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KING TYPE MODIFICATION OF q-BERNSTEIN-SCHURER OPERATORS

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Abstract. Very recently the q-Bernstein-Schurer operators which reproduce only constant function were introduced and studied by C. V. Muraru (2011). Inspired by J. P. King, Positive linear operators which preserve x^2 (2003), in this paper we modify q-Bernstein-Schurer operators to King type modification of q-Bernstein-Schurer operators, so that these operators reproduce constant as well as quadratic test functions x^2 and study the approximation properties of these operators. We establish a convergence theorem of Korovkin type. We also get some estimations for the rate of convergence of these operators by using modulus of continuity. Furthermore, we give a Voronovskaja-type asymptotic formula for these operators.

Keywords: King type operator; q-Bernstein-Schurer operator; Korovich type approximation theorem; rate of convergence; Voronovskaja-type result; modulus of continuity

MSC 2010: 41A10, 41A25, 41A36

1. Introduction

Let $q > 0$. For each nonnegative integer k, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} (1-q^k)/(1-q), & q \neq 1, \\ k, & q = 1 \end{cases}$$

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and

$$[k]_q! := \begin{cases} [k]_q[k-1]_q \dots [1]_q, & k \geqslant 1, \\ 1, & k = 0, \end{cases}$$

respectively.

Then for $q > 0$ and integers n, k, $n \geqslant k \geqslant 0$, we have

$$[k+1]_q = 1 + q[k]_q \quad \text{and} \quad [k]_q + q^k[n-k]_q = [n]_q.$$

For the integers n, k, $n \geqslant k \geqslant 0$, the q-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Let $q > 0$. For a nonnegative integer n, the q-analogue of $(x-a)^n$ is defined by

$$(x-a)_q^n = \begin{cases} 1, & n = 0, \\ (x-a)(x-qa)\dots(x-q^{n-1}a), & n \geqslant 1. \end{cases}$$

All of the previous concepts can be found in [7], [9].

In 1997 Phillips [15] introduced and studied the q analogue of Bernstein polynomials. After this, the applications of q-calculus in the approximation theory became one of the main areas of research, and many authors studied new classes of q-generalized operators (for instance, see [1], [2], [5], [8], [6], [11], [12], [14]). Very recently Muraru [13] introduced and studied the following q-Bernstein-Schurer operators for any fixed $p \in \mathbb{N} \cup \{0\}$:

$$S_{n,p}(f;q;x) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k} f([k]_q/[n]_q), \tag{1.1}$$

where $x \in [0,1]$, $n \in \mathbb{N}$, $0 < q < 1$ and $f \in C[0, 1+p]$.

The moments of these operators $S_{n,p}(f;q;x)$ were obtained as follows (see [13]):

Remark 1.1. For $S_{n,p}(t^j;q;x)$, $j = 0,1,2$, we have

$$S_{n,p}(1;q;x) = 1,\ S_{n,p}(t;q;x) = \frac{[n+p]_q x}{[n]_q},$$
$$S_{n,p}(t^2;q;x) = \frac{[n+p]_q}{[n]_q^2}([n+p]_q x^2 + x(1-x)).$$

It is well known that the classical Bernstein polynomials preserve constant as well as linear functions. To make the convergence faster, King [10] proposed a method of modified Bernstein polynomials as follows:

$$V_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} f\Big(\frac{k}{n}\Big),$$

where $f \in C[0,1]$, $0 \leqslant x \leqslant 1$, and $r_n^*(x)\colon\ [0,1] \to [0,1]$ are defined by

$$r_n^*(x) = \begin{cases} x^2, & n = 1, \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{n}{n-1}x^2 + \dfrac{1}{4(n-1)^2}}, & n = 2,3,\ldots \end{cases}$$

For $e_i(x) = x^i$, $i = 0,1,2$, these operators $V_n(f;x)$ preserve the test functions e_0, e_2 and $V_n(f;x) = r_n^*(x)$ holds. Replacing $r_n^*(x)$ by e_1, one reobtains Bernstein polynomials.

It can be observed from the above Remark 1.1 that these operators $S_{n,p}(f;q;x)$ reproduce only constant functions. Inspired by King, to make the convergence faster, we can modify these operators so that they reproduce constant as well as quadratic test functions e_2. For this purpose we propose the modification of these operators which were defined above by (1.1) to be

$$\tilde{S}_{n,p}(f;q;x) = \sum_{k=0}^{n+p} \begin{bmatrix} n+p \\ k \end{bmatrix}_q r_{n,p}^k(q,x)(1 - r_{n,p}(q,x))_q^{n+p-k} f([k]_q/[n]_q), \tag{1.2}$$

where $x \in [0,1]$, $n \in \mathbb{N}$, $0 < q < 1$, $f \in C[0,1+p]$, $p \in \mathbb{N} \cup \{0\}$ is fixed,

$$r_{1,p}(q,x) = \begin{cases} x^2, & \text{if } p = 0, \\ \dfrac{-[1+p]_q + \sqrt{[1+p]_q^2 + 4[1+p]_q([1+p]_q - 1)x^2}}{2[1+p]_q([1+p]_q - 1)}, & \text{if } p = 1,2,\ldots \end{cases}$$

$$r_{n,p}(q,x) = \frac{-[n+p]_q + \sqrt{[n+p]_q^2 + 4[n]_q^2[n+p]_q([n+p]_q - 1)x^2}}{2[n+p]_q([n+p]_q - 1)}, \quad n \geqslant 2.$$

Note that $0 \leqslant r_{n,p}(q,x) \leqslant 1$ for $0 < q < 1$, $n \in \mathbb{N}$, $x \in [0,1]$ and fixed $p \in \mathbb{N} \cup \{0\}$.

The aim of the present article is to study approximation properties of these operators $\tilde{S}_{n,p}(f;q;x)$ and to estimate the rate of convergence by using modulus of continuity. Furthermore, we give the quantitative Voronovskaja-type asymptotic formula.

In the paper, C is a positive constant. In different places, the value of C may be different. For $f \in C[0,1+p]$, we denote $\|f\| = \max\{|f(x)|; x \in [0,1+p]\}$.

In the sequel, we shall need the following auxiliary results.

Lemma 2.1. *For $\tilde{S}_{n,p}(t^j;q;x)$, $j=0,1,2,3,4$, we have*

(i) $\tilde{S}_{n,p}(1;q;x)=1$;

(ii) $\tilde{S}_{n,p}(t;q;x)=[n+p]_q r_{n,p}(q,x)/[n]_q$;

(iii) $\tilde{S}_{n,p}(t^2;q;x)=([n+p]_q/[n]_q^2)[[n+p]_q r_{n,p}^2(q,x)+r_{n,p}(q,x)(1-r_{n,p}(q,x))]=x^2$;

(iv) $\tilde{S}_{n,p}(t^3;q;x)=([n+p]_q/[n]_q^3)r_{n,p}(q,x)+((2q+q^2)/[n]_q^3)[n+p]_q[n+p-1]_q\times r_{n,p}^2(q,x)+(q^3/[n]_q^3)[n+p]_q[n+p-1]_q[n+p-2]_q r_{n,p}^3(q,x)$, *for* $n+p\geqslant 2$;

(v) $\tilde{S}_{n,p}(t^4;q;x)=([n+p]_q/[n]_q^4)r_{n,p}(q,x)+((3q+3q^2+q^3)/[n]_q^4)[n+p]_q[n+p-1]_q r_{n,p}^2(q,x)+((3q^3+2q^4+q^5)/[n]_q^4)[n+p]_q[n+p-1]_q[n+p-2]_q r_{n,p}^3(q,x)+(q^6/[n]_q^4)[n+p]_q[n+p-1]_q[n+p-2]_q[n+p-3]_q r_{n,p}^4(q,x)$, *for* $n+p\geqslant 3$.

P r o o f. In view of the definition given by (1.2) and Remark 1.1, we can easily obtain that identities (i), (ii), (iii) hold.

(iv) When $j=3$ and $n+p\geqslant 2$, in view of $[k+1]_q=1+q[k]_q$ we have

$$\begin{aligned}
\tilde{S}_{n,p}(t^3;q;x)&=\sum_{k=1}^{n+p}\begin{bmatrix}n+p\\k\end{bmatrix}_q r_{n,p}^k(q,x)(1-r_{n,p}(q,x))_q^{n+p-k}\Big(\frac{[k]_q}{[n]_q}\Big)^3\\
&=\frac{1}{[n]_q^3}\sum_{k=0}^{n+p-1}\frac{[n+p]_q!(1+2q[k]_q+q^2[k]_q^2)}{[k]_q![n+p-k-1]_q!}r_{n,p}^{k+1}(q,x)(1-r_{n,p}(q,x))_q^{n+p-k-1}\\
&=\frac{[n+p]_q}{[n]_q^3}r_{n,p}(q,x)\\
&\quad+\frac{2q+q^2}{[n]_q^3}\sum_{k=1}^{n+p-1}\frac{[n+p]_q!}{[k-1]_q![n+p-k-1]_q!}r_{n,p}^{k+1}(q,x)(1-r_{n,p}(q,x))_q^{n+p-k-1}\\
&\quad+\frac{q^3}{[n]_q^3}\sum_{k=2}^{n+p-1}\frac{[n+p]_q!}{[k-2]_q![n+p-k-1]_q!}r_{n,p}^{k+1}(q,x)(1-r_{n,p}(q,x))_q^{n+p-k-1}\\
&=\frac{[n+p]_q}{[n]_q^3}r_{n,p}(q,x)+\frac{2q+q^2}{[n]_q^3}[n+p]_q[n+p-1]_q r_{n,p}^2(q,x)\\
&\quad+\frac{q^3}{[n]_q^3}[n+p]_q[n+p-1]_q[n+p-2]_q r_{n,p}^3(q,x).
\end{aligned}$$

(v) When $j=4$ and $n+p\geqslant 3$, similarly to the case of $j=3$ and $n+p\geqslant 2$, by simple calculation we can get the stated result. □

Lemma 2.2. *Let* $0 < q < 1$, $x \in [0,1]$, $n \geqslant 2$ *we have*

(i) $0 \leqslant x - \tilde{S}_{n,p}(t;q;x) \leqslant [n+p]_q/(2[n]_q([n+p]_q - 1))$;

(ii) $\tilde{S}_{n,p}((t-x)^2;q;x) \leqslant [n+p]_q/([n]_q([n+p]_q - 1))$.

P r o o f. (i) For $0 < q < 1$, $n \in \mathbb{N}$, $x \in [0,1]$, by simple calculation we can easily obtain $x - \tilde{S}_{n,p}(t;q;x) = x - ([n+p]_q/[n]_q) r_{n,p}(q,x) \geqslant 0$.

On the other hand, for $n \geqslant 2$ we have

$$
\begin{aligned}
x - \tilde{S}_{n,p}(t;q;x) &= x - \frac{[n+p]_q}{[n]_q} r_{n,p}(q,x) \\
&= \frac{2[n]_q([n+p]_q - 1)x + [n+p]_q - \sqrt{[n+p]_q^2 + 4[n]_q^2[n+p]_q([n+p]_q - 1)x^2}}{2[n]_q([n+p]_q - 1)} \\
&= \frac{1}{2[n]_q([n+p]_q - 1)} \\
&\quad \times \frac{4[n]_q[n+p]_q([n+p]_q - 1)x - 4[n]_q^2([n+p]_q - 1)x^2}{2[n]_q([n+p]_q - 1)x + [n+p]_q + \sqrt{[n+p]_q^2 + 4[n]_q^2[n+p]_q([n+p]_q - 1)x^2}} \\
&\leqslant \frac{1}{2[n]_q([n+p]_q - 1)} \cdot \frac{4[n]_q[n+p]_q([n+p]_q - 1)}{2[n]_q([n+p]_q - 1) + \sqrt{4[n]_q^2([n+p]_q - 1)^2}} \\
&= \frac{[n+p]_q}{2[n]_q([n+p]_q - 1)}.
\end{aligned}
$$

(ii) In view of Lemma 2.1 and (i) above, for $x \in [0,1]$, $n \geqslant 2$ we have

$$
\begin{aligned}
\tilde{S}_{n,p}((t-x)^2;q;x) &= \tilde{S}_{n,p}(t^2;q;x) - 2x\tilde{S}_{n,p}(t;q;x) + x^2 \\
&= 2x(x - \tilde{S}_{n,p}(t;q;x)) \leqslant \frac{[n+p]_q}{[n]_q([n+p]_q - 1)}.
\end{aligned}
$$

□

Lemma 2.3. *For* $f \in C[0,1+p]$, $x \in [0,1]$ *and* $n \in \mathbb{N}$ *we have*

$$|\tilde{S}_{n,p}(f;q;x)| \leqslant \|f\|.$$

P r o o f. In view of the definition given by (1.2) and Lemma 2.1, we have

$$|\tilde{S}_{n,p}(f;q;x)| \leqslant \tilde{S}_{n,p}(1;q;x)\|f\| = \|f\|.$$

□

Let $W^2 = \{g \in C[0,1+p]\colon\ g',\ g'' \in C[0,1+p]\}$. For $\delta > 0$, $f \in C[0,1+p]$, Peetre's K-functional is defined as

$$K_2(f,\delta) = \inf\{\|f-g\| + \delta\|g''\|\colon\ g \in W^2\}.$$

Let $\delta > 0$, $f \in C[0,1+p]$. The second order modulus of smoothness for f is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\leqslant\sqrt{\delta}} \ \sup_{x,x+2h\in[0,1+p]} |f(x+2h) - 2f(x+h) + f(x)|,$$

the usual modulus of continuity for f is defined as

$$\omega(f,\delta) = \sup_{0<h\leqslant\delta} \ \sup_{x,x+h\in[0,1+p]} |f(x+h) - f(x)|.$$

For $f \in C[0,1+p]$, following [4, p. 177, Theorem 2.4], there exists a constant $C > 0$ such that

$$K_2(f,\delta) \leqslant C\omega_2(f,\sqrt{\delta}). \tag{2.1}$$

$\square$

Lemma 2.4. *Let $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For every $x \in (0,1]$ we have*

(i) $\lim\limits_{n\to\infty} [n]_{q_n} \tilde{S}_{n,p}(t-x; q_n; x) = (x-1)/2$;

(ii) $\lim\limits_{n\to\infty} [n]_{q_n} \tilde{S}_{n,p}((t-x)^2; q_n; x) = x(1-x)$;

(iii) $\lim\limits_{n\to\infty} [n]_{q_n} \tilde{S}_{n,p}((t-x)^4; q_n; x) = 0$.

Proof. Assume that $n \geqslant 3$, $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$.

(i) Denote

$$A_n(q_n;x) = \sqrt{[n+p]_{q_n}^2 + 4[n]_{q_n}^2[n+p]_{q_n}([n+p]_{q_n}-1)x^2},$$
$$B_n(q_n;x) = 2[n]_{q_n}([n+p]_{q_n}-1)x + [n+p]_{q_n}.$$

For every $x \in (0,1]$ we have

$$\lim_{n\to\infty} [n]_{q_n} \tilde{S}_{n,p}(t-x; q_n; x) = \lim_{n\to\infty} ([n+p]_{q_n} r_{n,p}(q_n,x) - [n]_{q_n} x)$$
$$= \lim_{n\to\infty} \frac{A_n(q_n;x) - B_n(q_n;x)}{2([n+p]_{q_n}-1)} = \lim_{n\to\infty} \frac{2[n]_{q_n}^2 x^2 - 2[n]_{q_n}[n+p]_{q_n} x}{A_n(q_n;x) + B_n(q_n;x)} = \frac{x-1}{2}.$$

(ii) Since $[n]_{q_n}\tilde{S}_{n,p}((t-x)^2;q_n;x) = -2x[n]_{q_n}\tilde{S}_{n,p}(t-x;q_n;x)$, so, by (i) above we obtain $\lim_{n\to\infty}[n]_{q_n}\tilde{S}_{n,p}((t-x)^2;q_n;x) = x(1-x)$.

(iii) In view of Lemma 2.1, using $[n+p]_{q_n} = [n]_{q_n} + q_n^n[p]_{q_n}$, $\lim_{n\to\infty} r_{n,p}(q_n,x) = x$ and $\lim_{n\to\infty}([n+p]_{q_n} r_{n,p}(q_n,x) - [n]_{q_n}x) = (x-1)/2$, we have

$$
\begin{aligned}
&\lim_{n\to\infty}[n]_{q_n}\tilde{S}_{n,p}((t-x)^4;q_n;x) = \lim_{n\to\infty}[n]_{q_n}[\tilde{S}_{n,p}(t^4;q_n;x) - 4x\tilde{S}_{n,p}(t^3;q_n;x)\\
&\quad + 6x^2\tilde{S}_{n,p}(t^2;q_n;x) - 4x^3\tilde{S}_{n,p}(t;q_n;x) + x^4]\\
&= \lim_{n\to\infty}\Big[\frac{[n+p]_{q_n}([n+p]_{q_n}-1)([n+p]_{q_n}-[2]_{q_n})}{[n]_{q_n}^2} r_{n,p}^4(q_n,x)\\
&\quad - 4x\frac{[n+p]_{q_n}([n+p]_{q_n}-1)}{[n]_{q_n}} r_{n,p}^3(q_n,x) + 6x^2\frac{[n+p]_{q_n}^2}{[n]_{q_n}} r_{n,p}^2(q_n,x)\\
&\quad - 4x^3[n+p]_{q_n} r_{n,p}(q_n,x) + x^4[n]_{q_n}\Big] - x^4 - 3apx^4\\
&= \lim_{n\to\infty}\Big[\frac{[n+p]_{q_n}([n+p]_{q_n}-1)[n+p]_{q_n}}{[n]_{q_n}^2} r_{n,p}^4(q_n,x) - 4x\frac{[n+p]_{q_n}^2}{[n]_{q_n}} r_{n,p}^3(q_n,x)\\
&\quad + 6x^2\frac{[n+p]_{q_n}^2}{[n]_{q_n}} r_{n,p}^2(q_n,x) - 4x^3[n+p]_{q_n} r_{n,p}(q_n,x) + x^4[n]_{q_n}\Big] + x^4 - 3apx^4\\
&= \lim_{n\to\infty}\Big[\frac{[n+p]_{q_n}([n+p]_{q_n}-1)}{[n]_{q_n}^2}([n+p]_{q_n} r_{n,p}(q_n,x) - [n]_{q_n}x) r_{n,p}^3(q_n,x)\\
&\quad - 3x\frac{[n+p]_{q_n}^2}{[n]_{q_n}} r_{n,p}^3(q_n,x) + 6x^2\frac{[n+p]_{q_n}^2}{[n]_{q_n}} r_{n,p}^2(q_n,x)\\
&\quad - 4x^3[n+p]_{q_n} r_{n,p}(q_n,x) + x^4[n]_{q_n}\Big] - 3apx^4\\
&= \lim_{n\to\infty}\Big[-3x\frac{[n+p]_{q_n}}{[n]_{q_n}}([n+p]_{q_n} r_{n,p}(q_n,x) - [n]_{q_n}x) r_{n,p}^2(q_n,x)\\
&\quad - 3[n+p]_{q_n}x^2 r_{n,p}^2(q_n,x) + 6x^2\frac{[n+p]_{q_n}}{[n]_{q_n}}([n+p]_{q_n} r_{n,p}(q_n,x) - [n]_{q_n}x) r_{n,p}(q_n,x)\\
&\quad + 2[n+p]_{q_n}x^3 r_{n,p}(q_n,x) + x^4[n]_{q_n}\Big] - 3apx^4 + \frac{x-1}{2}x^3\\
&= \lim_{n\to\infty}[-3x^2([n+p]_{q_n} r_{n,p}(q_n,x) - [n]_{q_n}x) r_{n,p}(q_n,x)\\
&\quad - x^3([n+p]_{q_n} r_{n,p}(q_n,x) - [n]_{q_n}x)] + 2x^4 - 2x^3 = 0.
\end{aligned}
$$

□

3. Main results

First we give the following convergence theorem for the sequence $\{\tilde{S}_{n,p}(f;q)\}$.

Theorem 3.1. *Let $q_n \in (0,1)$. Then the sequence $\{\tilde{S}_{n,p}(f;q_n)\}$ converges to f uniformly on $[0,1]$ for any $f \in C[0,1+p]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof. Let $q_n \in (0,1)$ and $\lim_{n\to\infty} q_n = 1$, then we have $[n]_{q_n} \to \infty$ as $n \to \infty$ (see [16]). Thus, by Lemma 2.1 and Lemma 2.2, we have $\lim_{n\to\infty} \|\tilde{S}_{n,p}(e_j;q_n;\cdot) - e_j\|_{C[0,1]} = 0$ for $e_j(x) = x^j$, $j = 0,1,2$, where $\|f\|_{C[0,1]} = \max\{|f(x)|; x \in [0,1]\}$. According to the well-known Bohman-Korovkin theorem ([3, p. 40, Theorem 1.9]), we get that the sequence $\{\tilde{S}_{n,p}(f;q_n)\}$ converges to f uniformly on $[0,1]$ for any $f \in C[0,1+p]$.

We prove the converse result by contradiction. If $\{q_n\}$ does not tend to 1 as $n \to \infty$, then it must contain a subsequence $\{q_{n_k}\} \subset (0,1)$ with $n_k \geqslant 2$, such that $\lim_{k\to\infty} q_{n_k} = q_0 \in [0,1)$. Thus

$$\lim_{k\to\infty} \frac{1}{[n_k]_{q_{n_k}}} = \lim_{k\to\infty} \frac{1-q_{n_k}}{1-(q_{n_k})^{n_k}} = 1 - q_0.$$

Taking $n = n_k$, $q = q_{n_k}$ in $\tilde{S}_{n,p}(t;q;x)$, by Lemma 2.1 we get

$$\begin{aligned}
\tilde{S}_{n_k,p}(t;q_{n_k};x) &= \frac{[n_k+p]_{q_{n_k}} r_{n_k,p}(q_{n_k},x)}{[n_k]_{q_{n_k}}} \\
&= \frac{-[n_k+p]_{q_{n_k}} + \sqrt{[n_k+p]_{q_{n_k}}^2 + 4[n_k]_{q_{n_k}}^2 [n_k+p]_{q_{n_k}}([n_k+p]_{q_{n_k}}-1)x^2}}{2[n_k]_{q_{n_k}}([n_k+p]_{q_{n_k}}-1)} \\
&\to -\frac{1-q_0}{2q_0} + \sqrt{\Big(\frac{1-q_0}{2q_0}\Big)^2 + \frac{x^2}{q_0}} \neq x, \quad \text{as } k \to \infty.
\end{aligned}$$

This leads to a contradiction, hence $\lim_{n\to\infty} q_n = 1$. Theorem is proved. □

Next we estimate the rate of convergence.

Theorem 3.2. *Let $f \in C[0,1+p]$, $x \in [0,1]$, $n \geqslant 2$, $q \in (0,1)$ we have $|\tilde{S}_{n,p}(f;q;x) - f(x)| \leqslant 2\omega(f,\delta_n)$, where $\delta_n = \sqrt{[n+p]_q/([n]_q([n+p]_q-1))}$.*

Proof. By Lemma 2.1 we have

$$|\tilde{S}_{n,p}(f;q;x) - f(x)| = |\tilde{S}_{n,p}(f(t)-f(x);q;x)| \leqslant \tilde{S}_{n,p}(|f(t)-f(x)|;q;x).$$

Since for $t \in [0,1+p]$, $x \in [0,1]$ and any $\delta > 0$ we have

$$|f(t) - f(x)| \leqslant (1 + \delta^{-2}(t-x)^2)\omega(f,\delta).$$

We get

$$|\tilde{S}_{n,p}(f;q;x)-f(x)|\leqslant[\tilde{S}_{n,p}(1;q;x)+\delta^{-2}\tilde{S}_{n,p}((t-x)^2;q;x)]\omega(f,\delta).$$

By Lemma 2.1 and Lemma 2.2, for $x\in[0,1]$, $n\geqslant 2$ we have

$$|\tilde{S}_{n,p}(f;q;x)-f(x)|\leqslant(1+\delta^{-2}\delta_n^2)\omega(f,\delta).$$

Taking $\delta=\delta_n$, then from the above inequality we obtain the desired result. □

Corollary 3.1. *Let $M>0$, $0<\alpha\leqslant 1$, $f\in\mathrm{Lip}_M^\alpha$ on $[0,1+p]$ and $n\geqslant 2$, $q\in(0,1)$. Then we have*

$$\|\tilde{S}_{n,p}(f;q;\cdot)-f\|_{C[0,1]}\leqslant 2M\delta_n^\alpha,$$

where δ_n is given in Theorem 3.2.

Proof. Let $M>0$, $0<\alpha\leqslant 1$, $f\in\mathrm{Lip}_M^\alpha$ on $[0,1+p]$. Then we have $f\in C[0,1+p]$. For any $\delta>0$, since $f\in\mathrm{Lip}_M^\alpha$ is equivalent to $\omega(f,\delta)\leqslant M\delta^\alpha$, thus, by Theorem 3.2, for $x\in[0,1]$, $n\geqslant 2$ we have

$$|\tilde{S}_{n,p}(f;q;x)-f(x)|\leqslant 2\omega(f,\delta_n)\leqslant 2M\delta_n^\alpha,$$

which completes the proof. □

Theorem 3.3. *Let $f\in C[0,1+p]$, $x\in[0,1]$, $n\geqslant 2$, $q\in(0,1)$. Then we have*

$$|\tilde{S}_{n,p}(f;q;x)-f(x)|\leqslant C\omega_2\Big(f,\delta_n\sqrt{1+\frac{\delta_n^2}{4}}\Big)+\omega\Big(f,\frac{\delta_n^2}{2}\Big),$$

where C is a positive constant and δ_n is given in Theorem 3.2.

Proof. For $f\in C[0,1+p]$, $x\in[0,1]$ we define

$$\hat{S}_{n,p}(f;q;x)=\tilde{S}_{n,p}(f;q;x)-f\Big(\frac{[n+p]_q r_{n,p}(q,x)}{[n]_q}\Big)+f(x). \tag{3.1}$$

By Lemma 2.1 we get $\hat{S}_{n,p}(1;q;x)=1$, $\hat{S}_{n,p}(t;q;x)=x$. Let $g\in W^2$, $t\in[0,1+p]$, $x\in[0,1]$. Then by Taylor's formula

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)\,\mathrm{d}u$$

we obtain

$$\hat{S}_{n,p}(g;q;x) = g(x) + \hat{S}_{n,p}\bigg(\int_x^t (t-u)g''(u)\,\mathrm{d}u;q;x\bigg).$$

By the definition given by (3.1) and Lemma 2.2, for $x \in [0,1]$, $n \geqslant 2$ we have

$$\begin{aligned}
|\hat{S}_{n,p}(g;q;x) - g(x)| &\leqslant \bigg|\tilde{S}_{n,p}\bigg(\int_x^t (t-u)g''(u)\,\mathrm{d}u;q;x\bigg)\bigg| \\
&\quad + \bigg|\int_x^{[n+p]_q r_{n,p}(q,x)/[n]_q} \Big(\frac{[n+p]_q r_{n,p}(q,x)}{[n]_q} - u\Big)g''(u)\,\mathrm{d}u\bigg| \\
&\leqslant \tilde{S}_{n,p}\bigg(\bigg|\int_x^t |t-u|\,|g''(u)|\,\mathrm{d}u\bigg|;q;x\bigg) \\
&\quad + \int_{[n+p]_q r_{n,p}(q,x)/[n]_q}^x \bigg|\frac{[n+p]_q r_{n,p}(q,x)}{[n]_q} - u\bigg|\,|g''(u)|\,\mathrm{d}u \\
&\leqslant \bigg[\tilde{S}_{n,p}((t-x)^2;q;x) + \Big(x - \frac{[n+p]_q r_{n,p}(q,x)}{[n]_q}\Big)^2\bigg]\|g''\| \leqslant \delta_n^2\Big(1+\frac{\delta_n^2}{4}\Big)\|g''\|.
\end{aligned}$$

On the other hand, by the definition given by (3.1) and Lemma 2.3 we have

$$|\hat{S}_{n,p}(f;q;x)| \leqslant |\tilde{S}_{n,p}(f;q;x)| + 2\|f\| \leqslant 3\|f\|,$$

thus, for $x \in [0,1]$, $n \geqslant 2$ we have

$$\begin{aligned}
|\tilde{S}_{n,p}(f;q;x) - f(x)| &\leqslant |\hat{S}_{n,p}(f-g;q;x)| \\
&\quad + |\hat{S}_{n,p}(g;q;x) - g(x)| + |g(x) - f(x)| + \bigg|f\Big(\frac{[n+p]_q r_{n,p}(q,x)}{[n]_q}\Big) - f(x)\bigg| \\
&\leqslant 4\|f-g\| + \delta_n^2\Big(1+\frac{\delta_n^2}{4}\Big)\|g''\| + \omega\Big(f,\frac{\delta_n^2}{2}\Big).
\end{aligned}$$

Hence, taking infimum on the right hand side over all $g \in W^2$, we can get

$$|\tilde{S}_{n,p}(f;q;x) - f(x)| \leqslant 4K_2\Big(f,\delta_n^2\Big(1+\frac{\delta_n^2}{4}\Big)\Big) + \omega\Big(f,\frac{\delta_n^2}{2}\Big).$$

By inequality (2.1), for every $q \in (0,1)$ we have

$$|\tilde{S}_{n,p}(f;q;x) - f(x)| \leqslant C\omega_2\Big(f,\delta_n\sqrt{1+\frac{\delta_n^2}{4}}\Big) + \omega\Big(f,\frac{\delta_n^2}{2}\Big).$$

□

Theorem 3.4. *Let $f \in C^1[0,1+p]$, $x \in [0,1]$, $n \geqslant 2$, $q \in (0,1)$. Then we have*

$$|\tilde{S}_{n,p}(f;q;x) - f(x)| \leqslant \|f'\| \frac{[n+p]_q}{2[n]_q([n+p]_q - 1)} + 2\delta_n \omega(f', \delta_n),$$

where $\|f'\| = \max\{|f'(x)|; x \in [0,1+p]\}$, and δ_n is given in Theorem 3.2.

P r o o f. Let $f \in C^1[0,1+p]$. Then for any $t \in [0,1+p]$, $x \in [0,1]$ we have

$$f(t) - f(x) - f'(x)(t-x) = \int_x^t (f'(u) - f'(x))\,\mathrm{d}u.$$

So, for any $\delta > 0$ we get

$$\begin{aligned}|f(t) - f(x) - f'(x)(t-x)| &\leqslant \left|\int_x^t |f'(u) - f'(x)|\,\mathrm{d}u\right| \leqslant \omega(f', |t-x|)|t-x| \\ &\leqslant \omega(f', \delta)(|t-x| + \delta^{-1}(t-x)^2),\end{aligned}$$

hence

$$\begin{aligned}&|\tilde{S}_{n,p}(f(t) - f(x) - f'(x)(t-x); q; x)| \\ &\qquad \leqslant \omega(f', \delta)(\tilde{S}_{n,p}(|t-x|; q; x) + \delta^{-1}\tilde{S}_{n,p}((t-x)^2; q; x)).\end{aligned}$$

Using the Cauchy-Schwartz inequality, we obtain

$$\tilde{S}_{n,p}(|t-x|; q; x) \leqslant \sqrt{\tilde{S}_{n,p}(1; q; x)}\sqrt{\tilde{S}_{n,p}((t-x)^2; q; x)},$$

so we have

$$\begin{aligned}&|\tilde{S}_{n,p}(f(t) - f(x) - f'(x)(t-x); q; x)| \\ &\quad \leqslant \omega(f', \delta)\Big(\sqrt{\tilde{S}_{n,p}(1; q; x)} + \delta^{-1}\sqrt{\tilde{S}_{n,p}((t-x)^2; q; x)}\Big)\sqrt{\tilde{S}_{n,p}((t-x)^2; q; x)}.\end{aligned}$$

Thus, by Lemma 2.1 and Lemma 2.2, for $x \in [0,1]$, $n \geqslant 2$ we get

$$\begin{aligned}|\tilde{S}_{n,p}(f; q; x) - f(x)| &\leqslant |f'(x)|\,|\tilde{S}_{n,p}(t-x; q; x)| \\ &\quad + \omega(f', \delta)(1 + \delta^{-1}\sqrt{\tilde{S}_{n,p}((t-x)^2; q; x)})\sqrt{\tilde{S}_{n,p}((t-x)^2; q; x)} \\ &\leqslant \|f'\| \frac{[n+p]_q}{2[n]_q([n+p]_q - 1)} + \omega(f', \delta)(1 + \delta^{-1}\delta_n)\delta_n.\end{aligned}$$

Taking $\delta = \delta_n$, then from the above inequality we obtain the desired result. □

Finally, we give the quantitative Voronovskaja-type asymptotic formula.

Theorem 3.5. *Let $x \in (0,1]$, $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a$ as $n \to \infty$. For any $f \in C^2[0,1+p]$, we have* $\lim_{n\to\infty} [n]_{q_n}(\tilde{S}_{n,p}(f;q_n;x) - f(x)) = \frac{1}{2}(x-1)(f'(x) - xf''(x))$.

Proof. Let $f \in C^2[0,1+p]$ and $x \in (0,1]$ be fixed. For any $t \in [0,1+p]$, by the Taylor formula we have

$$f(t) - f(x) = f'(x)(t-x) + \frac{f''(x)}{2}(t-x)^2 + r(t,x)(t-x)^2,$$

where $r(t,x) \in C[0,1+p]$ and $\lim_{t\to x} r(t,x) = 0$. By Lemma 2.1, we get

$$\begin{aligned}\tilde{S}_{n,p}(f;q_n;x) - f(x) &= f'(x)\tilde{S}_{n,p}((t-x);q_n;x) + \frac{f''(x)}{2}\tilde{S}_{n,p}((t-x)^2;q_n;x) \\ &\quad + \tilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x).\end{aligned} \tag{3.2}$$

In view of $\lim_{t\to x} r(t,x) = 0$, for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that when $t \in U_x(\delta) = \{t \mid t \in [0,1+p] \text{ and } |t-x| < \delta\}$, we have $|r(t,x)| < \varepsilon$. Denoting

$$\lambda_\delta(t,x) = \begin{cases} 1, & |t-x| \geqslant \delta, \\ 0, & |t-x| < \delta. \end{cases}$$

then $|r(t,x)(t-x)^2| \leqslant \varepsilon(t-x)^2 + \lambda_\delta(t,x)|r(t,x)|(t-x)^2$, $|\tilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x)| \leqslant \varepsilon\tilde{S}_{n,p}((t-x)^2;q_n;x) + \tilde{S}_{n,p}(\lambda_\delta(t,x)|r(t,x)|(t-x)^2;q_n;x)$.

Since $[0,1+p]\backslash U_x(\delta)$ is compact, also $r(t,x)$ is bounded on $[0,1+p]$. So, there exists a constant $L > 0$ such that for any $t \in [0,1+p]$, we obtain $\lambda_\delta(t,x)|r(t,x)|(t-x)^2 \leqslant L(t-x)^4$, thus

$$|\tilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x)| \leqslant \varepsilon\tilde{S}_{n,p}((t-x)^2;q_n;x) + L\tilde{S}_{n,p}((t-x)^4;q_n;x).$$

Note that $\varepsilon > 0$ being arbitrary, by Lemma 2.4 we obtain

$$\lim_{n\to\infty} [n]_{q_n} |\tilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x)| = 0,$$

so

$$\lim_{n\to\infty} [n]_{q_n} \tilde{S}_{n,p}(r(t,x)(t-x)^2;q_n;x) = 0. \tag{3.3}$$

By equalities (3.2), (3.3) and Lemma 2.4 we can obtain the desired result. □

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