Xiaoling Ma; Fei Wen

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Czechoslovak Mathematical Journal, Vol. 63 (2013), No. 3, 777–782

Persistent URL: http://dml.cz/dmlcz/143488

Czechoslovak Mathematical Journal, 63 (138) (2013), 777–782

ON THE SPECTRAL RADIUS OF ‡-SHAPE TREES

Xiaoling Ma, Fei Wen, Urumqi

(Received May 21, 2012)

Abstract. Let $A(G)$ be the adjacency matrix of G. The characteristic polynomial of the adjacency matrix A is called the characteristic polynomial of the graph G and is denoted by $\varphi(G,\lambda)$ or simply $\varphi(G)$. The spectrum of G consists of the roots (together with their multiplicities) $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_n(G)$ of the equation $\varphi(G,\lambda)=0$. The largest root $\lambda_1(G)$ is referred to as the spectral radius of G. A ‡-shape is a tree with exactly two of its vertices having maximal degree 4. We will denote by $G(l_1,l_2,\ldots,l_7)$ ($l_1 \geqslant 0$, $l_i \geqslant 1$, $i=2,3,\ldots,7$) a ‡-shape tree such that $G(l_1,l_2,\ldots,l_7)-u-v = P_{l_1}\cup P_{l_2}\cup\ldots\cup P_{l_7}$, where u and v are the vertices of degree 4. In this paper we prove that $3\sqrt{2}/2 < \lambda_1(G(l_1,l_2,\ldots,l_7)) < 5/2$.

Keywords: spectra of graphs; spectral radius; ‡-shape tree

MSC 2010: 05C50

1. Introduction

Let $G=(V,E)$ be a simple undirected connected graph with the vertex set V and the edge set E. For a vertex $v\in V$, we denote by $d(v)$ and Δ the degree of v and the maximum degree of vertices of G, respectively. Let $A(G)$ be the adjacency matrix of G. The characteristic polynomial of the adjacency matrix A is called the characteristic polynomial of the graph G and is denoted by $\varphi(G,\lambda)$ or simply $\varphi(G)$. The spectrum of G consists of the roots (together with their multiplicities) $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \ldots \geqslant \lambda_n(G)$ of the equation $\varphi(G,\lambda)=0$. The largest root $\lambda_1(G)$ is referred to as the spectral radius of G. Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real. The terminology concerning graphs will follow [2]; for all details on graph spectra, not given here, see [1].

A †-shape tree D_n ($n \geqslant 7$) is the coalescence of the star $K_{1,4}$ and the path P_{n-4} with respect to two pendant vertices (see Figure 1).

This work was supported by the Natural Science Foundation of Xinjiang University (XY110102).

A T-shape $T(k_1,k_2,k_3)$ is a tree with exactly one of its vertices having the maximal degree 3 such that $T(k_1,k_2,k_3)-v=P_{k_1}\cup P_{k_2}\cup P_{k_3}$, where P_{k_i} is the path on k_i $(i=1,2,3)$ vertices, and v is the vertex of degree 3.

A ‡-shape is a tree with exactly two of its vertices having the maximal degree 4. We will denote by $G(l_1,l_2,\ldots,l_7)$ ($l_1\geqslant 0$, $l_i\geqslant 1$, $i=2,3,\ldots,7$) a ‡-shape tree such that $G(l_1,l_2,\ldots,l_7)-u-v=P_{l_1}\cup P_{l_2}\cup\ldots\cup P_{l_7}$, where u and v are the vertices of degree 4 (see Figure 2).

Let W_n be a graph obtained from the path P_{n-2} (indexed in the natural order $1,2,\ldots,n-2$) by adding two pendant edges at vertices 2 and $n-3$ (see Figure 1).

Let S_n be a graph obtained from the path P_{n-4} (indexed in the natural order $1,2,\ldots,n-4$) by adding four pendant edges at vertices 2 and $n-5$, that is $S_n=G(n-8,1,1,1,1,1,1)$ (see Figure 1).

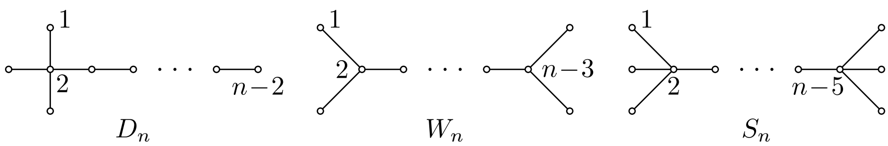

Figure 1

There are many results in literature concerning the largest eigenvalue of a graph and the graph structure (see [1], [7] and [4] for details). In this paper we are mainly interested in obtaining the lower and upper bounds for the largest eigenvalue of ‡-shape trees.

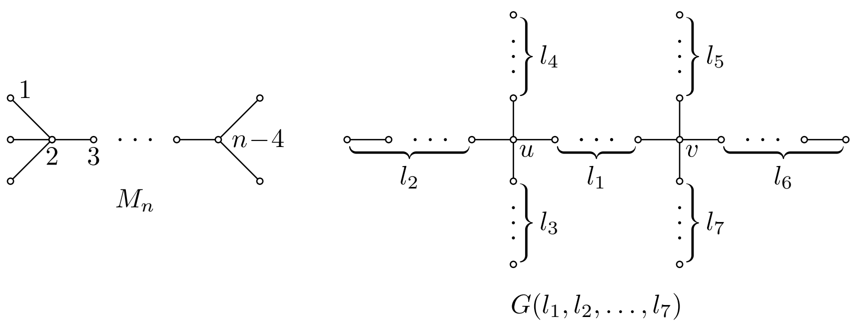

Figure 2

2. Main results

First some useful established results about the spectrum are presented, which will play an important role throughout this paper.

Lemma 2.1 ([5]). *The characteristic polynomial of a graph satisfies the following identities:*
(a) $\varphi(G \cup H, \lambda) = \varphi(G, \lambda)\varphi(H, \lambda)$;
(b) $\varphi(G, \lambda) = \varphi(G - e, \lambda) - \varphi(G - u - v, \lambda)$ *if* $e = uv$ *is a cut-edge of* G,
where $G - e$ *denotes the graph obtained from* G *by deleting the edge* e *and* $G - u - v$ *denotes the graph obtained from* G *by deleting the vertices* u, v *and the edges incident to them.*

Lemma 2.2 ([1]). *Let* P_n *denote the path on* n *vertices. Then*

$$\varphi(G, \lambda) = \prod_{j=1}^{n} \Big(\lambda - 2\cos\frac{\pi j}{n+1}\Big) = \frac{\sin((n+1)\arccos\frac{1}{2}\lambda)}{\sin(\arccos\frac{1}{2}\lambda)}.$$

Let $\lambda = 2\cos\theta$, set $t^{1/2} = \mathrm{e}^{\mathrm{i}\theta}$; it is useful to write the characteristic polynomial of P_n in the form

$$\varphi(P_n, t^{1/2} + t^{-1/2}) = \frac{t^{-n/2}(t^{n+1} - 1)}{(t-1)}.$$

Lemma 2.3 ([7]). *Let* $T_m = T(m, m, m)$. *Then*

$$\varphi(T_m, t^{1/2} + t^{-1/2}) = \frac{t^{-(m+1)/2}}{t-1}(t^{m+2} - 2t^{m+1} + 2t - 1)(\varphi(P_m, \lambda))^2.$$

Lemma 2.4. *Let* $G(0, 6l) = G(0, l, l, l, l, l, l)$. *Then*

$$\varphi(G(0, 6l), t^{1/2} + t^{-1/2}) = \frac{t^{-n/2}(t^{l+1} - 1)^4}{(t-1)^6}[(t^{l+2} - 2t^{l+1} + 2t - 1)^2 - t(t^{l+1} - 1)^2].$$

Proof. By Lemma 2.1 we get

$$\varphi(G(0, 6l), \lambda) = \varphi(T_l, \lambda)\varphi(T_l, \lambda) - (\varphi(P_l, \lambda))^6.$$

Let $\lambda = t^{1/2} + t^{-1/2}$, by Lemma 2.4 we have

$$\begin{aligned}\varphi(G(0, 6l), t^{1/2} + t^{-1/2}) &= \frac{t^{-(l+1)/2}}{t-1}(t^{l+2} - 2t^{l+1} + 2t - 1)(\varphi(P_l, \lambda))^2 - \varphi(P_l, \lambda)^6 \\ &= \frac{t^{-n/2}(t^{l+1} - 1)^4}{(t-1)^6}[(t^{l+2} - 2t^{l+1} + 2t - 1)^2 - t(t^{l+1} - 1)^2],\end{aligned}$$

where $6l + 2 = n$. □

Lemma 2.5 ([1]). *Let G be a connected graph and H a proper subgraph of G. Then $\lambda_1(H) < \lambda_1(G)$.*

Lemma 2.6 ([8]). *Let D_n be a †-shape tree. Then*

$$\lim_{n\to\infty} \lambda_1(D_n) = \frac{3\sqrt{2}}{2}.$$

Lemma 2.7. *Let S_n be a ‡-shape tree $G(n-8,1,1,1,1,1,1)$. Then*

$$\lambda_1(S_n) > \frac{3\sqrt{2}}{2}.$$

Proof. By the structure of the graphs S_n, M_{n_1} (see Figure 2) and D_{n_0}, we have that D_{n_0} is a subgraph of M_{n_1}, and S_n contains M_{n_1} as a subgraph for suitable $7 \leqslant n_0 < n_1 < n$. So we immediately obtain the following inequality from Lemma 2.5:

$$\lambda_1(S_n) > \lambda_1(M_{n_1}) > \lambda_1(D_{n_0}).$$

By Lemma 2.6, for $n_1 > n_0$ we have $\lambda_1(M_{n_1}) \geqslant \lim\limits_{n_0\to\infty} \lambda_1(D_{n_0}) = 3\sqrt{2}/2$, which implies that $\lambda_1(M_{n_1}) \geqslant 3\sqrt{2}/2$. Also by Lemma 2.5, we easily get $\lambda_1(S_n) > \lambda_1(M_{n_1}) \geqslant 3\sqrt{2}/2$. Thus for all $n > n_0$ we obtain

$$\lambda_1(S_n) > 3\sqrt{2}/2.$$

□

Hoffman and Smith [3] define an internal path in a graph, denoted by $v_0, v_1, \ldots, v_{k-1}, v_k$, as a path joining vertices v_0 and v_k which are both of degree greater than two (not necessarily distinct), while all other vertices (i.e. $v_1, \ldots, v_{k-1}$) are of degree equal to two.

Lemma 2.8 ([3]). *Let G be a connected graph that is not isomorphic to W_n. Let G_{uv} be the graph obtained from G by subdividing the edge uv of G. If uv lies on an internal path of G, then $\lambda_1(G_{uv}) < \lambda_1(G)$.*

Lemma 2.9 ([6]). *Let τ be a tree with the largest vertex degree Δ. Then*

$$\lambda_1(\tau) < 2\sqrt{\Delta - 1}. \tag{2.1}$$

Theorem 2.10. *Let $G = G(l_1, \ldots, l_7)$. Then*

$$\frac{3\sqrt{2}}{2} < \lambda_1(G) < \frac{5}{2}. \tag{2.2}$$

P r o o f. Let l be a positive integer such that $l_i < l$ $(i = 2, \ldots, 7)$. By Lemma 2.4 we have

$$\begin{aligned}\varphi(G(0,6l), t^{1/2}+t^{-1/2}) &= \frac{t^{-n/2}(t^{l+1}-1)^4}{(t-1)^6}[(t^{l+2}-2t^{l+1}+2t-1)^2 - t(t^{l+1}-1)^2] \\ &= \frac{t^{-n/2}(t^{l+1}-1)^4}{(t-1)^6}[((t-2)(t^{l+1}-1)+3(t-1))^2 - t(t^{l+1}-1)^2] \\ &=: \Phi(t).\end{aligned}$$

Let t_1 be the largest root of $\Phi(t)$, then $t_1 < 4$ since $\Phi(t) > 0$ for $t \geqslant 4$. Let $f(t) = t^{1/2} + t^{-1/2}$, then $f'(t) = t^{-3/2}(t-2)/2 \geqslant 0$ for $t \geqslant 1$, so $f(t)$ strictly increases in $[1, \infty)$. Thus $\lambda_1(G(0,6l)) = t_1^{1/2} + t_1^{-1/2} < 4^{1/2} + 4^{-1/2} = 5/2$.

On the one hand, by Lemmas 2.5 and 2.7 we have the inequality

$$\frac{3\sqrt{2}}{2} < \lambda_1(S_n) = \lambda_1(G(l_1,1,1,1,1,1,1)) \leqslant \lambda_1(G(l_1,l_2,l_3,l_4,l_5,l_6,l_7)). \tag{2.3}$$

On the other hand, by Lemmas 2.5 and 2.8, we obtain the inequality

$$\lambda_1(G(l_1,l_2,l_3,l_4,l_5,l_6,l_7)) < \lambda_1(G(l_1,l,l,l,l,l,l)) < \lambda_1(G(0,l,l,l,l,l,l)) < \frac{5}{2}. \tag{2.4}$$

Combining inequalities (2.3) and (2.4), we obtain the main result

$$\frac{3\sqrt{2}}{2} < \lambda_1(G(l_1,l_2,l_3,l_4,l_5,l_6,l_7)) < \frac{5}{2}.$$

□

Now we have $\lambda_1(G(l_1,l_2,l_3,l_4,l_5,l_6,l_7)) < 2\sqrt{3}$ by inequality (2.1). Here we see that the upper bound inequality (2.2) is better than the upper bound inequality (2.1).

References

[1] *D. Cvetiović, M. Doob, H. Sachs*: Spectra of Graphs. Theory and Applications. VEB Deutscher Verlag der Wissenschaften, Berlin, 1980.
[2] *F. Harary*: Graph Theory. Addison-Wesley Series in Mathematics. Addison-Wesley Publishing Company. IX, Reading, Mass.-Menlo Park, London, 1969.
[3] *A. J. Hoffman, J. H. Smith*: On the spectral radii of topologically equivalent graphs. Recent Adv. Graph Theory, Proc. Symp. Prague 1974. Academia, Praha, 1975, pp. 273–281.
[4] *S. B. Hu*: On the spectral radius of H-shape trees. Int. J. Comput. Math. *87* (2010), 976–979.
[5] *C. D. Godsil*: Algebraic Combinatorics. Chapman and Hall, New York, 1993.
[6] *C. D. Godsil*: Spectra of trees. Convexity and Graph Theory. Proc. Conf., Israel 1981, Ann. Discrete Math. 20. 1984, pp. 151–159.
[7] *W. Wang, C. X. Xu*: On the spectral characterization of T-shape trees. Linear Algebra Appl. *414* (2006), 492–501.
[8] *R. Woo, A. Neumaier*: On graphs whose spectral radius is bounded by $\frac{3}{2}\sqrt{2}$. Graphs Comb. *23* (2007), 713–726.

Authors' address: Xiaoling Ma (corresponding author), Fei Wen, College of Mathematics and System Sciences, Xinjiang University, Urumqi, Xinjiang 830046, P. R. China, e-mail: mathmxl115@xju.edu.cn, wenfei1998@126.com.