follows:

$$\mu_u(t) = |\{x \in \Omega\colon u(x) > t\}| = |u > t| \quad \forall t \in \mathbb{R}.$$

The decreasing rearrangement u_* of u is defined as the generalized inverse function of $\mu_u(t)$, i.e.

$$u_*(s) = \inf\{t \in \mathbb{R}\colon \mu_u(t) \leqslant s\}, \quad s \in \overline{\Omega}_* = [0, |\Omega|].$$

We shall use the following classical result about the decreasing rearrangement.

Lemma 2.1 (see [20] and the references therein)**.** *Let u and v be measurable functions in Ω, then the following assertions are true.*

(1) $\mu_u(u_*(s)) \leqslant s$, $\forall s \in \Omega_*$. *Moreover, if u has no flat regions, then μ_u is continuous and* $\mu_u(u_*(s)) = s$.

(2) *u and u_* are equimeasurable, i.e.* $|u > \theta| = |u_* > \theta|$ $\forall \theta \in \mathbb{R}$.

(3) *Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a Borel function such that $\varphi(u) \in L^1(\Omega)$. Then*

$$\int_\Omega \varphi(u(x))\,\mathrm{d}x = \int_{\Omega_*} \varphi(u_*(s))\,\mathrm{d}s. \tag{2.1}$$

(4) *If $u \leqslant v$ almost everywhere in Ω, then* $u_*(s) \leqslant v_*(s)$ $\forall s \in \Omega_*$.

(5) *The mapping $u \to u_*$ sends $L^p(\Omega)$ into $L^p(\Omega_*)$ and it is a contraction, i.e.* $\|u_* - v_*\|_{L^p(\Omega_*)} \leqslant \|u - v\|_{L^p(\Omega)}$.

(6) *Let $E \subset \Omega$ be a measurable subset. Then*

$$\int_E u(x)\,\mathrm{d}x \leqslant \int_0^{|E|} u_*(s)\,\mathrm{d}s.$$

(7) *Let ψ be a non-decreasing function. Then $\psi(u_*) = (\psi(u))_*$ a.e. in Ω.*

Lemma 2.2 (see [24])**.** *Let $F \in L^1_{\mathrm{loc}}(\Omega_*)$, $F \geqslant 0$ and $u \in W^{1,1}(\Omega)$ with $u \geqslant 0$. Then for all s and s' with $s \leqslant s'$ in $\overline{\Omega}_*$, we have*

$$\int_{u_*(s')}^{u_*(s)} F(\mu_u(\theta))(-\mu_u'(\theta))\,\mathrm{d}\theta \leqslant \int_s^{s'} F(\sigma)\,\mathrm{d}\sigma.$$

Now we recall another notion: the relative rearrangement.

Let now $v \in L^p(\Omega)$ with $1 \leqslant p \leqslant \infty$, we define a function w in Ω_* by

$$w(s) = \begin{cases} \int_{\{u>u_*(s)\}} v(x)\,\mathrm{d}x & \text{if } |u = u_*(s)| = 0,\\ \int_{\{u>u_*(s)\}} v(x)\,\mathrm{d}x + \int_0^{s-|u>u_*(s)|} (v|_{P_u(u_*(s))})_*(t)\,\mathrm{d}t & \text{otherwise,}\end{cases}$$

where $(v|_{P_u(u_*(s))})_*$ is the decreasing rearrangement of the restriction of v to $P_u(u_*(s))$.

Lemma 2.3 (see [22], [20], and [27])**.** *Let $u \in L^1(\Omega)$ and $v \in L^p(\Omega)$ with $1 \leqslant p \leqslant \infty$. Then $w \in W^{1,p}(\Omega_*)$ and $\|\mathrm{d}w/\mathrm{d}s\|_{L^p(\Omega_*)} \leqslant \|v\|_{L^p(\Omega)}$.*

Definition 2.1. The function $\mathrm{d}w/\mathrm{d}s$ is called the relative rearrangement of v with respect to u and is denoted by v_{*u}.

The relative rearrangement has several properties as follows (see also [8], [25], [22], [27]).

Lemma 2.4. *Assume that $u, v_1, v_2 \in L^1(\Omega)$. Then the following assertions hold:*

(i) *If $v_1 \leqslant v_2$ a.e. in Ω, then $v_{1*u} \leqslant v_{2*u}$ a.e. in Ω_*.*

(ii) *If F is a Borel function such that $F(u) \in L^1(\Omega)$, then $(v_1 + F(u))_{*u} = v_{1*u} + F(u_*)$. In particular, $(c + F(u))_{*u} = c + F(u_*)$, where $c \in \mathbb{R}$. Moreover, if $b \in L^\infty(\Omega)$ and $|P(u)| = 0$, then $(F(u)b)_{*u} = F(u_*)b_{*u}$.*

(iii) *The mapping $v \to v_{*u}$ sends $L^p(\Omega)$ into $L^p(\Omega_*)$ for any $1 \leqslant p \leqslant \infty$ and it is a contraction, i.e. $\|v_{1*u} - v_{2*u}\|_{L^p(\Omega_*)} \leqslant \|v_1 - v_2\|_{L^p(\Omega)}$. In particular,*

$$\|v_{*u}\|_{L^p(\Omega_*)} \leqslant \|v\|_{L^p(\Omega)}\ \forall v \in L^p(\Omega).$$

We also need the notion of a co-area regular function (see [1]):

Definition 2.2. Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$. For $\theta \in \mathbb{R}$ we set $\mu_{u,0}(\theta) := |\{x \in \Omega\colon u(x) > \theta \text{ and } \nabla u(x) = 0\}|$ and $\mu_{u,1}(\theta) = \mu_u(\theta) - \mu_{u,0}(\theta)$. We will say that u is a co-area regular function if the Radon measure $(\mu_{u,0})'$ is singular with respect to the Lebesgue measure on $\mathbb{R}$.

Lemma 2.5 (see [8] and [28])**.** *If $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for some $p > 1$, then u is a co-area regular function.*

Lemma 2.6 (see [8], [26], and [1])**.** *Let v be a co-area regular function of $W^{1,p}(\Omega)$, $1 \leqslant p \leqslant \infty$. If v_n is a bounded sequence of $W^{1,p}(\Omega)$ such that v_n converges to v strongly in $W^{1,p}(\Omega)$, then*

$$\begin{aligned} v'_{n*}(s) &\to v'_*(s) && \text{a.e. in } \Omega_*, \\ k(s)v'_{n*}(s) &\to k(s)v'_*(s) && \text{strongly in } L^p(\Omega_*), \end{aligned}$$

where k is defined as in (1.1)*.*

Now we recall the notion of the mean value operator introduced in [22] (see also [26]).

Definition 2.3. Let $u \in L^1(\Omega)$ and $\varphi \in L^1(\Omega_*)$, we define the mean value operator $\mathscr{M}_u(\varphi)$ by

$$\mathscr{M}_u(\varphi)(x) = \begin{cases} \varphi(|u > u(x)|) & \text{if } x \in \Omega \setminus P(u), \\ \dfrac{1}{|u = u(x)|} \displaystyle\int_{|u>u(x)|}^{|u \geqslant u(x)|} \varphi(\sigma)\,\mathrm{d}\sigma & \text{otherwise.} \end{cases}$$

Definition 2.4. Let $u, v \in L^1(\Omega)$ and $\varphi \in L^1(\Omega_*)$. We define the second category mean value operator $\mathscr{M}_{u,v}(\varphi)$ as the function

$$\mathscr{M}_{u,v}(\varphi)(x) = \begin{cases} \varphi(|u > u(x)|) & \text{if } x \in \Omega \setminus P(u), \\ \mathscr{M}_{v_i}(h_i)(x) & \text{if } x \in P_u(\theta_i), \end{cases}$$

where $v_i = v|_{P_u(\theta_i)}$ is the restriction of v to $P_u(\theta_i)$ and $h_i\colon (0, |P_u(\theta_i)|) \to \mathbb{R}$, $h_i(s) = \varphi(s + |u > \theta_i|)$.

Lemma 2.7 (see [9] and [22])**.** *Let $u \in L^1(\Omega)$ and $v \in L^p(\Omega)$, $1 \leqslant p \leqslant \infty$. For any $\varphi \in L^{p'}(\Omega_*)$, we have*

$$\int_{\Omega_*} v_{*u}(s)\varphi(s)\,\mathrm{d}s = \int_\Omega \mathscr{M}_{u,v}(\varphi)(x)v(x)\,\mathrm{d}x.$$

If $|P(u)|=0$ *the last equality is reduced to*

$$\int_{\Omega_*} v_{*u}(s)\varphi(s)\,\mathrm{d}s=\int_{\Omega}\varphi(|u>u(x)|)v(x)\,\mathrm{d}x.$$

Lemma 2.8 (see [24] and [31])**.** *Let* $u\in W_0^{1,p}(\Omega)$ *and* $v=|u|$, *where* $p>1$. *Then for a.e.* $\theta\in[0,\operatorname*{ess\,sup}_{\Omega} v]$,

$$-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\{v>\theta\}}|\nabla v|\,\mathrm{d}x\geqslant 2\pi^{1/2}\mu_v^{1/2}(\theta).$$

3. Some useful results

As mentioned in the introduction, the proof of Theorem 1.1 is based on the application of the Galerkin method. We have devoted this section to proving some results which are crucial for passing to the limit in the approximate problem.

Lemma 3.1. *Set* $L^1_+(\Omega)=\{w\in L^1(\Omega)\colon\ w\geqslant 0\}$. *Let* $u\in L^1(\Omega)$ *and let* F *be a Borel function such that* $F(u)\in L^1_+(\Omega)$, $v\in L^\infty(\Omega)$. *Then*

$$(F(u)v)_{*u}(s)=F(u_*(s))v_{*u}(s)\ \text{a.e. } s\in\Omega_*. \tag{3.1}$$

P r o o f. *Step 1*: We prove that (3.1) holds for all $v\in L^\infty(\Omega)$ with $|P(v)|=0$ (i.e. v has no flat region).

Using Lemma 2.7, we deduce that for all $\varrho\in L^\infty(\Omega_*)$,

$$\int_{\Omega_*}(F(u)v)_{*u}(s)\varrho(s)\,\mathrm{d}s=\int_{\Omega}\mathscr{M}_{u,F(u)v}(\varrho)(x)F(u(x))v(x)\,\mathrm{d}x.$$

By the definition of $\mathscr{M}_{u,F(u)v}$ (see Definition 2.4), we have

$$F(u(x))\mathscr{M}_{u,F(u)v}(\varrho)(x)=\begin{cases}F(u(x))\varrho(|u>u(x)|) & \text{if } x\in\Omega\setminus P(u),\\ 0 & \text{if } F(u(x))=0,\\ F(u(x))\varrho(|u>u(x)|+\nu(x)) & \text{otherwise,}\end{cases}$$

where $\nu(x)=|\{y\colon\ u(y)=u(x),F(u(x))v(y)>F(u(x))v(x)\}|$.

Since $F(u(x))>0$ in the third case, we have

$$\nu(x)=|\{y:\ u(y)=u(x),v(y)>v(x)\}|. \tag{3.2}$$

In view of this, we obtain

$$F(u(x))\mathscr{M}_{u,F(u)v}(\varrho)(x) = F(u(x))\mathscr{M}_{u,v}(\varrho)(x),$$

which leads to

$$\int_{\Omega_*} (F(u)v)_{*u}(s)\varrho(s)\,\mathrm{d}s = \int_\Omega \mathscr{M}_{u,v}(\varrho)(x)F(u(x))v(x)\,\mathrm{d}x. \tag{3.3}$$

On the other hand, we have

$$\int_{\Omega_*} F(u_*)v_{*u}(s)\varrho(s)\,\mathrm{d}s = \int_\Omega \mathscr{M}_{u,v}(F(u_*)\varrho)(x)v(x)\,\mathrm{d}x. \tag{3.4}$$

By the definition of $\mathscr{M}_{u,v}$, we have

$$\mathscr{M}_{u,v}(F(u_*)\varrho)(x) = \begin{cases} F(u_*(|u > u(x)|))\varrho(|u > u(x)|) & \text{if } x \in \Omega \setminus P(u), \\ F(u_*(|u > u(x)| + \nu(x)))\varrho(|u > u(x)| + \nu(x)) & \text{otherwise,} \end{cases}$$

where ν is defined as in (3.2).

If $x \in P(u)$, then $[|u > u(x)|, |u \geqslant u(x)|] \subseteq P(u_*)$ and $u_*(s) = u(x)$ for all $s \in [|u > u(x)|, |u \geqslant u(x)|]$. Hence, we have

$$u_*(|u > u(x)| + \nu(x)) = u(x) \quad \text{for a.e. } x \in P(u).$$

Moreover, by the definition of u_*, it is easy to see that

$$u_*(|u > u(x)|) = u(x) \text{ for a.e. } x \in \Omega \setminus P(u).$$

The above two relations show that

$$\mathscr{M}_{u,v}(F(u_*)\varrho)(x) = F(u)(x)\mathscr{M}_{u,v}(\varrho)(x) \quad \text{for a.e. } x \in \Omega. \tag{3.5}$$

It follows from (3.3)–(3.5) that

$$\begin{aligned} \int_{\Omega_*} (F(u)v)_{*u}(s)\varrho(s)\,\mathrm{d}s &= \int_\Omega \mathscr{M}_{u,v}(\varrho)(x)F(u)(x)v(x)\,\mathrm{d}x \\ &= \int_{\Omega_*} F(u_*)v_{*u}(s)\varrho(s)\,\mathrm{d}s, \quad \forall \varrho \in L^\infty(\Omega_*), \end{aligned} \tag{3.6}$$

which implies that $(F(u)v)_{*u} = F(u_*)v_{*u}$.

Step 2: We prove that (3.1) holds for all $v \in L^\infty(\Omega)$. Clearly, (3.1) holds if $v \equiv 0$. Now we assume that $v \not\equiv 0$.

Let φ_n be an eigenfunction of the Laplacian operator corresponding to λ_n (see Section 4 for detail). For all n, we know that φ_n is analytic in Ω (see [15] and [5]), and so $|P(\varphi_n)| = 0$ (i.e. φ_n does not have flat regions). Moreover, $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega)$, and so there exist $\{a_n\}_{n=1}^{\infty} \subseteq R$ such that $v = \sum\limits_{i=1}^{\infty} a_i\varphi_i$. Let $v_n = \sum\limits_{i=1}^{n} a_i\varphi_i$. Then $v_n \in L^\infty(\Omega)$ and

$$v_n \to v \quad \text{strongly in } L^2(\Omega) \text{ and a.e. in } \Omega.$$

Furthermore, v_n is analytic in Ω and $|P(v_n)| = 0$ as soon as n is large enough (recall that $v \not\equiv 0$). By the proof of Step 1, we have for $l > 0$

$$\int_{\Omega_*} (T_l(F(u))v_n)_{*u}(s)\varrho(s)\,\mathrm{d}s = \int_{\Omega_*} T_l(F(u_*))v_{n*u}(s)\varrho(s)\,\mathrm{d}s, \quad \forall \varrho \in L^\infty(\Omega_*). \tag{3.7}$$

By assertion (iii) of Lemma 2.4 and the Lebesgue dominated convergence theorem, we conclude that

$$(T_l(F(u))v_n)_{*u} \to (T_l(F(u))v)_{*u} \quad \text{strongly in } L^2(\Omega),$$

and

$$v_{n*u} \to v_{*u} \quad \text{strongly in } L^1(\Omega).$$

The above two convergence results together with (3.7) lead to

$$\int_{\Omega_*} (T_l(F(u))v)_{*u}(s)\varrho(s)\,\mathrm{d}s = \int_{\Omega_*} T_l(F(u_*))v_{*u}(s)\varrho(s)\,\mathrm{d}s \quad \forall \varrho \in L^\infty(\Omega_*). \tag{3.8}$$

Let $l \to \infty$ in (3.8). We find that

$$\int_{\Omega_*} (F(u)v)_{*u}(s)\varrho(s)\,\mathrm{d}s = \int_{\Omega_*} F(u_*)v_{*u}(s)\varrho(s)\,\mathrm{d}s \quad \forall \varrho \in L^\infty(\Omega_*),$$

which implies that $(F(u)v)_{*u} = F(u_*)v_{*u}$.

Thus, the proof of Lemma 3.1 is completed. □

Remark 3.1. Since $L^\infty(\Omega)$ is dense in $L^p(\Omega)$ for $1 < p < \infty$, we deduce that (3.1) holds for $F(u) \in L_+^{p'}(\Omega)$, $v \in L^p(\Omega)$. In contrast to Lemma 2.4 (see also Lemma 11 in [9]), we do not need the condition $|P(u)| = 0$ but require that $F(u) \in L_+^1(\Omega)$. We also point out that the special case $F(u) = u \in L_+^1(\Omega)$ has already been studied in [12].

Lemma 3.2. *Let $v \in W^{2,p}(\Omega)$ and $v_n \in W^{1,p}(\Omega)$ be such that*

$$v_n \to v \quad \text{strongly in } W^{1,p}(\Omega) \text{ with } p > 1. \tag{3.9}$$

Then we have

$$k(|v_n > v_n(\cdot)|)v'_{n*}(|v_n > v_n(\cdot)|)\chi_{\Omega\setminus P(v_n)}\chi_{\{v_n>0\}} \\ \to k(|v > v(\cdot)|)v'_*(|v > v(\cdot)|)\chi_{\Omega\setminus P(v)}\chi_{\{v>0\}} \quad \text{strongly in } L^p(\Omega),$$

where $\chi_{\Omega\setminus P(v_n)}$ and $\chi_{\Omega\setminus P(v)}$ are the characteristic functions of $\Omega\setminus P(v_n)$ and $\Omega\setminus P(v)$, respectively.

Proof. By the equimeasurability, we obtain that

$$\begin{aligned} &\int_\Omega |k(|v_n > v_n(x)|)v'_{n*}(|v_n > v_n(x)|)\chi_{\Omega\setminus P(v_n)}\chi_{\{v_n>0\}}|^p \,\mathrm{d}x \\ &= \int_{\Omega\setminus P(v_n)} |k(|v_n > v_n(x)|)v'_{n*}(|v_n > v_n(x)|)\chi_{\{v_n>0\}}|^p \,\mathrm{d}x \\ &= \int_{\Omega_*\setminus P(v_{n*})} |k(s)v'_{n*}(s)\chi_{\{v_{n*}>0\}}|^p \,\mathrm{d}s = \int_{\Omega_*} |k(s)v'_{n*}(s)\chi_{\{v_{n*}>0\}}|^p \,\mathrm{d}s \\ &= \int_{\Omega_*\setminus P(v_*)} |k(s)v'_{n*}(s)\chi_{\{v_{n*}>0\}}|^p \,\mathrm{d}s + \int_{P(v_*)} |k(s)v'_{n*}(s)\chi_{\{v_{n*}>0\}}|^p \,\mathrm{d}s. \end{aligned} \tag{3.10}$$

Similarly, we have

$$\int_\Omega |k(|v > v(x)|)v'_*(|v > v(x)|)\chi_{\Omega\setminus P(v)}\chi_{\{v>0\}}|^p \,\mathrm{d}x \\ = \int_{\Omega_*} |k(s)v'_*(s)\chi_{\{v_*>0\}}|^p \,\mathrm{d}s. \tag{3.11}$$

Since $\chi_{\{v_{n*}>0\}}$ converges to $\chi_{\{v_*>0\}}$ a.e. in $\Omega_*\setminus P(v_*)$, using the Lebesgue dominated convergence theorem, Lemma 2.5, Lemma 2.6, and (3.9), we conclude that

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega_*\setminus P(v_*)} |k(s)v'_{n*}(s)\chi_{\{v_{n*}>0\}}|^p \,\mathrm{d}s &= \int_{\Omega_*\setminus P(v_*)} |k(s)v'_*(s)\chi_{\{v_*>0\}}|^p \,\mathrm{d}s \\ &= \int_{\Omega_*} |k(s)v'_*(s)\chi_{\{v_*>0\}}|^p \,\mathrm{d}s \end{aligned} \tag{3.12}$$

and

$$\lim_{n\to\infty} \int_{P(v_*)} |k(s)v'_{n*}(s)\chi_{\{v_{n*}>0\}}|^p \,\mathrm{d}s = 0. \tag{3.13}$$

Passing to the limit as n tends to ∞ in (3.9) and using (3.10)–(3.13), we have

$$\lim_{n\to\infty}\int_\Omega |k(|v_n > v_n(x)|)v'_{n*}(|v_n > v_n(x)|)\chi_{\Omega\setminus P(v_n)}\chi_{\{v_n>0\}}|^p\,\mathrm{d}x = \int_\Omega |k(|v > v(x)|)v'_*(|v > v(x)|)\chi_{\Omega\setminus P(v)}\chi_{\{v>0\}}|^p\,\mathrm{d}x. \tag{3.14}$$

If $x \in P(v_n)$, then $[|v_n > v_n(x)|, |v_n \geqslant v_n(x)|] \subset P(v_{n*})$ and so $k(s)v'_{n*}(s) = 0$ for a.e. $s \in [|v_n > v_n(x)|, |v_n \geqslant v_n(x)|]$, which implies that $\mathscr{M}_{v_n,w}(kv'_{n*})(x) = 0$ for a.e. $x \in P(v_n)$ and any $w \in L^{p'}(\Omega)$. If $x \notin P(v_n)$, then $\mathscr{M}_{v_n,w}(kv'_{n*})(x) = k(|v_n > v_n(x)|)v'_{n*}(|v_n > v_n(x)|)$ for any $w \in L^{p'}(\Omega)$. Thus, we obtain that

$$\mathscr{M}_{v_n,w}(kv'_{n*})(x) = k(|v_n > v_n(x)|)v'_{n*}(|v_n > v_n(x)|)\chi_{\Omega\setminus P(v_n)}. \tag{3.15}$$

Applying the same argument, we get

$$\mathscr{M}_{v,w}(kv'_*)(x) = k(|v > v(x)|)v'_*(|v > v(x)|)\chi_{\Omega\setminus P(v)}. \tag{3.16}$$

Let $\psi = \chi_{P(v_*)}$ be the characteristic function of $P(v_*)$. We deduce from Lemma 2.7 and (3.15) that

$$\begin{aligned}\int_\Omega k(|v_n > v_n(x)|)&v'_{n*}(|v_n > v_n(x)|)\chi_{\Omega\setminus P(v_n)}\chi_{\{v_n>0\}}w(x)\,\mathrm{d}x\\ &= \int_\Omega \mathscr{M}_{v_n,w}(kv'_{n*})(x)\chi_{\{v_n>0\}}w(x)\,\mathrm{d}x\\ &= \int_{\Omega_*} k(s)v'_{n*}(s)(\chi_{\{v_n>0\}}w)_{*v_n}(s)\,\mathrm{d}s\\ &= \int_{\Omega_*} k(s)v'_{n*}(s)(\chi_{\{v_n>0\}}w)_{*v_n}(s)(1-\psi(s))\,\mathrm{d}s\\ &\quad+\int_{\Omega_*} k(s)v'_{n*}(s)(\chi_{\{v_n>0\}}w)_{*v_n}(s)\psi(s)\,\mathrm{d}s = I_n + J_n.\end{aligned} \tag{3.17}$$

By Remark 3.1, we obtain

$$(\chi_{\{v_n>0\}}w)_{*v_n} = \chi_{\{v_{n*}>0\}}w_{*v_n} \text{ and } (\chi_{\{v>0\}}w)_{*v} = \chi_{\{v_*>0\}}w_{*v}. \tag{3.18}$$

Arguing as in Lemma 3.2 of [30], we deduce that

$$(1-\psi)w_{*v_n} \rightharpoonup (1-\psi)w_{*v} \quad \text{weakly in } L^{p'}(\Omega_*). \tag{3.19}$$

On the other hand, we have

$$\chi_{\{v_{n*}>0\}} \to \chi_{\{v_*>0\}} \quad \text{a.e. in } \Omega_* \setminus P(v_*). \tag{3.20}$$

It follows from Lemma 2.5–Lemma 2.7, (3.16) and (3.18)–(3.20) that

$$\begin{aligned}\lim_{n\to\infty} I_n &= \lim_{n\to\infty}\int_{\Omega_*\setminus P(v_*)} k(s)v'_{n*}(s)\chi_{\{v_{n*}>0\}}w_{*v_n}(s)(1-\psi(s))\,\mathrm{d}s \\ &= \int_{\Omega_*\setminus P(v_*)} k(s)v'_*(s)\chi_{\{v_*>0\}}w_{*v}(s)(1-\psi(s))\,\mathrm{d}s \\ &= \int_{\Omega_*} k(s)v'_*(s)\chi_{\{v_*>0\}}w_{*v}(s)\,\mathrm{d}s \\ &= \int_{\Omega_*} k(s)v'_*(s)(\chi_{\{v>0\}}w)_{*v}(s)\,\mathrm{d}s \\ &= \int_{\Omega} \mathscr{M}_{v,w}(kv'_*)(x)\chi_{\{v>0\}}w(x)\,\mathrm{d}x \\ &= \int_{\Omega} k(|v>v(x)|)v'_*(|v>v(x)|)\chi_{\Omega\setminus P(v)}\chi_{\{v>0\}}w(x)\,\mathrm{d}x.\end{aligned} \tag{3.21}$$

We conclude from Lemma 2.5 and Lemma 2.6 that

$$kv'_{n*}\psi \to 0 \quad \text{strongly in } L^p(\Omega_*). \tag{3.22}$$

By the Hölder inequality, we get

$$|J_n| \leqslant \left(\int_{\Omega_*} |k(s)v'_{n*}(s)\psi(s)|^p\,\mathrm{d}s\right)^{1/p}\left(\int_{\Omega_*} |(\chi_{\{v_n>0\}}w)_{*v_n}|^{p'}\,\mathrm{d}s\right)^{1/p'}. \tag{3.23}$$

From Lemma 2.4 it is easy to see that the sequence $\{(\chi_{\{v_n>0\}}w)_{*v_n}\}$ is bounded uniformly in $L^{p'}(\Omega_*)$ with respect to n. Thus from (3.22) and (3.23) we have

$$\lim_{n\to\infty} J_n = 0. \tag{3.24}$$

By (3.17), (3.21) and (3.24), we obtain that

$$\begin{aligned}\lim_{n\to\infty}\int_{\Omega} k(|v_n>v_n(x)|)v'_{n*}(|v_n>v_n(x)|)\chi_{\Omega\setminus P(v_n)}\chi_{\{v_n>0\}}w(x)\,\mathrm{d}x \\ = \int_{\Omega} k(|v>v(x)|)v'_*(|v>v(x)|)\chi_{\Omega\setminus P(v)}\chi_{\{v>0\}}w(x)\,\mathrm{d}x,\end{aligned}$$

which implies that

$$\begin{aligned}k(|v_n>v_n(x)|)v'_*(|v_n>v_n(\cdot)|)\chi_{\Omega\setminus P(v_n)}\chi_{\{v_n>0\}} \\ \rightharpoonup k(|v>v(x)|)v'_*(|v>v(\cdot)|)\chi_{\Omega\setminus P(v)}\chi_{\{v>0\}}\end{aligned}$$

weakly in $L^p(\Omega)$. Since $L^p(\Omega)(1<p<\infty)$ is a uniformly convex space, the conclusion of Lemma 3.2 follows immediately from the above relation and (3.14). □

Lemma 3.3. *Let k be the function defined by* (1.1), $\psi \in C^+(\mathbb{R})$ *and* $v \in W^{1,p}(\Omega)$ *with* $p > 1$. *Then for any* $\theta \in \mathbb{R}$ *and* $h \in \mathbb{R}^+$,

$$\int_{\{\theta < v_* \leqslant \theta + h\}} \psi(v_*)|k(s)v_*'(s)|^p \,\mathrm{d}s \leqslant \frac{1}{(2\pi^{1/2})^p} \int_{\{\theta < v \leqslant \theta + h\}} \psi(v)|\nabla v|^p \,\mathrm{d}x. \tag{3.25}$$

P r o o f. We introduce two functions $\overline{\psi}$ and $\widehat{\psi}$ defined as follows:

$$\overline{\psi}(s) = \begin{cases} \psi(s), & \theta < s \leqslant \theta + h, \\ 0, & s \leqslant \theta \text{ or } s > \theta + h, \end{cases}$$

and

$$\widehat{\psi}(s) = \int_0^s (\overline{\psi}(t))^{1/p} \,\mathrm{d}t.$$

Let $w = \widehat{\psi}(v)$. Then it is easy to see that $w \in W^{1,p}(\Omega)$. By Theorem 1.2 in [31], we get

$$\int_{\Omega_*} |k(s)w_*'(s)|^p \,\mathrm{d}s \leqslant \frac{1}{(2\pi^{1/2})^p} \int_{\Omega} |\nabla w|^p \,\mathrm{d}x. \tag{3.26}$$

Since $\widehat{\psi}$ is a nondecreasing function, we get $w_*(s) = (\widehat{\psi}(v))_*(s) = \widehat{\psi}(v_*(s))$ for a.e. $s \in \Omega_*$. Thus, we have

$$w_*'(s) = v_*'(s)(\overline{\psi}(v_*(s)))^{1/p} = v_*'(s)(\psi(v_*))^{1/p}\chi_{\{\theta < v_* \leqslant \theta + h\}} \quad \text{for a.e. } s \in \Omega_*. \tag{3.27}$$

Note that

$$\nabla w = \widehat{\psi}'(v)\nabla v = (\overline{\psi}(v))^{1/p}\nabla v = (\psi(v))^{1/p}\nabla v \chi_{\{\theta < v \leqslant \theta + h\}},$$

thus the conclusion (3.25) follows immediately from (3.26) and (3.27). □

R e m a r k 3.2. Lemma 3.3 is an extension of the Pólya-Szegö inequality for monotone rearrangement. Using this lemma, we may obtain L^∞ estimates on u_{h+} (Theorem 4.3 and Lemma 4.4). Moreover, if we let h tend to infinity in (3.25), then

$$\int_{\{v_* > \theta\}} \psi(v_*)|k(s)v_*'(s)|^p \,\mathrm{d}s \leqslant \frac{1}{(2\pi^{1/2})^p} \int_{\{v > \theta\}} \psi(v)|\nabla v|^p \,\mathrm{d}x.$$

Here and in what follows, we use the following notation. For any $v \in H_0^1(\Omega)$ and $h > 0$, $\varepsilon > 0$,

$$(4.1)\ G_{h\varepsilon}(x, v, v'_*) = \Big[\lambda\varphi(T_h(v_+))\frac{|k(|v > v(x)|)v'_*(|v > v(x)|)|^q}{1 + \varepsilon|k(|v > v(x)|)v'_*(|v > v(x)|)|^q} + \frac{g(x)}{1 + \varepsilon g(x)}\Big]\chi_{\{v>0\}},$$

$$(4.2)\ \widetilde{G}_{h\varepsilon}(x, v, v'_*) = \Big[\lambda\varphi(T_h(v_+))\frac{|k(|v > v(x)|)v'_*(|v > v(x)|)|^q\chi_{\Omega\setminus P(v)}}{1 + \varepsilon|k(|v > v(x)|)v'_*(|v > v(x)|)|^q} + \frac{g(x)}{1 + \varepsilon g(x)}\Big]\chi_{\{v>0\}},$$

$$(4.3)\ G_h(x, v, v'_*) = [\lambda\varphi(T_h(v_+))|k(|v > v(x)|)v'_*(|v > v(x)|)|^q + g(x)]\chi_{\{v>0\}},$$

$$(4.4)\ \widetilde{G}_h(x, v, v'_*) = [\lambda\varphi(T_h(v_+))|k(|v > v(x)|)v'_*(|v > v(x)|)|^q\chi_{\Omega\setminus P(v)} + g(x)]\chi_{\{v>0\}},$$

where T_h is the truncation function defined as $T_h(s) = \min\{h, \max\{s, -h\}\}$.

In order to avoid the lack of regularity of the term $\lambda\varphi(u_+)|k(|u > u(x)|)u'_*(|u > u(x)|)|^q$, rather than looking for solutions of $(\mathscr{P})$ directly, we shall consider a truncated problem $(\mathscr{P}_h)$ defined as

$$(\mathscr{P}_h)\qquad \begin{cases} -\Delta u_h = G_h(x, u_h, u'_{h*}) \quad \text{in } \Omega,\\ u_h = \gamma_h \text{ (a negative constant to be determined)} \quad \text{on } \partial\Omega,\\ -\int_{\partial\Omega} \partial u_h/\partial n = I \text{ (a given positive constant).} \end{cases}$$

The existence result to problem $(\mathscr{P}_h)$ is stated as follows.

Theorem 4.1. *Let $h > 0$ be fixed and let q be a positive constant with $q < 2$, $0 \leqslant g \in L^r(\Omega)$. Suppose that φ: $[0, \infty] \to [0, \infty]$ is a continuous function and $\int_\Omega g(x)\,\mathrm{d}x > I$. Then there exists at least one solution $u_h \in W^{2,\alpha}(\Omega) \cap H^1(\Omega)$ to problem $(\mathscr{P}_h)$ in the following sense:*

(i) $-\Delta u_h = \widetilde{G}_h(x, u_h, u'_{h*})$ *in* Ω.

(ii) $u_h|_{\partial\Omega} = \gamma_h$ *(a negative constant).*

(iii) $-\int_{\partial\Omega} \partial u_h/\partial n = I$.

To prove Theorem 4.1, we shall first consider a family of approximate problems $(\mathscr{P}_{h\varepsilon})$:

$$(\mathscr{P}_{h\varepsilon})\qquad \begin{cases} -\Delta u_{h\varepsilon} = G_{h\varepsilon}(x, u_{h\varepsilon}, u'_{h\varepsilon*}) & \text{in } \Omega,\\ u_{h\varepsilon} = \gamma_\varepsilon \text{ (a constant to be determined)} & \text{on } \partial\Omega,\\ -\int_{\partial\Omega} \partial u_{h\varepsilon}/\partial n = I \text{ (a given positive constant).} \end{cases}$$

The existence of solutions to problem $(\mathscr{P}_{h\varepsilon})$ is stated as follows.

Theorem 4.2. *Suppose that the assumption of Theorem* 4.1 *holds. Then there exists at least one solution $u_{h\varepsilon} \in H^2(\Omega)$ to problem $(\mathscr{P}_{h\varepsilon})$ in the following sense:*

(i) $-\Delta u_{h\varepsilon} = \widetilde{G}_{h\varepsilon}(x, u_{h\varepsilon}, u'_{h\varepsilon *})$ *in* Ω,

(ii) $u_{h\varepsilon}|_{\partial\Omega} = \gamma_{h\varepsilon}$ *(a constant),*

(iii) $-\int_{\partial\Omega} \partial u_{h\varepsilon}/\partial n = I$.

4.1. Proof of Theorem 4.1 and Theorem 4.2. We shall first prove Theorem 4.2 by the Galerkin method and topological degree theory. The idea of this proof comes from [2], [8] and [10].

Let

$$V = \{v \in H^1(\Omega) \colon\ v \equiv \text{ constant on } \partial\Omega\}$$

be endowed with the scalar product $[u, v] = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + u_{|\partial\Omega} v_{|\partial\Omega}$ for $u, v \in V$ and

$$\widetilde{V} = \{v \in V \colon\ |\{x \in \Omega \colon\ \nabla v(x) = 0\}| = 0\}.$$

Let $(\lambda_i, \varphi_i)_{i\geqslant 1}$ be the eigenvalues and eigenfunctions associated to the problem

$$\begin{cases} -\Delta\varphi_i = \lambda_i\varphi_i & \text{in } \Omega,\\ \varphi_i = 0 & \text{on } \partial\Omega,\\ \int_\Omega \varphi_i^2(x)\,\mathrm{d}x = 1. \end{cases}$$

Let $V_m = \operatorname{span}\{1, \varphi_1, \varphi_2, \ldots, \varphi_m\}$. Since for $v \in V$ we have

$$v = [v, 1] + \sum_{j=1}^{\infty} [v, \varphi_j] \frac{\varphi_j}{\lambda_j},$$

and thus $\overline{\bigcup_{i=1}^{\infty} V_i} = V$.

For all $t \in [0, 1]$ and fixed $h > 0$, set

$$G_{t\varepsilon} \colon\ v \in V \mapsto tG_{h\varepsilon}(x, v, v'_*) + (1-t)\lambda_1 v_+, \tag{4.5}$$

$$E_{t\varepsilon} = \{v \in \widetilde{V}, j_{t\varepsilon}(v) = 0, e_{t\varepsilon}(v) = 0\}, \tag{4.6}$$

where

$$e_{t\varepsilon}(v) = -I + \int_\Omega G_{t\varepsilon}(v)\,\mathrm{d}x, \quad j_{t\varepsilon}(v) = \int_\Omega |\nabla v|^2\,\mathrm{d}x + Iv_{|\partial\Omega} - \int_\Omega G_{t\varepsilon}(v) v_+\,\mathrm{d}x.$$

Then we have the following result.

Lemma 4.1. *Let $0 \leqslant g \in L^r(\Omega)$ and let $\varphi\colon [0,\infty] \to [0,\infty]$ be a continuous function. If $\int_\Omega g(x)\,\mathrm{d}x > I$, then for fixed $h > 0$ there exists $\varepsilon_0 > 0$ such that for any ε with $0 < \varepsilon < \varepsilon_0$ and for all $v \in \bigcup_{t\in[0,1]} E_{t\varepsilon}$ the following assertions hold.*

(i) *There exists a positive constant C independent of t and v such that*

$$\|v_+\|_{L^p(\Omega)} \leqslant C(\|\nabla v\|_{L^2(\Omega)} + 1) \quad \forall p > 1.$$

(ii) *For every $\delta > 0$ there exists a constant $C_\delta > 0$ independent of t and v such that*

$$-v_{|\partial\Omega} \leqslant \delta\|\nabla v\|_{L^2(\Omega)} + C_\delta.$$

(iii) *There exists a positive constant $C_{h\varepsilon}$ independent of t and v such that*

$$\|v\|_{H^1(\Omega)} < C_{h\varepsilon}.$$

P r o o f of (i). Define $g_\varepsilon(x) = g(x)/(1+\varepsilon g(x))$. It is easy to see that

$$\int_\Omega g_\varepsilon(x)\,\mathrm{d}x = \int_\Omega \frac{g(x)}{1+\varepsilon g(x)}\,\mathrm{d}x \to \int_\Omega g\,\mathrm{d}x.$$

Since $\int_\Omega g(x)\,\mathrm{d}x > I$, we deduce from the above inequality that there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_0$,

$$\int_\Omega g_\varepsilon(x)\,\mathrm{d}x \geqslant \int_\Omega g_{\varepsilon_0}(x)\,\mathrm{d}x > \int_\Omega g(x)\,\mathrm{d}x - \frac{1}{2}\Big(\int_\Omega g(x)\,\mathrm{d}x - I\Big) > I. \tag{4.7}$$

Let $\eta = 1/2 + I/2\|g_{\varepsilon_0}\|_{L^1(\Omega)}$. By (4.7), it is easy to see that $0 < \eta < 1$.

If $v \in \bigcup_{t\in[0,\eta]} E_{t\varepsilon}$, then there exists $t \in [0,\eta]$ such that $v \in E_{t\varepsilon}$ and $e_{t\varepsilon}(v) = 0$. By the definition of $G_{t\varepsilon}$ we get $G_{t\varepsilon}(v) \geqslant (1-\eta)\lambda_1 v_+$. Since $e_{t\varepsilon}(v) = 0$, we get $I = \int_\Omega G_{t\varepsilon}(v)\,\mathrm{d}x \geqslant (1-\eta)\lambda_1 \int_\Omega v_+\,\mathrm{d}x$, i.e.

$$\int_\Omega v_+ dx \leqslant \frac{I}{(1-\eta)\lambda_1}. \tag{4.8}$$

Thus, using (4.8) and the Poincaré inequality, we obtain

$$\|v_+\|_{L^p(\Omega)} \leqslant C_1(\|\nabla v\|_{L^2(\Omega)} + 1),$$

where C_1 is a positive constant independent of t and v.

If $v \in \bigcup\limits_{t\in[\eta,1]} E_{t\varepsilon}$, then there exists $t \in [\eta,1]$ such that $v \in E_{t\varepsilon}$ and $e_{t\varepsilon}(v) = 0$. By the definition of $G_{t\varepsilon}$, we have $G_{t\varepsilon}(v) \geqslant \eta g_\varepsilon(x)\chi_{\{v>0\}}$. Since $e_{t\varepsilon}(v) = 0$ and $g_{\varepsilon_0} \leqslant g_\varepsilon$ for $0 < \varepsilon \leqslant \varepsilon_0$, we get

$$\eta \int_\Omega g_{\varepsilon_0}(x)\chi_{\{v>0\}}\,\mathrm{d}x \leqslant \eta \int_\Omega g_\varepsilon(x)\chi_{\{v>0\}}\,\mathrm{d}x \leqslant I.$$

In view of (4.7), we get $\eta \int_\Omega g_{\varepsilon_0}(x)\,\mathrm{d}x > I$. Thus using the same argument as in Lemma 3.1 in [2], we deduce that there exists a positive constant C_2 independent of v, t, and ε such that

$$\|v_+\|_{L^p(\Omega)} \leqslant C_2(\|\nabla v\|_{L^2(\Omega)} + 1) \quad \forall p \in [1,\infty) \text{ and } \forall v \in \bigcup_{t\in[\eta,1]} E_{t\varepsilon}. \tag{4.9}$$

Setting $C = \max\{C_1, C_2\}$, the conclusion (i) follows immediately.

P r o o f of (ii) and (iii). Since the proofs of (ii) and (iii) are similar to the proofs of Lemma 3.2 and Lemma 3.4 in [2] (see also [29] and [10]), we omit the details here. □

As in [3], we introduce a Galerkin approximation method to problem $(\mathscr{P}_{h\varepsilon})$.

Define a family of operators $T_m^t\colon V_m \longrightarrow V_m$ such that if $v \in V_m$ then $T_m^t v$ is the unique solution of the problem

$$a(T_m^t v, w) = -Iw_{|\partial\Omega} + \int_\Omega G_{t\varepsilon}(v)(x)w(x)\,\mathrm{d}x + \int_\Omega v(x)w(x)\,\mathrm{d}x \ \forall w \in V_m, \tag{4.10}$$

where

$$a(v,w) = \int_\Omega \nabla v(x)\nabla w(x)\,\mathrm{d}x + \int_\Omega v(x)w(x)\,\mathrm{d}x.$$

By the Lax-Milgram theorem, it is easy to see that T_m^t is well defined.

Set

$$E_m^t = \{u_m^t \in V_m\colon T_m^t u_m^t = u_m^t\}, \quad \mathbb{E}_m = \bigcup_{t\in[0,1]} E_m^t,$$

and

$$c_m = \inf_{v\in\mathbb{E}_m} \|v\|_{H^1(\Omega)}.$$

We have the following result.

Lemma 4.2. *There exists a constant $m_\varepsilon > 0$ such that for all $m \geqslant m_\varepsilon$ we have $\mathbb{E}_m \neq \emptyset$ and $c_m > 0$.*

In order to prove Lemma 4.2, we need the following result. Let $L\colon L^2(\Omega) \to V$ be the operator defined by

$$L\omega = u \Leftrightarrow \begin{cases} -\Delta u + u = \omega & \text{in } \Omega, \\ u \in V, \\ -\int_{\partial\Omega} \partial u/\partial n = 0, \end{cases}$$

where $\omega \in L^2(\Omega)$.

Let ξ_0 be the solution of the problem

$$\begin{cases} -\Delta \xi_0 + \xi_0 = 0 & \text{in } \Omega, \\ \xi_0 \in V, \\ -\int_{\partial\Omega} \partial \xi_0/\partial n = I. \end{cases}$$

For any $r > 0$, let B_r be the ball of $H^1(\Omega)$ centered at the origin with radius r.

Define the operator $\psi_1\colon V \to V$ as

$$\psi_1(v) = L(v + \lambda_1 v_+) + \xi_0.$$

Then by [2] (see also [29], [10]), we have the following result.

Proposition 4.1. *The topological degree $d(\mathbb{I} - \psi_1, B_{C_{h\varepsilon}}, 0) = -1$, where $C_{h\varepsilon}$ is the constant as in* (iii) *of Lemma* 4.1 *and $\mathbb{I}$ is the identity operator. Moreover, the operator ψ_1 has a unique fixed point $w_1 \in V$, i.e. $\psi_1(w_1) = w_1$ and w_1 is also the unique solution of the problem*

$$(Q)\begin{cases} -\Delta w_1 = \lambda_1 w_{1+} & \text{in } \Omega, \\ w_1 = 0 & \text{on } \partial\Omega, \\ -\int_{\partial\Omega} \partial w_1/\partial n = I. \end{cases}$$

Proof of Lemma 4.2. *Step 1*: We prove that there exists a constant $m_\varepsilon > 0$ such that $\mathbb{E}_m \neq \emptyset$ for all $m \geqslant m_\varepsilon$.

By the result of [10] (see also [2]), we have

$$T_m^0(v) = \widehat{P}_m[L(v + \lambda_1 v_+) + \xi_0] = \widehat{P}_m[\psi_1(v)],$$

where $\widehat{P}_m$ is the orthogonal projection of V onto V_m.

Since $\widehat{P}_m\psi_1(\cdot)$ is a uniform compact perturbation of the operator $\psi_1(\cdot)$ on $B_{h\varepsilon}$ (see [10]), by the above equality and Proposition 4.1 we conclude that there exists $m_\varepsilon > 0$ such that

$$\begin{aligned} d(\mathbb{I} - T_m^0, V_m \cap B_{C_{h\varepsilon}}, 0) &= d(\mathbb{I} - \widehat{P}_m\psi_1(v), V_m \cap B_{C_{h\varepsilon}}, 0) \\ &= d(\mathbb{I} - \psi_1(v), B_{C_{h\varepsilon}}, 0) = -1. \end{aligned}$$

Thus by the Kronecker existence theorem, we deduce that there exists a function $u_m^0 \in V_m$ such that $T_m^0 u_m^0 = u_m^0$. Hence, $\mathbb{E}_m \neq \emptyset$ for all $m \geqslant m_\varepsilon$.

Step 2: We prove that $c_m > 0$ for all $m \geqslant m_\varepsilon$.

We argue by contradiction. Assume that there exists $m \geqslant m_\varepsilon$ such that $c_m = 0$. Note that $\mathbb{E}_m \neq \emptyset$. By the definition of c_m we conclude that for any $0 < \sigma < 1$ there exist $t_{m\sigma} \in [0,1]$ and $u_m^{t_{m\sigma}} \in E_m^{t_{m\sigma}}$ such that

$$0 \leqslant \|u_m^{t_{m\sigma}}\|_{H^1(\Omega)} \leqslant \sigma. \tag{4.11}$$

Since $u_m^{t_{m\sigma}} \in E_m^{t_{m\sigma}}$, taking $t = t_{m\sigma}$ in (4.10), we get by definition

$$\int_\Omega \nabla u_m^{t_{m\sigma}}(x)\nabla w(x)\,\mathrm{d}x = -Iw_{|\partial\Omega} + \int_\Omega G_{t_{m\sigma}\varepsilon}(x, u_m^{t_{m\sigma}}, (u_{m*}^{t_{m\sigma}})')w(x)\,\mathrm{d}x. \tag{4.12}$$

Thus by (4.11) we deduce that there exist a subsequence of $\{t_{m\sigma}\}$ (still denoted by $\{t_{m\sigma}\}$) and $t_m \in [0,1]$ such that $t_{m\sigma} \to t_m$ and

$$u_m^{t_{m\sigma}} \to 0 \quad \text{strongly in } H^1(\Omega) \text{ and a.e. in } \Omega, \text{ as } \sigma \to 0. \tag{4.13}$$

Since $g_\varepsilon(x)\chi_{\{u_m^{t_{m\sigma}}>0\}}$ is bounded uniformly in $L^\infty(\Omega)$ with respect to σ, there exists a function $\tilde{g}_{m\varepsilon} \in L^\infty(\Omega)$ such that

$$g_\varepsilon(x)\chi_{\{u_m^{t_{m\sigma}}>0\}} \rightharpoonup \tilde{g}_{m\varepsilon}(x) \quad \text{weakly} * \text{ in } L^\infty(\Omega). \tag{4.14}$$

Let σ tend to zero in (4.12). By Lemma 3.2, (4.13) and (4.14), we can conclude that

$$t_m \int_\Omega \tilde{g}_{m\varepsilon}(x)w(x)\,\mathrm{d}x = Iw_{|\partial\Omega} \quad \forall w \in V_m. \tag{4.15}$$

We distinguish the cases $t_m = 0$ and $t_m \neq 0$.

Case (i): $t_m = 0$.

Choosing $w \equiv 1$ in (4.15), we get $I = 0$ which is a contradiction.

Case (ii): $t_m \neq 0$.

We have $0 < t_m \leqslant 1$. Since $g_\varepsilon(x)\chi_{\{u_m^{t_m\sigma}>0\}} \geqslant 0$ in Ω, by (4.14) we deduce that $\tilde{g}_{m\varepsilon}(x) \geqslant 0$ in Ω. Choosing $w \equiv 1$ in (4.15), we obtain

$$\int_\Omega \tilde{g}_{m\varepsilon}(x)\,\mathrm{d}x = \frac{I}{t_m} > 0. \tag{4.16}$$

Choosing $w = \varphi_1$ in (4.15), we get

$$\int_\Omega \tilde{g}_{m\varepsilon}(x)\varphi_1\,\mathrm{d}x = 0 > 0. \tag{4.17}$$

Using the facts that $\varphi_1(x) > 0$ and $\tilde{g}_{m\varepsilon}(x) \geqslant 0$ in Ω, we conclude that (4.16) contradicts (4.17).

Hence, the proof of Lemma 4.2 is completed. □

Obviously, $\|w_1\|_{H^1(\Omega)} > 0$ since $w_1 \not\equiv 0$ is the unique solution of problem (Q). Set

$$\tilde{c}_m = \min\Big\{\frac{c_m}{2}, \frac{1}{2}\|w_1\|_{H^1(\Omega)}\Big\}.$$

We conclude that $\tilde{c}_m < C_{h\varepsilon}$, where $C_{h\varepsilon}$ is the constant as in (iii) of Lemma 4.1. We have the following lemma.

Lemma 4.3. *The topological degree $d(\mathbb{I} - T_m^t, V_m \cap B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}, 0)$ is well defined and*

$$d(\mathbb{I} - T_m^t, V_m \cap B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}, 0) = -1 \quad \textit{for } m \geqslant m_\varepsilon,$$

where T_m^t is defined in (4.10) *and m_ε is defined as in Lemma* 4.2.

Proof. The proof is divided into four steps. In Step 1–Step 3, we will check that $d(\mathbb{I} - T_m^t, V_m \cap B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}, 0)$ is well defined. In Step 4, we complete the proof.

Step 1: we prove that no fixed point $u_m^t \in V_m$ of T_m^t can be a constant. We argue by contradiction. Suppose that $u_m^t \equiv c$ is a fixed point of T_m^t, where $c \in \mathbb{R}$. Choosing $w = 1$ in (4.10), we have $\int_\Omega G_{t\varepsilon}(c)(x)\,\mathrm{d}x = I > 0$, which implies that $c > 0$.

Taking φ_1 as a test function in (4.10), we get

$$\int_\Omega G_{t\varepsilon}(c)(x)\varphi_1(x)\,\mathrm{d}x = \int_\Omega tg_\varepsilon(x)\varphi_1(x) + (1-t)\lambda_1 c\varphi_1\,\mathrm{d}x = 0. \tag{4.18}$$

Case 1.1. Let $t \in (1/2, 1]$.

Since φ_1 is positive in Ω, using the fact that $g_\varepsilon \geqslant 0$ and (4.7), we have

$$\int_\Omega tg_\varepsilon(x)\varphi_1(x) + (1-t)\lambda_1 c\varphi_1\,\mathrm{d}x > \frac{1}{2}\int_\Omega g_\varepsilon(x)\varphi_1(x) > 0,$$

which is a contradiction with (4.18).

Case 1.2. Let $t \in [0, 1/2]$.

We also have

$$\int_\Omega t g_\varepsilon(x)\varphi_1(x) + (1-t)\lambda_1 c\varphi_1 \,\mathrm{d}x \geqslant \frac{c\lambda_1}{2}\int_\Omega \varphi_1 \,\mathrm{d}x > 0,$$

which is a contradiction with (4.18).

Step 2: Setting $E_m = V_m \cap B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}$, we prove that T_m^t is a continuous and compact operator in $\overline{E}_m$.

Let $\{v_n\} \subset \overline{E}_m$ and let $v \in \overline{E}_m$ be such that $v_n \to v$ in V_m. By the definition of V_m and the fact that $\varphi_1, \varphi_2, \ldots, \varphi_m$ are analytic functions, it is easy to see that $V_m \subseteq \widetilde{V} \cup R$. In order to prove the continuity of the operator T_m^t, we distinguish the case $v \in \overline{E}_m \cap \widetilde{V}$ from the case $v \in \overline{E}_m \cap R$.

Case 2.1. Let $v \in \overline{E}_m \cap \widetilde{V}$.

In this case, we have

$$\chi_{\{v_n>0\}} \to \chi_{\{v>0\}} \quad \text{a.e. in } \Omega. \tag{4.19}$$

Using the above relations and Lemma 3.2, we can conclude that

$$G_{t\varepsilon}(v_n) \to G_{t\varepsilon}(v) \quad \text{strongly in } L^{2/q}(\Omega). \tag{4.20}$$

Case 2.2. Let $v \in \overline{E}_m \cap R$.

In this case, we have $v \equiv c$, where c is a constant such that $\tilde{c}_m \leqslant |c| \leqslant C_{h\varepsilon}$. Since $v_n \to v$ in V_m, there exists $n_0 > 0$ such that for all $n > n_0$,

$$v_n > \frac{\tilde{c}_m}{2} \quad \text{if } c > 0$$

and

$$v_n < -\frac{\tilde{c}_m}{2} \quad \text{if } c < 0.$$

For $w \in V$, define

$$H_\varepsilon(w) = \lambda\varphi(T_h(w_+))\frac{|k(|w > w(x)|)w_*'(|w > w(x)|)|^q\chi_{\{w>0\}}}{1 + \varepsilon|k(|w > w(x)|)w_*'(|w > w(x)|)|^q}.$$

If $c > 0$, then $\{v_n > 0\} = \Omega = \{v > 0\}$ for all $n > n_0$, which implies that (4.19) holds true for $v \in \overline{E}_m \cap R$. On the other hand, by Lemma 2.1 and Remark 3.2 we have $H_\varepsilon(v_n) = 0 = H_\varepsilon(c)$ for $v_n \in \overline{E}_m \cap R$ and

$$\|H_\varepsilon(v_n)\|_{L^{2/q}(\Omega)} \leqslant c_h\|\nabla v_n\|_{L^2(\Omega)}^{q/2} \quad \text{for } v_n \in \overline{E}_m \cap \widetilde{V},$$

where c_h is a positive constant depending only on h. Hence, we have that $H_\varepsilon(v_n)$ converges strongly to $H_\varepsilon(c)$. In view of this and (4.19), we then conclude (4.20) also holds for $v = c > 0$.

If $c < 0$ then $\chi_{\{v_n>0\}} = \chi_{\{v>0\}} = 0$ for all $n > n_0$. The conclusion (4.20) follows immediately.

The continuity of T_m^t follows from (4.20) immediately. Using the same argument, we obtain the compactness of T_m^t.

Step 3: We prove that $[\mathbb{I} - T_m^t](v) \neq 0$ for all $v \in \partial(B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}) \cap V_m$.

Indeed, if $[\mathbb{I} - T_m^t](v) = 0$, then using Lemma 4.1 and Lemma 4.2, we have $\tilde{c}_m < \|v\|_{H^1(\Omega)} < C_{h\varepsilon}$. Hence, for all $v \in \partial(B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}) \cap V_m$ we have $[\mathbb{I} - T_m^t](v) \neq 0$.

By Step 1–Step 3, we obtain that the topological degree $d(\mathbb{I} - T_m^t, V_m \cap B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}, 0)$ is well defined.

Step 4: Since the rest of the proof is similar to Theorem 1.3 in [10] (see also [2]), we only sketch it here.

By invariance under homotopy and Step 1 of the proof of Lemma 4.2, we obtain

$$\begin{aligned} d(\mathbb{I} - T_m^t, V_m \cap B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}, 0) &= d(\mathbb{I} - T_m^0, V_m \cap B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}, 0) \\ &= d(I - \psi_1(v), B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}, 0) = d(I - \psi_1(v), B_{C_{h\varepsilon}}, 0) = -1. \end{aligned} \tag{4.21}$$

Hence, the proof is complete. □

P r o o f of Theorem 4.2. By Lemma 4.3, we know that for all $t \in [0,1]$ there exists at least one sequence $\{u_m^t\}$ such that

$$T_m^t u_m^t = u_m^t, \quad u_m^t \in V_m \cap B_{C_{h\varepsilon}} \setminus \overline{B}_{\tilde{c}_m}.$$

Taking $t = 1$ in (4.10) and denoting $u_m = u_m^1$, we get by definition

$$\begin{aligned} &\int_\Omega \nabla u_m(x) \nabla w(x)\,\mathrm{d}x \\ &\qquad = -I w_{|\partial\Omega} + \int_\Omega G_{h\varepsilon}(x, u_m, u'_{m*}) w(x)\,\mathrm{d}x \quad \forall w \in V_m\ \forall m \in \mathbb{N} \end{aligned} \tag{4.22}$$

and

$$\|u_m\|_{H^1(\Omega)} \leqslant C_{h\varepsilon} \quad \forall m \in \mathbb{N}. \tag{4.23}$$

Setting $\widetilde{V}_m = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$, from (4.22) we easily have

$$\int_\Omega [-\Delta u_m(x) - \widetilde{P}_m G_{h\varepsilon}(x, u_m, u'_{m*})] w(x)\,\mathrm{d}x = 0 \quad \forall w \in \widetilde{V}_m,$$

where $\widetilde{P}_m$ is the orthogonal projection operator from $L^2(\Omega)$ onto $\widetilde{V}_m$.

The above relation implies that $-\Delta u_m(x) = \widetilde{P}_m G_{h\varepsilon}(x, u_m, u'_{m*})$. By (4.23), we get

$$\|-\Delta u_m\|_{L^2(\Omega)} = \|\widetilde{P}_m G_{h\varepsilon}(x, u_m, u'_{m*})\|_{L^2(\Omega)} \leqslant \widetilde{C}_{h\varepsilon} \quad \forall m \in \mathbb{N}, \tag{4.24}$$

where $\widetilde{C}_{h\varepsilon}$ is a positive constant independent of m.

From (4.23) and (4.24), by standard regularity results, we deduce that $\{u_m\}$ is uniformly bounded in $H^2(\Omega)$ with respect to m. Thus by using the Sobolev embedding theorem, we deduce that there exists a subsequence of $\{u_m\}$ (still denoted by $\{u_m\}$) and a function $u_{h\varepsilon} \in H^2(\Omega)$ such that

$$u_m \rightharpoonup u_{h\varepsilon} \quad \text{weakly in } H^2(\Omega) \tag{4.25}$$

and

$$u_m \to u_{h\varepsilon} \quad \text{strongly in } H^1(\Omega) \text{ and a.e. in } \Omega. \tag{4.26}$$

Since $g_\varepsilon(x)\chi_{\{u_m>0\}}$ is bounded uniformly in $L^2(\Omega)$ with respect to m, there exist a subsequence of $\{g_\varepsilon(x)\chi_{\{u_m>0\}}\}$ (still denoted by $\{g_\varepsilon(x)\chi_{\{u_m>0\}}\}$) and a function $\tilde{g}_\varepsilon \in L^2(\Omega)$ such that

$$g_\varepsilon(x)\chi_{\{u_m>0\}} \rightharpoonup \tilde{g}_\varepsilon(x) \quad \text{weakly in } L^2(\Omega). \tag{4.27}$$

By Lemma 3.2, (4.22), (4.26), and (4.27), we conclude that $u_{h\varepsilon}$ satisfies

$$\begin{aligned} \int_\Omega \nabla u_{h\varepsilon}(x)\nabla w(x)\,\mathrm{d}x &= \int_\Omega \lambda\varphi(T_h(u_{h\varepsilon+})) \\ &\quad\times \frac{|k(|u_{h\varepsilon} > u_{h\varepsilon}(x)|)u'_{h\varepsilon*}(|u_{h\varepsilon} > u_{h\varepsilon}(x)|)|^q \chi_{\Omega\setminus P(u_{h\varepsilon})}\chi_{\{u_{h\varepsilon}>0\}}}{1+\varepsilon|k(|u_{h\varepsilon} > u_{h\varepsilon}(x)|)u'_{h\varepsilon*}(|u_{h\varepsilon} > u_{h\varepsilon}(x)|)|^q} w(x)\,\mathrm{d}x \\ &\quad+ \int_\Omega \tilde{g}_\varepsilon(x)w(x)\,\mathrm{d}x - Iw_{|\partial\Omega} \quad \forall w \in V. \end{aligned} \tag{4.28}$$

Now we analyze the term $\tilde{g}_\varepsilon$ of (4.27). Note that $u_{h\varepsilon} \in H^2(\Omega)$ and it follows from (4.28) that

$$\begin{aligned} -\Delta u_{h\varepsilon} &= \lambda\varphi(T_h(u_{h\varepsilon+})) \\ &\quad\times \frac{|k(|u_{h\varepsilon} > u_{h\varepsilon}(x)|)u'_{h\varepsilon*}(|u_{h\varepsilon} > u_{h\varepsilon}(x)|)|^q \chi_{\Omega\setminus P(u_{h\varepsilon})}\chi_{\{u_{h\varepsilon}>0\}}}{1+\varepsilon|k(|u_{h\varepsilon} > u_{h\varepsilon}(x)|)u'_{h\varepsilon*}(|u_{h\varepsilon} > u_{h\varepsilon}(x)|)|^q} \\ &\quad+ \tilde{g}_\varepsilon(x) \text{ a.e. in } \Omega. \end{aligned} \tag{4.29}$$

In view of (4.29), we have

$$\tilde{g}_\varepsilon(x) = 0 \quad \text{a.e. in } P(u_{h\varepsilon}). \tag{4.30}$$

On the other hand, it is easy to see that

$$\{u_{h\varepsilon} > 0\} \subseteq \overline{\lim_{m\to\infty}} \{u_m > 0\} \subseteq \{u_{h\varepsilon} \geqslant 0\}.$$

Using the fact that $\overline{\lim}_{m\to\infty} \chi_{\{u_m>0\}} = \chi_{\overline{\lim}_{m\to\infty}\{u_m>0\}}$ and the above relation, we obtain

$$\chi_{\{u_{h\varepsilon}>0\}} \leqslant \overline{\lim_{m\to\infty}} \chi_{\{u_m>0\}} \leqslant \chi_{\{u_{h\varepsilon}\geqslant 0\}}. \tag{4.31}$$

Since $g_\varepsilon \geqslant 0$, (4.27) and (4.31) make it possible to conclude that

$$g_\varepsilon(x)\chi_{\{u_{h\varepsilon}>0\}} \leqslant \tilde{g}_\varepsilon(x) \leqslant g_\varepsilon(x)\chi_{\{u_{h\varepsilon}\geqslant 0\}} \quad \text{a.e. in } \Omega. \tag{4.32}$$

Due to (4.30) and (4.32), we deduce that

$$\tilde{g}_\varepsilon(x) = g_\varepsilon(x)\chi_{\{u_{h\varepsilon}>0\}} \quad \text{a.e. in } \Omega. \tag{4.33}$$

Taking $w = 1$ in (4.28), recalling (4.2), (4.29), and (4.33) makes it possible to obtain

$$-\int_{\partial\Omega} \frac{\partial u_{h\varepsilon}}{\partial n} = \int_\Omega \widetilde{G}_{h\varepsilon}(x, u_{h\varepsilon}, u'_{h\varepsilon *})\,\mathrm{d}x = I. \tag{4.34}$$

In view of (4.29), (4.33), and (4.34), the conclusion of Theorem 4.2 follows immediately. □

P r o o f of Theorem 4.1. Similarly to the proof of Lemma 4.1, we conclude that there exists a positive constant C_h independent of ε such that

$$\|u_{h\varepsilon}\|_{H^1(\Omega)} \leqslant C_h. \tag{4.35}$$

By standard regularity results, (4.29), and (4.35), we deduce that $\{u_{h\varepsilon}\}$ is uniformly bounded in $W^{2,\alpha}(\Omega) \cap H^1(\Omega)$ with respect to ε. Thus there exists a subsequence of $\{u_{h\varepsilon}\}$ (still denoted by $\{u_{h\varepsilon}\}$) and a function $u_h \in W^{2,\alpha}(\Omega) \cap H^1(\Omega)$ such that

$$u_{h\varepsilon} \rightharpoonup u_h \quad \text{weakly in } W^{2,\alpha}(\Omega) \cap H^1(\Omega) \tag{4.36}$$

and

$$u_{h\varepsilon} \to u_h \quad \text{strongly in } W^{1,\alpha}(\Omega) \text{ and a.e. in } \Omega. \tag{4.37}$$

Similarly to the proof of (4.27) and (4.33), we deduce that there exists a sequence of $\{g_\varepsilon(x)\chi_{\{u_{h\varepsilon}>0\}}\}$ (still denoted by $\{g_\varepsilon(x)\chi_{\{u_{h\varepsilon}>0\}}\}$) such that

$$g_\varepsilon(x)\chi_{\{u_{h\varepsilon}>0\}} \rightharpoonup g(x)\chi_{\{u_h>0\}} \quad \text{weakly in } L^r(\Omega). \tag{4.38}$$

By Lemma 3.1, (4.28), (4.36)–(4.38), we can conclude that u_h satisfies

$$\int_\Omega \nabla u_h(x)\nabla w(x)\,\mathrm{d}x = -Iw_{|\partial\Omega} + \int_\Omega \widetilde{G}_h(x,u_h,u'_{h*})w(x)\,\mathrm{d}x \quad \forall w\in V, \tag{4.39}$$

where $\widetilde{G}_h$ is defined as in (4.4). Note that $u_h \in W^{2,\alpha}(\Omega)$, and the conclusion of Theorem 4.1 follows immediately from (4.39). □

4.2. Proof of the main results. In order to prove the main results, we shall first look for some uniform estimates of the sequence $\{u_h\}$, where u_h is a solution of $(\mathscr{P}_h)$ given in Theorem 4.1. We have the following results.

Theorem 4.3. *Under the same assumptions as in Theorem* 1.1, *if* u_h *is a solution of* $(\mathscr{P}_h)$ *given in Theorem* 4.1, *then there exists a positive constant* M *independent of* h *such that*

$$\|u_{h+}\|_{L^\infty(\Omega)} \leqslant M.$$

Before giving the proof of Theorem 4.3, we need the following lemma.

Lemma 4.4. *Under the same assumptions as in Theorem* 1.1, *if* u_h *is the solution of problem* $(\mathscr{P}_h)$ *given in Theorem* 4.1, *then*

$$u_{h|\partial\Omega} = \gamma_h < 0 \tag{4.40}$$

and

$$W_h(u_{h+*}(0)) \leqslant \frac{1}{4\pi}\int_0^{|\Omega|}\frac{1}{\theta}\int_0^\theta \Big(g_*(s) + \frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s\,\mathrm{d}\theta, \tag{4.41}$$

where the function W_h *is defined by*

$$W_h(s) = \int_0^s \exp\Big(-\frac{q}{8\pi}\int_\theta^s \varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\Big)\,\mathrm{d}\theta \quad \textit{for } s\in\mathbb{R}. \tag{4.42}$$

P r o o f. First of all, we prove that $u_{h|\partial\Omega} = \gamma_h < 0$. We argue by contradiction. If $\gamma_h \geqslant 0$, using the maximum principle it is easy to see that $u_h > 0$ in Ω. Then we have

$$I = -\int_{\partial\Omega}\frac{\partial u_h}{\partial n} = \int_\Omega \widetilde{G}_h(x,u_h,u'_{h*})\,\mathrm{d}x \geqslant \int_\Omega g(x)\,\mathrm{d}x > I,$$

which is a contradiction. Thus, $u_{h|\partial\Omega} = \gamma_h < 0$.

Now we prove that (4.41) holds. Let $S_{\theta,t}$ be a real function defined for $\theta > 0$, $t > 0$ by

$$S_{\theta,t}(r) = \begin{cases} 1, & r > \theta + t, \\ \dfrac{r-\theta}{t}, & \theta \leqslant r \leqslant \theta + t, \\ 0, & r \leqslant \theta. \end{cases}$$

Since $u_h \in H^1(\Omega) \cap W^{2,\alpha}(\Omega)$ and $u_{h|\partial\Omega} = \gamma_h \leqslant 0$, it is easy to see that $S_{\theta,t}(u_{h+}) \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Multiplying the equation in problem $(\mathscr{P}_h)$ by $S_{\theta,t}(u_{h+})$ and integrating by parts, we have

$$\frac{1}{t}\int_{\{\theta < u_{h+} \leqslant \theta + t\}} |\nabla u_{h+}|^2 \,\mathrm{d}x = \int_{\{u_{h+} > \theta\}} \widetilde{G}_h(x, u_h, u'_{h*}) S_{\theta,t}(u_{h+}) \,\mathrm{d}x. \tag{4.43}$$

It is easy to see that

$$u_{h+*}(s) = u_{h*}(s) \geqslant 0 \quad \text{for } s \in [0, |u_h > 0|],$$

which implies that

$$\begin{aligned} &|k(|u_h > u_h(x)|) u'_{h*}(|u_h > u_h(x)|)|^q \chi_{\Omega \setminus P(u_h)} \chi_{\{u_h > 0\}} \\ &\quad = |k(|u_{h+} > u_{h+}(x)|) u'_{h+*}(|u_{h+} > u_{h+}(x)|)|^q \chi_{\Omega \setminus P(u_{h+})} \chi_{\{u_{h+} > 0\}} \\ &\qquad \text{a.e. } x \in \Omega. \end{aligned} \tag{4.44}$$

Let us define $v_h = u_{h+}$. From (4.43) and (4.44), we have

$$\begin{aligned} \frac{1}{t}\int_{\{\theta < v_h \leqslant \theta + t\}} |\nabla v_h|^2 \,\mathrm{d}x &= \int_{\{v_h > \theta\}} \widetilde{G}_h(x, v_h, v'_{h*}) S_{\theta,t}(v_h) \,\mathrm{d}x \\ &\leqslant \int_{\{v_h > \theta\}} \widetilde{G}_h(x, v_h, v'_{h*}) \,\mathrm{d}x. \end{aligned} \tag{4.45}$$

By the definition of $\widetilde{G}_h$, applying the Hölder inequality and Young's inequality in (4.45), we have

$$\begin{aligned} &\frac{1}{t}\int_{\{\theta < v_h \leqslant \theta + t\}} |\nabla v_h|^2 \,\mathrm{d}x \\ &\quad \leqslant \frac{q}{2}\int_{\{v_h > \theta\}} (\varphi(T_h(v_h)))^{2/q} |k(|v_h > v_h(x)|) v'_{h*}(|v_h > v_h(x)|)|^2 \chi_{\Omega \setminus P(v_h)} \,\mathrm{d}x \\ &\qquad + \int_{\{v_h > \theta\}} \Big(g(x) + \frac{(2-q)\lambda^{2/(2-q)}}{2}\Big) \,\mathrm{d}x. \end{aligned} \tag{4.46}$$

Let t tend to zero in (4.46). By Lemma 2.1, we get

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\{v_h>\theta\}}|\nabla v_h|^2\,\mathrm{d}x
&\leqslant \frac{q}{2}\int_{\{v_h>\theta\}}(\varphi(T_h(v_h)))^{2/q}|k(|v_h>v_h(x)|)v_{h*}'(|v_h>v_h(x)|)|^2\chi_{\Omega\setminus P(v_h)}\,\mathrm{d}x\\
&\quad+\int_{\{v_h>\theta\}}\Big(g(x)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}x\\
&\leqslant \frac{q}{2}\int_{\{v_{h*}>\theta\}}(\varphi(T_h(v_{h*})))^{2/q}|k(s)v_{h*}'(s)|^2\chi_{\Omega\setminus P(v_{h*})}\,\mathrm{d}s\\
&\quad+\int_0^{|v_h>\theta|}\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
(4.47)\qquad -\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\{v_h>\theta\}}|\nabla v_h|^2\,\mathrm{d}x
&\leqslant \frac{q}{2}\int_\theta^\infty\Big(-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\{v_{h*}>\tau\}}(\varphi(T_h(v_{h*})))^{2/q}|k(s)v_{h*}'(s)|^2\,\mathrm{d}s\Big)\,\mathrm{d}\tau\\
&\quad+\int_0^{|v_h>\theta|}\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s.
\end{aligned}
$$

By Lemma 3.3 (here $p=2$) we have

$$
\begin{aligned}
\frac{1}{t}\int_{\{\tau<v_{h*}\leqslant\tau+t\}}&(\varphi(T_h(v_{h*})))^{2/q}|k(s)v_{h*}'(s)|^2\,\mathrm{d}s\\
&\leqslant \frac{1}{t4\pi}\int_{\{\tau<v_h\leqslant\tau+t\}}(\varphi(T_h(v_h)))^{2/q}|\nabla v_h|^2\,\mathrm{d}x,
\end{aligned}
$$

and thus

$$
\begin{aligned}
(4.48)\qquad -\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\{v_{h*}>\tau\}}&(\varphi(T_h(v_{h*})))^{2/q}|k(s)v_{h*}'(s)|^2\,\mathrm{d}s\\
&\leqslant -\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\{v_h>\tau\}}(\varphi(T_h(v_h)))^{2/q}|\nabla v_h|^2\,\mathrm{d}x.
\end{aligned}
$$

Inequality (4.47) and (4.48) show that

$$
\begin{aligned}
(4.49)\qquad -\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\{v_h>\theta\}}|\nabla v_h|^2\,\mathrm{d}x &\leqslant \frac{q}{8\pi}\int_\theta^\infty(\varphi(T_h(\tau)))^{2/q}\Big(-\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\{v_h>\tau\}}|\nabla v_h|^2\,\mathrm{d}x\Big)\,\mathrm{d}\tau\\
&\quad+\int_0^{|v_h>\theta|}\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s.
\end{aligned}
$$

Using Gronwall's Lemma and Lemma 2.2, we then get

(4.50)

$$\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\{v_h>\theta\}}|\nabla v_h|^2\,\mathrm{d}x\\
&\quad\leqslant \int_\theta^\infty \exp\left(\frac{q}{8\pi}\int_\theta^s(\varphi(T_h(\tau)))^{2/q}\,\mathrm{d}\tau\right)\Big(g_*(\mu_{v_h}(s))+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)(-\,\mathrm{d}\mu_{v_h}(s))\\
&\quad\leqslant \int_0^{\mu_{v_h}(\theta)} \exp\left(\frac{q}{8\pi}\int_\theta^{v_{h*}(s)}(\varphi(T_h(\tau)))^{2/q}\,\mathrm{d}\tau\right)\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\exp\left(\frac{q}{8\pi}\int_0^\theta \varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\Big(-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\{v_h>\theta\}}|\nabla v_h|^2\,\mathrm{d}x\Big)\\
&\quad\leqslant \int_0^{\mu_{v_h}(\theta)} \exp\left(\frac{q}{8\pi}\int_0^{v_{h*}(s)}\varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s.
\end{aligned}$$

The above inequality and the Hölder inequality imply that

(4.51)

$$\begin{aligned}
&\exp\left(\frac{q}{8\pi}\int_0^\theta \varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\Big(-\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\{v_h>\theta\}}|\nabla v_h|\,\mathrm{d}x\Big)^2\\
&\quad\leqslant -\mu_{v_h}'(\theta)\int_0^{\mu_{v_h}(\theta)} \exp\left(\frac{q}{8\pi}\int_0^{v_{h*}(s)}\varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s.
\end{aligned}$$

We deduce from (4.51) and Lemma 2.8 that

$$\begin{aligned}
&4\pi\exp\left(\frac{q}{8\pi}\int_0^\theta \varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\\
&\quad\leqslant \frac{-\mu_{v_h}'(\theta)}{\mu_{v_h}(\theta)}\int_0^{\mu_{v_h}(\theta)} \exp\left(\frac{q}{8\pi}\int_0^{v_{h*}(s)}\varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s.
\end{aligned}$$

By Lemma 2.2 and integrating the above inequality between $0=v_{h*}(|\Omega|)$ and $v_{h*}(r)$ we find that

$$\begin{aligned}
&4\pi\int_0^{v_{h*}(r)}\exp\left(\frac{q}{8\pi}\int_0^\theta \varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\mathrm{d}\theta\\
&\quad\leqslant \int_r^{|\Omega|}\frac{1}{\theta}\int_0^\theta \exp\left(\frac{q}{8\pi}\int_0^{v_{h*}(s)}\varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\quad\leqslant \exp\left(\frac{q}{8\pi}\int_0^{v_{h*}(0)}\varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\right)\int_0^{|\Omega|}\frac{1}{\theta}\int_0^\theta\Big(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Big)\,\mathrm{d}s\,\mathrm{d}\theta
\end{aligned}$$

for any $r\in[0,|\Omega|]$.

By the above inequality, we obtain

$$4\pi\,\frac{\int_0^{v_{h*}(r)}\exp\bigl(q(8\pi)^{-1}\int_0^\theta \varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\bigr)\,\mathrm{d}\theta}{\exp\bigl(q(8\pi)^{-1}\int_0^{v_{h*}(0)}\varphi^{2/q}(T_h(\tau))\,\mathrm{d}\tau\bigr)} \leqslant \int_0^{|\Omega|}\frac{1}{\theta}\int_0^\theta\Bigl(g_*(s)+\frac{(2-q)\lambda^{2/(2-q)}}{2}\Bigr)\,\mathrm{d}s\,\mathrm{d}\theta.$$

Assertion (4.41) follows immediately from the above inequality. □

P r o o f of Theorem 4.3. Similarly to the proof of Proposition 3.4 in [36], using Lemma 4.5 and condition (1.3) we deduce that there exists a positive constant M independent of h such that $\|u_{h+}\|_{L^\infty(\Omega)} \leqslant M$. We omit the details here. □

P r o o f of Theorem 1.1. Fix $h > M$ and denote $u = u_h$, where u_h is given in Theorem 4.1 and M is defined as in Theorem 4.3. By Theorem 4.3, we find that $u = u_h \leqslant h$ and

$$\widetilde{G}_h(x,u,u'_*) = [\lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\Omega\setminus P(u)} + g(x)]\chi_{\{u>0\}} \quad \text{a.e. } x\in\Omega.$$

Thus we deduce that u satisfies the equation

$$(4.52)\quad -\Delta u = [\lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\Omega\setminus P(u)} + g(x)]\chi_{\{u>0\}} \quad \text{in } \Omega.$$

It is easy to see that

$$\lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\Omega\setminus P(u)} \in \lambda\varphi(u_+)|k(\beta(u(x)))u'_*(\beta(u(x)))|^q.$$

Moreover, we see that u satisfies (2) and (3) in Theorem 1.1. Thus we complete the proof of Theorem 1.1. □

P r o o f of Theorem 1.2. Fix $h > M$ and denote $u = u_h$, where u_h is given in Theorem 4.1 and M is defined as in Theorem 4.3. As before, we see that u satisfies equation (4.52). Now we prove that u is a solution to problem ($\mathscr{P}$) in the standard sense since $g(x) > 0$.

First of all, we claim that

$$(4.53)\qquad |\Omega_1| = |\{x\colon\ u(x) > 0 \text{ and } \nabla u(x) = 0\}| = 0.$$

We argue by contradiction. Supposing that $|\Omega_1| \neq 0$, by the fact that u belongs to $W^{2,\alpha}(\Omega)$ and using the classic result (see §6.18 and §6.19 in [16]), we deduce that $\Delta u = 0$ in Ω_1. Thus, by (4.52) we have

$$0 = -\Delta u(x) = g(x)\chi_{\{u>0\}} = g(x) > 0 \quad \text{in } \Omega_1,$$

which is a contradiction. Thus, (4.53) holds.

By (4.53), we get

$$\begin{aligned}(4.54)\quad &\lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\Omega\setminus P(u)}\chi_{\{u>0\}}\\ &\qquad = \lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\{u>0\}} \quad \text{in } \{u>0\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(4.55)\quad &\lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\Omega\setminus P(u)}\chi_{\{u>0\}}\\ &\qquad = \lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\{u>0\}} = 0 \quad \text{in } \{u\leqslant 0\}.\end{aligned}$$

It follows from (4.54) and (4.55) that

$$\begin{aligned}(4.56)\quad &\lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\Omega\setminus P(u)}\chi_{\{u>0\}}\\ &\qquad = \lambda\varphi(u_+)|k(|u>u(x)|)u'_*(|u>u(x)|)|^q\chi_{\{u>0\}} \quad \text{in } \Omega.\end{aligned}$$

Since $u=u_h$ is a solution to problem $(\mathscr{P}_h)$, by Lemma 4.4 we have $u_{|\partial\Omega}=\gamma<0$ and $-\int_{\partial\Omega}\partial u/\partial n = I$. Thus, from (4.52) and (4.56), we deduce that u is a solution to problem $(\mathscr{P})$ in the standard sense. □

P r o o f of Theorem 1.3. Clearly, under the assumptions of Theorem 1.3, problem $(\mathscr{P}_h)$ admits a solution $u_h\in W^{2,\alpha}(\Omega)\cap H^1(\Omega)$ in the sense of Theorem 4.1. Moreover, it is easy to see that (4.40) holds, i.e. $u_h|_{\partial\Omega}=\gamma_h<0$. Now we prove that $u_{h+}\in L^\infty(\Omega)$. We use the ideas of [4].

Let $\psi_l(s)=s-T_l(s)$ and $\varphi(s)=\int_0^s\varphi^{2/q}(s)$, where $l>0$. Multiplying the equation in problem $(\mathscr{P}_h)$ by $\mathrm{e}^{\varphi(u_{h+})}\psi_l(u_{h+})$ and integrating by parts, we have

$$\begin{aligned}(4.57)\quad &\int_\Omega\varphi^{2/q}(u_{h+})\mathrm{e}^{\varphi(u_{h+})}\psi_l(u_{h+})|\nabla u_{h+}|^2\,\mathrm{d}x+\int_{\{u_{h+}>l\}}\mathrm{e}^{\varphi(u_{h+})}|\nabla u_{h+}|^2\,\mathrm{d}x\\ &\qquad=\int_{\{u_{h+}>l\}}\widetilde{G}_h(x,u_h,u'_{h*})\mathrm{e}^{\varphi(u_{h+})}\psi_l(u_{h+})\,\mathrm{d}x.\end{aligned}$$

Let us define $v=u_{h+}$. Proceeding as in (4.44), we deduce that $\widetilde{G}_h(x,u_h,u'_{h*})=\widetilde{G}_h(x,v,v'_*)$. Then equality (4.57) can be written as

$$\begin{aligned}(4.58)\quad &\int_\Omega\varphi^{2/q}(v)\mathrm{e}^{\varphi(v)}\psi_l(v)|\nabla v|^2\,\mathrm{d}x+\int_{\{v>l\}}\mathrm{e}^{\varphi(v)}|\nabla v|^2\,\mathrm{d}x\\ &\qquad=\int_{\{v>l\}}\widetilde{G}_h(x,v,v'_*)\mathrm{e}^{\varphi(v)}\psi_l(v)\,\mathrm{d}x.\end{aligned}$$

By Lemma 2.1, Remark 3.2 and Young's inequality, we have

$$\int_{\{v>l\}} \widetilde{G}_h(x,v,v'_*)\mathrm{e}^{\varphi(v)}\psi_l(v)\,\mathrm{d}x = \int_{\{v_*>l\}} \lambda\varphi(v_*)|k(s)v'_*(s)|^q\mathrm{e}^{\varphi(v_*)}\psi_l(v_*)\,\mathrm{d}s$$
$$+ \int_{\{v>l\}} g(x)\mathrm{e}^{\varphi(v)}\psi_l(v)\,\mathrm{d}x \leqslant \frac{1}{(2\pi^{1/2})^q}\int_{\{v>l\}} \lambda\varphi(v)|\nabla v|^q\mathrm{e}^{\varphi(v)}\psi_l(v)\,\mathrm{d}x$$
$$+ \int_{\{v>l\}} g(x)\mathrm{e}^{\varphi(v)}\psi_l(v)\,\mathrm{d}x \leqslant \int_{\{v>l\}} \varphi^{2/q}(v)|\nabla v|^2\mathrm{e}^{\varphi(v)}\psi_l(v)\,\mathrm{d}x$$
$$+ \int_{\{v>l\}} [g(x)+C_4\lambda^{2/(2-q)}]\mathrm{e}^{\varphi(v)}\psi_l(v)\,\mathrm{d}x,$$

where C_4 is a positive constant which depends only on q and π.

The above inequality together with (4.58) imply that

$$\int_{\{v>l\}} \mathrm{e}^{\varphi(v)}|\nabla v|^2\,\mathrm{d}x \leqslant \int_{\{v>l\}} [g(x)+C_4\lambda^{2/(2-q)}]\mathrm{e}^{\varphi(v)}\psi_l(v)\,\mathrm{d}x. \tag{4.59}$$

Recalling that $\varphi \in L^{2/q}[0,\infty)$, we obtain that $\mathrm{e}^{\varphi(v)}$ is bounded. Therefore, from (4.59) we deduce that

$$\int_\Omega |\nabla\psi_l(v)|^2\,\mathrm{d}x = \int_{\{v>l\}} |\nabla v|^2\,\mathrm{d}x \leqslant C_5\int_{\{v>l\}} [g(x)+\lambda^{2/(2-q)}]\psi_l(v)\,\mathrm{d}x, \tag{4.60}$$

where C_5 is a positive constant depending only on C_4 and $\mathrm{e}^{\int_0^\infty \varphi(s)\,\mathrm{d}s}$.

Taking into account Stampacchia procedure (see [32]), we conclude that there exists a positive constant C_6 such that

$$\|v\|_{L^\infty(\Omega)} \leqslant C_6.$$

Fixing $h > C_6$ and denoting $u = u_h$, we conclude that $u_+ \in L^\infty(\Omega)$. Arguing as in the proof of Theorem 1.1, we deduce that $u \in W^{2,\alpha}(\Omega)\cap H^1(\Omega)$ is a solution to problem ($\mathscr{P}$) in the sense of Theorem 1.1. Moreover, using the same argument as in Theorem 1.2, we find that the function u is a solution to problem ($\mathscr{P}$) in the standard sense. □

Proof of Theorem 1.4. Since $\varphi \equiv c_0$, we have $\varphi(T_h(s)) = c_0 = \varphi(s)$ for all $h > 0$. Hence, Theorem 1.4 is a direct consequence of Theorem 4.1 and Theorem 1.2. □

[1] *F. J. Almgren Jr., E. H. Lieb*: Symmetric decreasing rearrangement is sometimes continuous. J. Am. Math. Soc. *2* (1989), 683–773.
[2] *H. Berestycki, H. Brézis*: On a free boundary problem arising in plasma physics. Nonlinear Anal., Theory Methods Appl. *4* (1980), 415–436.
[3] *J. Blum*: Numerical Simulation and Optimal Control in Plasma Physics. With Applications to Tokamaks. Wiley/Gauthier-Villars Series in Modern Applied Mathematics. Wiley, Chichester, 1989.
[4] *L. Boccardo, S. Segura de León, C. Trombetti*: Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term. J. Math. Pures Appl. *80* (2001), 919–940.
[5] *R. Courant, D. Hilbert*: Methods of Mathematical Physics Vol. I. Translated and revised from the German original. First English ed. Interscience Publishers, New York, 1953.
[6] *J. I. Díaz, G. Galiano, J. F. Padial*: On the uniqueness of solutions of a nonlinear elliptic problem arising in the confinement of a plasma in a stellarator device. Appl. Math. Optimization *39* (1999), 61–73.
[7] *J. I. Díaz, M. B. Lerena, J. F. Padial, J. M. Rakotoson*: An elliptic-parabolic equation with a nonlocal term for the transient regime of a plasma in a stellarator. J. Differ. Equations *198* (2004), 321–355.
[8] *J. I. Díaz, J. F. Padial, J. M. Rakotoson*: Mathematical treatment of the magnetic confinement in a current carrying stellarator. Nonlinear Anal., Theory Methods Appl. *34* (1998), 857–887.
[9] *J. I. Díaz, J. M. Rakotoson*: On a nonlocal stationary free-boundary problem arising in the confinement of a plasma in a stellarator geometry. Arch. Ration. Mech. Anal. *134* (1996), 53–95.
[10] *A. Ferone, M. Jalal, J. M. Rakotoson, R. Volpicelli*: A topological approach for generalized nonlocal models for a confined plasma in a tokamak. Commun. Appl. Anal. *5* (2001), 159–181.
[11] *A. Ferone, M. Jalal, J. M. Rakotoson, R. Volpicelli*: Nonlocal generalized models for a confined plasma in a tokamak. Appl. Math. Lett. *12* (1999), 43–46.
[12] *A. Fiorenza, J. M. Rakotoson, L. Zitouni*: Relative rearrangement method for estimating dual norms. Indiana Univ. Math. J. *58* (2009), 1127–1150.
[13] *H. Gourgeon, J. Mossino*: Sur un problème à frontière libre de la physique des plasmas. Ann. Inst. Fourier *29* (1979), 127–141. (In French.)
[14] *H. Grad, P. N. Hu, D. C. Stevens*: Adiabatic evolution of plasma equilibrium. Proc. Nat. Acad. Sci. USA *72* (1975), 3789–3793.
[15] *A. Henrot*: Extremum Problems for Eigenvalues of Elliptic Operators. Frontiers in Mathematics. Birkhäuser, Basel, 2006.
[16] *E. H. Lieb, M. Loss*: Analysis. 2nd ed. Graduate Studies in Mathematics 14. American Mathematical Society, Providence, 2001.
[17] *C. Mercier*: The Magnetohydrodynamic Approach to the Problem of a Plasma Confinement in Closed Magnetic Configurations. EURATOM-CEA, Comm. of the European Communities, Luxembourg, 1974.
[18] *J. Mossino*: A priori estimates for a model of Grad-Mercier type in plasma confinement. Appl. Anal. *13* (1982), 185–207.
[19] *J. Mossino*: Application des inéquations quasi-variationnelles à quelques problèmes non linéaires de la physique des plasmas. Isr. J. Math. *30* (1978), 14–50. (In French.)
[20] *J. Mossino*: Inégalités isopérimétriques et applications en physique. Travaux en Cours. Hermann, Paris, 1984. (In French.)

[21] *J. Mossino*: Some nonlinear problems involving a free boundary in plasma physics. J. Differ. Equations *34* (1979), 114–138.
[22] *J. Mossino, R. Temam*: Directional derivative of the increasing rearrangement mapping and application to a queer differential equation in plasma physics. Duke Math. J. *48* (1981), 475–495.
[23] *J. Mossino, R. Temam*: Free boundary problems in plasma physics: review of results and new developments. Free Boundary Problems, Theory and Applications Vol. II. Proc. interdisc. Symp., Montecatini/Italy 1981, Res. Notes Math. 79 (A. Fasano, eds.). Pitman, 1983, pp. 672–681.
[24] *J. M. Rakotoson*: Existence of bounded solutions of some degenerate quasilinear elliptic equations. Commun. Partial Differ. Equations *12* (1987), 633–676.
[25] *J. M. Rakotoson*: Galerkin approximation, strong continuity of the relative rearrangement map and application to plasma physics equations. Differ. Integral Equ. *12* (1999), 67–81.
[26] *J. M. Rakotoson*: Multivalued fixed point index and nonlocal problems involving relative rearrangement. Nonlinear Anal., Theory Methods Appl. *66* (2007), 2470–2499.
[27] *J. M. Rakotoson*: Relative Rearrangement. An Estimation Tool for Boundary Problems. (Réarrangement relatif. Un instrument d'estimations dans les problèmes aux limites). Mathématiques & Applications 64, Springer, Berlin, 2008. (In French.)
[28] *J. M. Rakotoson*: Relative rearrangement for highly nonlinear equations. Nonlinear Anal., Theory Methods Appl. *24* (1995), 493–507.
[29] *J. M. Rakotoson*: Un modèle non local en physique des plasmas: résolution par une méthode de degré topologique. (A nonlocal model in plasma physics: solution by the method of topological degree). Acta Appl. Math. *4* (1985), 1–14. (In French.)
[30] *J. M. Rakotoson, M. L. Seoane*: Numerical approximations of the relative rearrangement: the piecewise linear case. Application to some nonlocal problems. M2AN, Math. Model. Numer. Anal. *34* (2000), 477–499.
[31] *J. M. Rakotoson, R. Temam*: A co-area formula with applications to monotone rearrangement and to regularity. Arch. Ration. Mech. Anal. *109* (1990), 213–238.
[32] *G. Stampacchia*: Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. Ann. Inst. Fourier *15* (1965), 189–257. (In French.)
[33] *R. Temam*: A non-linear eigenvalue problem: the shape at equilibrium of a confined plasma. Arch. Ration. Mech. Anal. *60* (1975), 51–73.
[34] *R. Temam*: Monotone rearrangement of a function and the Grad-Mercier equation of plasma physics. Recent methods in non-linear analysis, Proc. Int. Meet., Rome 1978. Pitagora, Bologna, 1979, pp. 83–98.
[35] *R. Temam*: Remarks on a free boundary value problem arising in plasma physics. Commun. Partial Differ. Equations *2* (1977), 563–585.
[36] *C. Trombetti*: Non-uniformly elliptic equations with natural growth in the gradient. Potential Anal. *18* (2003), 391–404.
[37] *W. Zou, F. Li, B. Lv*: On a nonlocal elliptic problem arising in the confinement of a plasma in a current carrying stellarator. Mathematical Methods in the Applied Sciences *36* (2013), 2128-2144.

Authors' addresses: *Weilin Zou*, College of Mathematics and Information Science, Nanchang Hangkong University, Nanchang, 330063, China, e-mail: `zwl267@yahoo.com.cn`; *Fengquan Li*, School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, China, e-mail: `fqli@dlut.edu.cn`; *Boqiang Lv*, College of Mathematics and Information Science, Nanchang Hangkong University, Nanchang, 330063, China, e-mail: `lbqxlnc@163.com`.