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COHOMOLOGY OF HOM-LIE SUPERALGEBRAS AND q-DEFORMED WITT SUPERALGEBRA

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Abstract. Hom-Lie algebra (superalgebra) structure appeared naturally in q-deformations, based on σ-derivations of Witt and Virasoro algebras (superalgebras). They are a twisted version of Lie algebras (superalgebras), obtained by deforming the Jacobi identity by a homomorphism. In this paper, we discuss the concept of α^k-derivation, a representation theory, and provide a cohomology complex of Hom-Lie superalgebras. Moreover, we study central extensions. As application, we compute derivations and the second cohomology group of a twisted $\operatorname{osp}(1,2)$ superalgebra and q-deformed Witt superalgebra.

Keywords: Hom-Lie superalgebra; derivation; cohomology; q-deformed superalgebra

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Introduction

Hom-Lie algebras and other Hom-algebras structures have been widely investigated during the last years. They were introduced and studied in [5], [7], [8], [9], [10], motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields. The paradigmatic examples are q-deformations of Witt and Virasoro algebras based on σ-derivation [1], [5], [6], [11]. Hom-Lie superalgebras were studied in [3]. Cohomology theory of Hom-Lie algebras was studied in [2], [15], [19], see also [12], [13], [14], [20], [21], [22], [23] for other important results about Hom-algebras. The purpose of this paper is to study representations and cohomology of Hom-Lie superalgebras. As application, we provide some calculations for q-deformed Witt superalgebra.

The paper is organized as follows. In the first section we give the definitions and some key constructions of Hom-Lie superalgebras. Section 2 is dedicated to the representation theory of Hom-Lie superalgebras, including adjoint and coadjoint representation. In Section 3 we construct a family of cohomologies of Hom-Lie superalgebras. In Section 4, we discuss extensions of Hom-Lie superalgebras and their connection to cohomology. In the last section we compute the derivations and the scalar second cohomology group of the q-deformed Witt superalgebra.

1. Hom-Lie superalgebras

In this section, we review the theory of Hom-Lie superalgebras established in [3] and generalize some results of [4]. For classical definitions and results we refer to [16], [17], [18]. Let $\mathcal{G}$ be a linear superspace over a field $\mathbb{K}$ that is a $\mathbb{Z}_2$-graded linear space with a direct sum $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$. The elements of $\mathcal{G}_j$, $j \in \mathbb{Z}_2$, are said to be homogeneous of parity j. The parity of a homogeneous element x is denoted by $|x|$. The space $\operatorname{End}(\mathcal{G})$ is $\mathbb{Z}_2$-graded with a direct sum $\operatorname{End}(\mathcal{G}) = (\operatorname{End}(\mathcal{G}))_0 \oplus (\operatorname{End}(\mathcal{G}))_1$ where $(\operatorname{End}(\mathcal{G}))_j = \{f \in \operatorname{End}(\mathcal{G})/f(\mathcal{G}_i) \subset \mathcal{G}_{i+j}\}$. Elements of $(\operatorname{End}(\mathcal{G}))_j$ are said to be homogeneous of parity j.

Definition 1.1. A Hom-Lie superalgebra is a triple $(\mathcal{G}, [\cdot,\cdot], \alpha)$ consisting of a superspace $\mathcal{G}$, an even bilinear map $[\cdot,\cdot]\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and an even superspace homomorphism $\alpha\colon \mathcal{G} \to \mathcal{G}$ satisfying

$$[x, y] = -(-1)^{|x||y|}[y, x], \tag{1.1}$$

$$(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|y||x|}[\alpha(y), [z, x]] = 0 \tag{1.2}$$

for all homogeneous elements x, y, z in $\mathcal{G}$.

Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ and $(\mathcal{G}', [\cdot,\cdot]', \alpha')$ be two Hom-Lie superalgebras. An even homomorphism $f\colon \mathcal{G} \to \mathcal{G}'$ is said to be a *morphism of Hom-Lie superalgebras* if

$$[f(x), f(y)]' = f([x, y]) \quad \forall x, y \in \mathcal{G}, \tag{1.3}$$

$$f \circ \alpha = \alpha' \circ f. \tag{1.4}$$

Remark 1.2. We recover the classical Lie superalgebra when $\alpha = \mathrm{id}$.
The Hom-Lie algebra is obtained when the part of parity one is trivial.

Example 1.3. Let $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ be a 3-dimensional superspace where $\mathcal{G}_0$ is generated by e_1 and $\mathcal{G}_1$ is generated by e_2, e_3. The triple $(\mathcal{G}, [\cdot,\cdot], \alpha)$ is a Hom-Lie superalgebra defined by $[e_1, e_2] = 2e_2, [e_1, e_3] = 2e_3$ and $[e_2, e_3] = e_1$, with $\alpha(e_1) = e_1$, $\alpha(e_2) = e_3$, $\alpha(e_3) = -e_2$.

Definition 1.4. Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra. A Hom-Lie superalgebra is called

- ▷ multiplicative if for all $x, y \in \mathcal{G}$ we have $\alpha([x,y]) = [\alpha(x), \alpha(y)]$;
- ▷ regular if α is an automorphism;
- ▷ involutive if α is an involution, that is $\alpha^2 = \mathrm{id}$.

The center of the Hom-Lie superalgebra, denoted $\mathcal{Z}(\mathcal{G})$, is defined by

$$\mathcal{Z}(\mathcal{G}) = \{x \in \mathcal{G} \colon\ [x,y] = 0,\ \forall y \in \mathcal{G}\}.$$

The next theorem generalizes the twisting principle stated in [3], [21] in the following sense: starting from a Hom-Lie superalgebra and an even Lie superalgebra endomorphism, we construct a new Hom-Lie superalgebra.

Theorem 1.5. *Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra, and $\beta\colon \mathcal{G} \to \mathcal{G}$ an even Lie superalgebra endomorphism. Then $(\mathcal{G}, [\cdot,\cdot]_\beta, \beta\circ\alpha)$, where $[x,y]_\beta = \beta([x,y])$, is a Hom-Lie superalgebra.*

Moreover, suppose that $(\mathcal{G}', [\cdot,\cdot]')$ is a Lie superalgebra and $\alpha'\colon \mathcal{G}' \to \mathcal{G}'$ is a Lie superalgebra endomorphism. If $f\colon \mathcal{G} \to \mathcal{G}'$ is a Lie superalgebra morphism that satisfies $f\circ\beta = \alpha'\circ f$ then

$$f\colon\ (\mathcal{G}, [\cdot,\cdot]_\beta, \beta\circ\alpha) \longrightarrow (\mathcal{G}', [\cdot,\cdot]', \alpha')$$

is a morphism of Hom-Lie superalgebras.

Proof. We show that $(\mathcal{G}, [\cdot,\cdot]_\beta, \beta\circ\alpha)$ satisfies the graded Hom-Jacobi identity (1.2). Indeed,

$$\begin{aligned} \circlearrowleft_{x,y,z} (-1)^{|x||z|}[\beta\circ\alpha(x), [y,z]_\beta]_\beta &= \circlearrowleft_{x,y,z} (-1)^{|x||z|}[\beta\circ\alpha(x), \beta([y,z])]_\beta \\ &= \beta^2(\circlearrowleft_{x,y,z} (-1)^{|x||z|}[\alpha(x), [y,z]]) \\ &= 0. \end{aligned}$$

The second assertion follows from

$$\begin{aligned} f([x,y]_\beta) &= f([\beta(x), \beta(y)]) = [f\circ\beta(x), f\circ\beta(y)]' \\ &= [\alpha'\circ f(x), \alpha'\circ f(y)]' = [f(x), f(y)]'_{\alpha'}. \end{aligned}$$

□

Example 1.6. We derive the following particular cases:

(1) If $(\mathcal{G}, [\cdot,\cdot], \alpha)$ is a multiplicative Hom-Lie superalgebra then, for any $n \in \mathbb{N}$, $(\mathcal{G}, \alpha^n \circ [\cdot,\cdot], \alpha^{n+1})$ is a multiplicative Hom-Lie superalgebra.

(2) If $(\mathcal{G}, [\cdot,\cdot])$ is a Lie superalgebra and a self-map α on $\mathcal{G}$ is an even Lie superalgebra morphism then $(\mathcal{G}, [\cdot,\cdot]_\alpha, \alpha)$ is a multiplicative Hom-Lie superalgebra.

(3) If $(\mathcal{G}, [\cdot,\cdot], \alpha)$ is a regular Hom-Lie superalgebra, then $(\mathcal{G}, \alpha^{-1} \circ [\cdot,\cdot])$ is a Lie superalgebra.

In the following we construct Hom-Lie superalgebras involving elements of the centroid of Lie superalgebras. Let $(\mathcal{G}, [\cdot,\cdot])$ be a Lie superalgebra. The centroid is defined by

$$\begin{aligned}\mathrm{Cent}(\mathcal{G}) &= \{\theta \in \mathrm{End}(\mathcal{G})\colon\ \theta([x,y]) = [\theta(x), y],\ \forall x, y \in \mathcal{G}\}\\ &= (\mathrm{Cent}(\mathcal{G}))_0 \oplus (\mathrm{Cent}(\mathcal{G}))_1.\end{aligned}$$

The centroid $\mathrm{Cent}(\mathcal{G})$ is a subsuperpace of $\mathrm{End}(\mathcal{G})$.

Proposition 1.7. *Let $(\mathcal{G}, [\cdot,\cdot])$ be a Lie superalgebra and $\theta \in (\mathrm{Cent}(\mathcal{G}))_0 \subset (\mathrm{End}(\mathcal{G}))_0$. Set for $x, y \in \mathcal{G}$*

$$\{x, y\} = \theta([x,y]).$$

Then $(\mathcal{G}, \{\cdot,\cdot\}, \theta)$ is a Hom-Lie superalgebra.

Proof. For $\theta \in (\mathrm{Cent}(\mathcal{G}))_0$ we have

$$\{x,y\} = \theta([x,y]) = -(-1)^{|x||y|}\theta([y,x]) = -(-1)^{|x||y|}[\theta(y), x] = [x, \theta(y)].$$

Then $\{x,y\} = [x,\theta(y)] = (-1)^{|x||y|}[\theta(y), x] = -(-1)^{|x||y|}\{y, x\}$.

Also we have

$$\{\theta(x), \{y,z\}\} = \{\theta(x), [y, \theta(z)]\} = [\theta(x), \theta([y,\theta(z)])] = [\theta(x), [\theta(y), \theta(z)]].$$

It follows that

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}\{\theta(x), \{y,z\}\} = \circlearrowleft_{x,y,z} (-1)^{|\theta(x)||\theta(z)|}[\theta(x), [\theta(y), \theta(z)]] = 0.$$

Since $(\mathcal{G}, [\cdot,\cdot])$ is a Lie superalgebra, the super Hom-Jacobi identity is satisfied. Thus $(\mathcal{G}, \{\cdot,\cdot\}, \theta)$ is a Hom-Lie superalgebra. □

2. Derivations of Hom-Lie superalgebras

We provide in the following a graded version of the study of derivations of Hom-Lie algebras stated in [19]. Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra, denote by α^k the k-times composition of α, i.e. $\alpha^k = \alpha \circ \ldots \circ \alpha$ (k-times). In particular, $\alpha^{-1} = 0$, $\alpha^0 = \mathrm{Id}$ and $\alpha^1 = \alpha$.

Definition 2.1. For any $k \geqslant -1$, we call $D \in (\mathrm{End}(\mathcal{G}))_i$ where $i \in \mathbb{Z}_2$, an α^k-derivation of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$ if $\alpha \circ D = D \circ \alpha$ and

$$D([x,y]) = [D(x), \alpha^k(y)] + (-1)^{|x||D|}[\alpha^k(x), D(y)] \quad \text{for all homogeneous elements } x, y \in \mathcal{G}.$$

We denote by $\mathrm{Der}_{\alpha^k}(\mathcal{G}) = (\mathrm{Der}_{\alpha^k}(\mathcal{G}))_0 \oplus (\mathrm{Der}_{\alpha^k}(\mathcal{G}))_1$ the set of α^k-derivations of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$, and

$$\mathrm{Der}(\mathcal{G}) = \bigoplus_{k \geqslant -1} \mathrm{Der}_{\alpha^k}(\mathcal{G}).$$

For any homogeneous element $a \in \mathcal{G}$ satisfying $\alpha(a) = a$, define $\mathrm{ad}_k(a) \in \mathrm{End}(\mathcal{G})$ by

$$\mathrm{ad}_k(a)(x) = [a, \alpha^k(x)], \ \forall x \in \mathcal{G}.$$

Notice that $\mathrm{ad}_k(a)$ and a are of the same parity.

Proposition 2.2. *Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. Then $\mathrm{ad}_k(a)$ is an α^{k+1}-derivation, which we call inner α^{k+1}-derivation.*

Proof. Indeed, we have

$$\mathrm{ad}_k(a) \circ \alpha(x) = [a, \alpha^{k+1}(x)] = [\alpha(a), \alpha^{k+1}(x)] = \alpha([a, \alpha^k(x)]) = \alpha \circ \mathrm{ad}_k(a)(x)$$

and

$$\begin{aligned}
\mathrm{ad}_k(a)([x,y]) &= [a, \alpha^k([x,y])] = [\alpha(a), [\alpha^k(x), \alpha^k(y)]] \\
&= -(-1)^{|a||y|}((-1)^{|x||a|}[\alpha^{k+1}(x), [\alpha^k(y), a]] \\
&\quad + (-1)^{|y||x|}[\alpha^{k+1}(y), [a, \alpha^k(x)]]) \\
&= (-1)^{|a||y|}((-1)^{|x||a|}(-1)^{|y||a|}[\alpha^{k+1}(x), [a, \alpha^k(y)]] \\
&\quad + (-1)^{|y||x|}(-1)^{|y||[a,x]|}[[a, \alpha^k(x)], \alpha^{k+1}(y)]) \\
&= [[a, \alpha^k(x)], \alpha^{k+1}(y)] + (-1)^{|x||a|}[\alpha^{k+1}(x), [a, \alpha^{k+1}(y)]] \\
&= [\mathrm{ad}_k(a)(x), \alpha^{k+1}(y)] + (-1)^{|x||a|}[\alpha^{k+1}(x), \mathrm{ad}_k(a)(y)].
\end{aligned}$$

Therefore, ad_k is an α^{k+1}-derivation. We denote by $\mathrm{Inn}_{\alpha^k}(\mathcal{G})$ the set of inner α^k-derivations, i.e.

$$\mathrm{Inn}_{\alpha^k}(\mathcal{G}) = \{[a, \alpha^{k-1}(\cdot)]/a \in \mathcal{G}_0 \cup \mathcal{G}_1,\ \alpha(a) = a\}.$$

For any $D \in \mathrm{Der}(\mathcal{G})$ and $D' \in \mathrm{Der}(\mathcal{G})$, define their commutator $[D, D']$ as usual:

$$[D, D'] = D \circ D' - (-1)^{|D||D'|} D' \circ D. \tag{2.1}$$

□

Lemma 2.3. *For any $D \in (\mathrm{Der}_{\alpha^k}(\mathcal{G}))_i$ and $D' \in (\mathrm{Der}_{\alpha^s}(\mathcal{G}))_j$, where $k + s \geqslant -1$ and $(i, j) \in \mathbb{Z}_2^2$, we have*

$$[D, D'] \in (\mathrm{Der}_{\alpha^{k+s}}(\mathcal{G}))_{|D|+|D'|}.$$

Proof. For any $x, y \in \mathcal{G}$ we have

$$\begin{aligned}
[D, D']([x, y]) &= D \circ D'([x, y]) - (-1)^{|D||D'|} D' \circ D([x, y]) \\
&= D([D'(x), \alpha^s(y)] + (-1)^{|x||D'|}[\alpha^s(x), D'(y)]) \\
&\quad - (-1)^{|D||D'|} D'([D(x), \alpha^k(y)] + (-1)^{|x||D|}[\alpha^k(x), D(y)]) \\
&= [DD'(x), \alpha^{k+s}(y)] + (-1)^{|D||D'(x)|}[\alpha^k D'(x), D\alpha^s(y)] \\
&\quad + (-1)^{|x||D'|}([D\alpha^s(x), \alpha^k D'(y)] + (-1)^{|x||D|}[\alpha^{k+s}(x), DD'(y)]) \\
&\quad - (-1)^{|D||D'|}([D'D(x), \alpha^{k+s}(y)] + (-1)^{|D'||D(x)|}[\alpha^s D(x), D'\alpha^k(y)]) \\
&\quad - (-1)^{|D||D'|}(-1)^{|x||D|}([D'\alpha^k(x), \alpha^s D(y)] \\
&\quad + (-1)^{|x||D'|}[\alpha^{k+s}(x), D'D(y)]).
\end{aligned}$$

Since D and D' satisfy $D \circ \alpha = \alpha \circ D$ and $D' \circ \alpha = \alpha \circ D'$, we have

$$\begin{aligned}
[D, D']([x, y]) &= [DD'(x) - (-1)^{|D||D'|} D'D(x), \alpha^{k+s}(y)] \\
&\quad + (-1)^{|x||D'|}(-1)^{|x||D|}[\alpha^{k+s}(x), DD'(y) - (-1)^{|D||D'|} D'D(y)] \\
&= [[D, D'](x), \alpha^{k+s}(y)] + (-1)^{|[D,D']||x|}[\alpha^{k+s}(x), [D, D'](y)].
\end{aligned}$$

It is easy to verify that $\alpha \circ [D, D'] = [D, D'] \circ \alpha$, which leads to $[D, D'] \in \mathrm{Der}_{\alpha^{k+s}}(\mathcal{G})$.

□

Remark 2.4. Obviously, we have

$$\mathrm{Der}_{\alpha^{-1}} = \{D \in \mathrm{End}(\mathcal{G}) \colon\ D \circ \alpha = \alpha \circ D, D([x, y]) = 0,\ \forall x, y \in \mathcal{G}\}.$$

Thus for any $D, D' \in \mathrm{Der}_{\alpha^{-1}}(\mathcal{G})$, we have $[D, D'] \in \mathrm{Der}_{\alpha^{-1}}(\mathcal{G})$.

By Lemma 2.3, obviously we have

Proposition 2.5. *With the above notation,* $\mathrm{Der}(\mathcal{G})$ *is a Lie superalgebra, in which the bracket is given by* (2.1).

Proposition 2.6. *If we consider on* $\mathrm{Der}(\mathcal{G})$ *the endomorphism* $\tilde{\alpha}$ *defined by* $\tilde{\alpha}(D) = \alpha \circ D$*, then* $(\mathrm{Der}(\mathcal{G}), [\cdot,\cdot], \tilde{\alpha})$ *is a Hom-Lie superalgebra where* $[\cdot,\cdot]$ *is given by* (2.1).

Now, we consider extensions of a Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$ using derivations. For any $D \in (\mathrm{End}(\mathcal{G}))_i$, consider the vector spaces $\widetilde{\mathcal{G}}_0 = \mathcal{G}_0 \oplus \mathbb{R}D$, $\widetilde{\mathcal{G}}_1 = \mathcal{G}_1$ and $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_0 \oplus \widetilde{\mathcal{G}}_1$. Define a skew-symmetric bilinear bracket operation $[\cdot,\cdot]_D$ on $\widetilde{\mathcal{G}}$ by

$$[g + \gamma D, h + \lambda D]_D = [g, h] - \lambda D(g) + \gamma D(h), \ \forall g, h \in \mathcal{G}.$$

Define $\alpha_D \in \mathrm{End}(\mathcal{G} \oplus \mathbb{R}D)$ by $\alpha_D(g + \lambda D) = \alpha(g) + \lambda D$.

Proposition 2.7. *With the above notation,* $(\widetilde{\mathcal{G}}, [\cdot,\cdot]_D, \alpha_D)$ *is a Hom-Lie superalgebra if and only if* D *is a derivation of the Hom-Lie superalgebra* $(\mathcal{G}, [\cdot,\cdot], \alpha)$.

3. Representations and cohomology of Hom-Lie superalgebras

In this section we study representations of Hom-Lie superalgebras, see [19], [4] for the nongraded case, and define a family of cohomologies by providing a family of coboundary operators defining cohomology complexes.

3.1. Representations of Hom-Lie superalgebras. Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra and $V = V_0 \oplus V_1$ an arbitrary vector superspace. Let $\beta \in \mathcal{G}l(V)$ be an arbitrary even linear self-map on V and let

$$[\cdot,\cdot]_V \colon \mathcal{G} \times V \to V,$$
$$(g, v) \mapsto [g, v]_V$$

be a bilinear map satisfying $[\mathcal{G}_i, V_j]_V \subset V_{i+j}$ where $i, j \in \mathbb{Z}_2$.

Definition 3.1. The triple $(V, [\cdot,\cdot]_V, \beta)$ is called a Hom-module on the Hom-Lie superalgebra $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_1$ or $\mathcal{G}$-Hom-module V if the even bilinear map $[\cdot,\cdot]_V$ satisfies

$$[\alpha(x), \beta(v)]_V = \beta([x, v]_V) \tag{3.1}$$

and

$$[[x, y], \beta(v)]_V = [\alpha(x), [y, v]_V]_V - (-1)^{|x||y|}[\alpha(y), [x, v]_V]_V \tag{3.2}$$

for all homogeneous elements $x, y \in \mathcal{G}$ and $v \in V$.

Hence, we say that $(V, [\cdot,\cdot]_V, \beta)$ is a representation of $\mathcal{G}$.

Example 3.2. Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra and $\mathrm{ad}\colon \mathcal{G} \to \mathrm{End}(\mathcal{G})$ an operator defined for $x \in \mathcal{G}$ by $\mathrm{ad}(x)(y) = [x, y]$. Then $(\mathcal{G}, \mathrm{ad}, \alpha)$ is a representation of $\mathcal{G}$.

Example 3.3. Given a representation $(V, [\cdot,\cdot]_V, \beta)$ of a Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$ denote $\widetilde{\mathcal{G}} = \mathcal{G} \oplus V$ and $\widetilde{\mathcal{G}}_k = \mathcal{G}_k \oplus V_k$. If $x \in \mathcal{G}_i$ and $v \in V_i$ $(i \in \mathbb{Z}_2)$, we denote $|(x, v)| = |x|$.

Define a super skew-symmetric bracket $[\cdot,\cdot]_{\widetilde{\mathcal{G}}}\colon \wedge^2(\mathcal{G} \oplus V) \to \mathcal{G} \oplus V$ by

$$[(x, u), (y, v)]_{\widetilde{\mathcal{G}}} = ([x, y], [x, v]_V - (-1)^{|x||y|}[y, u]_V).$$

Define $\tilde{\alpha}\colon \mathcal{G} \oplus V \to \mathcal{G} \oplus V$ by $\tilde{\alpha}(x, v) = (\alpha(x), \beta(v))$. Then $(\mathcal{G} \oplus V, [\cdot,\cdot]_{\widetilde{\mathcal{G}}}, \tilde{\alpha})$ is a Hom-Lie superalgebra, which we call the semi-direct product of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$ by V.

Remark 3.4. When $[\cdot,\cdot]_V$ is the zero-map, we say that the module V is trivial.

3.2. Cohomology of Hom-Lie superalgebras. Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. Let $x_1, \ldots, x_k$ be k homogeneous elements of $\mathcal{G}$, we denote by $|(x_1, \ldots, x_k)| = |x_1| + \ldots + |x_k|$ the parity of an element $(x_1, \ldots, x_k)$ in $\mathcal{G}^k$.

The set $C^k(\mathcal{G}, V)$ of k-cochains on the space $\mathcal{G}$ with values in V is the set of k-linear maps $f\colon \bigotimes^k \mathcal{G} \to V$ satisfying

$$f(x_1, \ldots, x_i, x_{i+1}, \ldots, x_k) = -(-1)^{|x_i||x_{i+1}|} f(x_1, \ldots, x_{i+1}, x_i, \ldots, x_k) \quad \text{for } 1 \leqslant i \leqslant k-1.$$

For $k = 0$ we have $C^0(\mathcal{G}, V) = V$.

The map f is called even (odd) when $f(x_1, \ldots, x_k) \in V_0$ $(f(x_1, \ldots, x_k) \in V_1)$ for all even (odd) elements $(x_1, \ldots, x_k) \in \mathcal{G}^k$.

A k-hom-cochain on $\mathcal{G}$ with values in V is defined to be a k-cochain $f \in C^k(\mathcal{G}, V)$ such that it is compatible with α and β in the sense that $\beta \circ f = f \circ \alpha$, i.e.

$$\beta \circ f(x_1, \ldots, x_k) = f(\alpha(x_1), \ldots, \alpha(x_k)).$$

Denote by $C^k_{\alpha,\beta}(\mathcal{G}, V)$ the set of k-hom-cochains:

$$C^k_{\alpha,\beta}(\mathcal{G}, V) = \{f \in C^k(\mathcal{G}, V)\colon\ \beta \circ f = f \circ \alpha\}. \tag{3.3}$$

For a given positive integer r, we define a map $\delta_r^k\colon C^k(\mathcal{G},V)\to C^{k+1}(\mathcal{G},V)$ by setting

$$(3.4)\qquad \begin{aligned}&\delta_r^k(f)(x_0,\ldots,x_k)\\ &=\sum_{0\leqslant s<t\leqslant k}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)}\\ &\quad\times f(\alpha(x_0),\ldots,\alpha(x_{s-1}),[x_s,x_t],\alpha(x_{s+1}),\ldots,\widehat{x_t},\ldots,\alpha(x_k))\\ &\quad+\sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)}[\alpha^{k+r-1}(x_s),f(x_0,\ldots,\widehat{x_s},\ldots,x_k)]_V,\end{aligned}$$

where $f\in C^k(\mathcal{G},V)$, $|f|$ is the parity of f, $x_0,\ldots,x_k\in\mathcal{G}$ and $\widehat{x_i}$ means that x_i is omitted.

In the sequel we assume that the Hom-Lie superalgebra $(\mathcal{G},[\cdot,\cdot],\alpha)$ is multiplicative.

Lemma 3.5. *With the above notation, for any $f\in C^k_{\alpha,\beta}(\mathcal{G},V)$ we have*

$$\delta_r^k(f)\circ\alpha=\beta\circ\delta_r^k(f).$$

Thus, we obtain a well-defined map

$$\delta_r^k\colon C^k_{\alpha,\beta}(\mathcal{G},V)\to C^{k+1}_{\alpha,\beta}(\mathcal{G},V).$$

Proof. Let $f\in C^k_{\alpha,\beta}(\mathcal{G},V)$ and $(x_0,\ldots,x_k)\in\mathcal{G}^{k+1}$. Then

$$\begin{aligned}&\delta_r^k(f)\circ\alpha(x_0,\ldots,x_k)=\delta^k(f)(\alpha(x_0),\ldots,\alpha(x_k))\\ &=\sum_{0\leqslant s<t\leqslant k}(-1)^{t+|x_t|(|f|+|x_{s+1}|+\ldots+|x_{t-1}|)}\\ &\quad\times f(\alpha^2(x_0),\ldots,\alpha^2(x_{s-1}),[\alpha(x_s),\alpha(x_t)],\alpha^2(x_{s+1}),\ldots,\widehat{x_t},\ldots,\alpha^2(x_k))\\ &\quad+\sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)}[\alpha^{k+r}(x_s),f(\alpha(x_0),\ldots,\widehat{x_s},\ldots,\alpha(x_k))]_V\\ &=\sum_{0\leqslant s<t\leqslant k}(-1)^{t+|x_t|(|f|+|x_{s+1}|+\ldots+|x_{t-1}|)}\\ &\quad\times f\circ\alpha(\alpha(x_0),\ldots,\alpha(x_{s-1}),[x_s,x_t],\alpha(x_{s+1}),\ldots,\widehat{x_t},\ldots,\alpha(x_k))\\ &\quad+\sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)}[\alpha^{k+r}(x_s),f\circ\alpha(x_0,\ldots,\widehat{x_s},\ldots,x_k)]_V\end{aligned}$$

$$= \sum_{0\leqslant s<t\leqslant k} (-1)^{t+|x_t|(|f|+|x_{s+1}|+\ldots+|x_{t-1}|)}$$
$$\times \beta\circ f(\alpha(x_0),\ldots,\alpha(x_{s-1}),[x_s,x_t],\alpha(x_{s+1}),\ldots,\widehat{x_t},\ldots,\alpha(x_k))$$
$$+ \sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)}[\alpha^{k+r}(x_s),\beta\circ f(x_0,\ldots,\widehat{x_s},\ldots,x_k)]_V$$
$$= \sum_{0\leqslant s<t\leqslant k} (-1)^{t+|x_t|(|f|+|x_{s+1}|+\ldots+|x_{t-1}|)}$$
$$\times \beta\circ f(\alpha(x_0),\ldots,\alpha(x_{s-1}),[x_s,x_t],\alpha(x_{s+1}),\ldots,\widehat{x_t},\ldots,\alpha(x_k))$$
$$+ \sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)}\beta([\alpha^{k+r-1}(x_s);f(x_0,\ldots,\widehat{x_s},\ldots,x_k)]_V)$$
$$= \beta\circ\delta_r^k(k)(x_0,\ldots,x_k),$$

which completes the proof. □

Theorem 3.6. *Let $(\mathcal{G},[\cdot,\cdot],\alpha)$ be a multiplicative Hom-Lie superalgebra and $(V,[\cdot,\cdot]_V,\beta)$ a $\mathcal{G}$-Hom-module.*

For a given integer $r\geqslant 1$, the pair $\Big(\bigoplus_{k>0} C^k_{\alpha,\beta}(\mathcal{G},V),\{\delta_r^k\}_{k>0}\Big)$ defines a cohomology complex, that is $\delta_r^k\circ\delta_r^{k-1}=0$.

Proof. For any $f\in C^{k-1}_{\alpha,\beta}(\mathcal{G},V)$ we have

$$\delta_r^k\circ\delta_r^{k-1}(f)(x_0,\ldots,x_k)=\sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} \tag{3.5}$$
$$\times\,\delta^{k-1}(f)(\alpha(x_0),\ldots,\alpha(x_{s-1}),[x_s,x_t],\alpha(x_{s+1}),\ldots,\widehat{x_t},\ldots,\alpha(x_k))$$
$$+\sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)} \tag{3.6}$$
$$\times\,[\alpha^{k+r-1}(x_s),\delta_r^{k-1}(f)(x_0,\ldots,\widehat{x_s},\ldots,x_k)]_V.$$

We evaluate the term (3.5):

$$\delta^{k-1}(f)(\alpha(x_0),\ldots,\alpha(x_{s-1}),[x_s,x_t],\alpha(x_{s+1}),\ldots,\widehat{x_t},\ldots,\alpha(x_k))$$
$$=\sum_{s'<t'<s}(-1)^{t'+|x_{t'}|(|x_{s'+1}|+\ldots+|x_{t'-1}|)} \tag{3.7}$$
$$\times f(\alpha^2(x_0),\ldots,\alpha^2(x_{s'-1}),[\alpha(x_{s'}),\alpha(x_{t'})],\alpha^2(x_{s'+1}),\ldots,$$
$$\widehat{x_{t'}},\ldots,\alpha^2(x_{s-1}),\alpha([x_s,x_t]),\alpha^2(x_{s+1}),\ldots,\widehat{x_t},\ldots,\alpha^2(x_k))$$

$$(3.8)\quad + \sum_{s'<s} (-1)^{s+|x_s|(|x_{s'+1}|+\dots+|x_{s-1}|)}$$
$$\times f(\alpha^2(x_0), \dots, \alpha^2(x_{s'-1}), [\alpha(x_{s'-1}), [x_s, x_t]],$$
$$\alpha^2(x_{s'+1}), \dots, \widehat{x_{s,t}}, \dots, \alpha^2(x_k))$$
$$(3.9)\quad + \sum_{s'<s<t'<t} (-1)^{t'+|x_{t'}|(|x_{s'+1}|+\dots+|[x_s,x_t]|+\dots+|x_{t'-1}|)}$$
$$\times f(\alpha^2(x_0), \dots, \alpha^2(x_{s'-1}), [\alpha(x_{s'}), \alpha(x_{t'})], \alpha^2(x_{s'+1}), \dots,$$
$$\alpha([x_s, x_t]), \dots, \widehat{x_{t'}}, \dots, \alpha^2(x_k))$$
$$(3.10)\quad + \sum_{s'<s<t<t'} (-1)^{t'+|x_{t'}|(|x_{s'+1}|+\dots+|x_{s-1}|+|[x_s,x_t]|+|x_{s+1}|+\dots+\widehat{|x_t|}+\dots+|x_{t'-1}|)}$$
$$\times f(\alpha^2(x_0), \dots, \alpha^2(x_{s'-1}), [\alpha(x_{s'}), \alpha(x_{t'})], \alpha^2(x_{s'+1}), \dots,$$
$$\alpha([x_s, x_t]), \dots, \widehat{x}_t, \dots, \widehat{x_{t'}}, \dots, \alpha^2(x_k))$$
$$(3.11)\quad + \sum_{s<t'<t} (-1)^{t'+|x_{t'}|(|x_{s+1}|+\dots+|x_{t'-1}|)}$$
$$\times f(\alpha^2(x_0), \dots, [[x_s, x_t], \alpha(x_{t'})], \alpha^2(x_{s+1}), \dots, \widehat{x_{t,t'}}, \dots, \alpha^2(x_k))$$
$$(3.12)\quad + \sum_{s<t<t'} (-1)^{t'-1+|x_{t'}|(|x_{s+1}|+\dots+\widehat{|x_t|}+\dots+|x_{t'-1}|)}$$
$$\times f(\alpha^2(x_0), \dots, \alpha^2(x_{s-1}), [[x_s, x_t], \alpha(x_{t'})], \alpha^2(x_{s+1}), \dots, \widehat{x_{t,t'}}, \dots, \alpha^2(x_k))$$
$$(3.13)\quad + \sum_{s<s'<t'<t} (-1)^{t'+|x_{t'}|(|x_{s'+1}|+\dots+|x_{t'-1}|)}$$
$$\times f(\alpha^2(x_0), \dots, \alpha^2(x_{s-1}), \alpha([x_s, x_t]), \dots,$$
$$[\alpha(x_{s'}), \alpha(x_{t'})], \dots, \widehat{x_{t'}}, \dots, \widehat{x}_t, \dots, \alpha^2(x_k))$$
$$(3.14)\quad + \sum_{s<s'<t<t'} (-1)^{t'-1+|x_{t'}|(|x_{s'+1}|+\dots+\widehat{x}_t+\dots+|x_{t'-1}|)}$$
$$\times f(\alpha^2(x_0), \dots, \alpha^2(x_{s-1}), \alpha([x_s, x_t]), \alpha^2(x_{s+1}), \dots,$$
$$[\alpha(x_{s'}), \alpha(x_{t'})], \dots, \widehat{x}_t, \dots, \widehat{x_{t'}}, \alpha^2(x_k))$$
$$(3.15)\quad + \sum_{t<s'<t'} (-1)^{t'+|x_{t'}|(|x_{s'+1}|+\dots+\widehat{x_{t,t'}}+\dots+|x_{t'-1}|)}$$
$$\times f(\alpha^2(x_0), \dots, \alpha^2(x_{s-1}), \alpha([x_s, x_t]), \alpha^2(x_{s+1}), \dots,$$
$$\widehat{x}_t, \dots, [\alpha(x_{s'}), \alpha(x_{t'})], \dots, \widehat{x_{t'}}, \dots, \alpha^2(x_k))$$
$$(3.16)\quad + \sum_{0\leqslant s'<s} (-1)^{s'+|x_{s'}|(|f|+|x_0|+\dots+|x_{s'-1}|)}$$
$$\times [\alpha^{k+r-1}(x_{s'}), f(\alpha(x_0), \dots, \widehat{x_{s'}}, \dots, [x_s, x_t], \dots, \widehat{x}_t, \dots, \alpha(x_k))]_V$$
$$(3.17)\quad + (-1)^{s+|[x_s,x_s]|(|f|+|x_0|+\dots+|x_{s-1}|)}$$
$$\times [\alpha^{k+r-2}([x_s, x_t]), f(\alpha(x_0), \dots, \widehat{[x_s, x_t]}, \alpha(x_{s+1}), \dots, \widehat{x}_t, \dots, \alpha(x_k))]_V$$

$$(3.18)\quad + \sum_{s<s'<t} (-1)^{s'+|x_{s'}|(|f|+|x_0|+\ldots+|[x_s,x_t]|+\ldots+|x_{s'-1}|)} \times [\alpha^{k+r-1}(x_{s'}), f(\alpha(x_0),\ldots,[x_s,x_t],\ldots,\widehat{x_{s',t}},\ldots,\alpha(x_k))]_V$$

$$(3.19)\quad + \sum_{t<s'} (-1)^{s'+|x_{s'}|(|f|+|x_0|+\ldots+|[x_s,x_t]|+\ldots+\widehat{|x_t|}+\ldots,|x_{s'-1}|)} \times [\alpha^{k+r-1}(x_{s'}), f(\alpha(x_0),\ldots,[x_s,x_t],\ldots,\widehat{x_{t,s'}},\ldots,\alpha(x_k))]_V.$$

The term (3.6) implies that

$$[\alpha^{k+r-1}(x_s), \delta^{k-1}(f)(x_0,\ldots,\widehat{x_s},\ldots,x_k)]_V$$

$$(3.20)= \Bigg[\alpha^{k+r-1}(x_s), \sum_{s'<t'<s} (-1)^{t'+|x_{t'}|(|x_{s'+1}|+\ldots+|x_{t'-1}|)} f(\alpha(x_0),\ldots,\alpha(x_{s'-1}), [x_{s'},x_{t'}],\alpha(x_{s'+1}),\ldots,\widehat{x_{s',t',s}},\alpha(x_{s+1}),\ldots,\alpha(x_k))\Bigg]_V$$

$$(3.21)\quad + \Bigg[\alpha^{k+r-1}(x_s), \sum_{s'<s<t'} (-1)^{t'-1+|x_{t'}|(|x_{s'+1}|+\ldots+\widehat{|x_s|}+\ldots+|x_{t'-1}|)} \times f(\alpha(x_0),\ldots,\alpha(x_{s'-1}),[x_{s'},x_{t'}],\alpha(x_{s'+1}),\ldots,\widehat{x_{t,s'}},\ldots,\alpha(x_k))\Bigg]_V$$

$$(3.22)\quad + \Bigg[\alpha^{k+r-1}(x_s), \sum_{s<s'<t'} (-1)^{t'+|x_{t'}|(|x_{s'+1}|+\ldots+|x_{t'-1}|)} \times f(\alpha(x_0),\ldots,\widehat{x_s},\ldots,\alpha(x_{s'-1}),[x_{s'},x_{t'}],\alpha(x_{s'+1}),\ldots,\widehat{x_{t'}},\ldots,\alpha(x_k))\Bigg]_V$$

$$(3.23)\quad + \Bigg[\alpha^{k+r-1}(x_s), \sum_{s'=0}^{s-1} (-1)^{s'+|x_{s'}|(|c|+|x_0|+\ldots+|x_{s'-1}|)} \times [\alpha^{k+r-2}(s'), f(x_0,\ldots,\widehat{x_{s',s}},\ldots,x_k)]_V\Bigg]_V$$

$$(3.24)\quad + \Bigg[\alpha^{k+r-1}(x_s), \sum_{s'=s+1}^{k} (-1)^{s'-1+|x_{s'}|(|f|+|x_0|+\ldots+\widehat{|x_s|}+\ldots+|x_{s'-1}|)} \times [\alpha^{k+r-2}(s'), f(x_0,\ldots,\widehat{x_{s',s}},\ldots,x_k)]_V\Bigg]_V.$$

Super-Hom-Jacobi identity leads to

$$\sum_{s<t} (-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)}((3.8)+(3.11)+(3.12)) = 0.$$

Using (3.2) and (3.3), we obtain by (3.17)

$$
\begin{aligned}
&= [\alpha^{k+r-2}([x_s, x_t]);\ f(\alpha(x_0), \ldots, \alpha(x_{s-1}),\\
&\quad \alpha(\widehat{[x_s, x_t]}), \alpha(x_{s+1}), \ldots, \widehat{x}_t, \ldots, \alpha(x_k))]_V\\
&= [\alpha^{k+r-1}(x_s), [\alpha^{k+r-2}(x_t), f(x_0, \ldots, \widehat{x_{s,t}}, \ldots, x_k)]_V]_V\\
&\quad - [\alpha^{k+r-1}(x_t), [\alpha^{k+r-2}(x_s), f(x_0, \ldots, \widehat{x_{s,t}}, \ldots, x_k)]_V]_V.
\end{aligned} \tag{3.25}
$$

Thus by (3.17), (3.23), and (3.24)

$$
\begin{aligned}
&\sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} + \sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)}\\
&\qquad + \sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)} = 0.
\end{aligned}
$$

By a simple calculation, we get by (3.16), (3.22), (3.18), (3.21), (3.19), and (3.20)

$$
\sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} + \sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)} = 0,
$$

$$
\sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} + \sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)} = 0,
$$

$$
\sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} + \sum_{s=0}^{k}(-1)^{s+|x_s|(|f|+|x_0|+\ldots+|x_{s-1}|)} = 0,
$$

and ((3.9)+(3.14))

$$
\begin{aligned}
&\sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)}\\
&= \sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} \sum_{s'<s<t'<t}(-1)^{t'+|x_{t'}|(|x_{s'+1}|+\ldots+|[x_s,x_t]|+\ldots+|x_{t'-1}|)}\\
&f(\alpha^2(x_0), \ldots, \alpha^2(x_{s'-1}), [\alpha(x_{s'}), \alpha(x_{t'})], \alpha^2(x_{s'+1}), \ldots, \alpha([x_s, x_t]), \ldots, \widehat{x_{t'}}, \ldots, \alpha^2(x_k))\\
&+ \sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} \sum_{s<s'<t<t'}(-1)^{t'-1+|x_{t'}|(|x_{s'+1}|+\ldots+\widehat{x}_t+\ldots+|x_{t'-1}|)}\\
&f(\alpha^2(x_0), \ldots, \alpha^2(x_{s-1}), \alpha([x_s, x_t]), \alpha^2(x_{s+1}), \ldots, \widehat{x_{t,t'}}, \ldots, [\alpha(x_{s'}), \alpha(x_{t'})] \ldots, \alpha^2(x_k))\\
&= 0.
\end{aligned}
$$

Similarly, $\sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} = 0$ ((3.7)+(3.15)) and ((3.10)+(3.13))

$$
\sum_{s<t}(-1)^{t+|x_t|(|x_{s+1}|+\ldots+|x_{t-1}|)} = 0.
$$

Therefore $\delta_r^k \circ \delta_r^{k-1} = 0$. □

The previous theorem shows that we may have infinitely many cohomology complexes.

Remark 3.7. From the proof of Theorem 3.6 we can deduce that if $[\cdot,\cdot]_V = 0$ then $\delta_r^k \circ \delta_r^{k-1}(f) = 0,\ f \in C^k(\mathcal{G}, V)$.

The corresponding cocycles, coboundaries and cohomology groups are defined as follows.

Definition 3.8. Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra and $(V, [\cdot,\cdot]_V, \beta)$ a Hom-module. With respect to the r-cohomology defined by the coboundary operators

$$\delta_r^k\colon\ C^k_{\alpha,\beta}(\mathcal{G}, V) \to C^{k+1}_{\alpha,\beta}(\mathcal{G}, V),$$

we have:

- ▷ The k-cocycles space is defined as $Z_r^k(\mathcal{G}, V) = \ker \delta_r^k$. The even or odd k-cocycles space is defined as $Z_{r,0}^k(\mathcal{G}, V) = Z_r^k(\mathcal{G}, V) \cap (C^k_{\alpha,\beta}(\mathcal{G}, V))_0$ or $Z_{r,1}^k(\mathcal{G}, V) = Z_r^k(\mathcal{G}, V) \cap (C^k_{\alpha,\beta}(\mathcal{G}, V))_1$, respectively.
- ▷ The k-coboundaries space is defined as $B_r^k(\mathcal{G}, V) = \operatorname{Im} \delta_r^{k-1}$. The even or odd k-coboundaries space is $B_{r,0}^k(\mathcal{G}, V) = B_r^k(\mathcal{G}, V) \cap (C^k_{\alpha,\beta}(\mathcal{G}, V))_0$ or $B_{r,1}^k(\mathcal{G}, V) = B_r^k(\mathcal{G}, V) \cap (C^k_{\alpha,\beta}(\mathcal{G}, V))_1$, respectively.
- ▷ The k^{th} cohomology space is the quotient $H_r^k(\mathcal{G}, V) = Z_r^k(\mathcal{G}, V)/B_r^k(\mathcal{G}, V)$. It decomposes as well as the even and odd k^{th} cohomology spaces.

Finally, we denote by $H_r^k(\mathcal{G}, V) = H_{r,0}^k(\mathcal{G}, V) \oplus H_{r,1}^k(\mathcal{G}, V)$ the k^{th} r-cohomology space and by $\bigoplus_{k\geqslant 0} H_r^k(\mathcal{G}, V)$ the r-cohomology group of the Hom-Lie superalgebra $\mathcal{G}$ with values in V.

Remark 3.9. The $Z_r^1(\mathcal{G}, \mathcal{G})$ is the set of α^r-derivations of $\mathcal{G}$.

Example 3.10. In this example we compute the second scalar cohomology group of the Hom-Lie superalgebra $\operatorname{osp}(1,2)_\lambda$ constructed in [3].

Let $\operatorname{osp}(1,2) = V_0 \oplus V_1$ be the vector superspace where V_0 is generated by

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and V_1 is generated by

$$F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $\lambda \in \mathbb{R}^*$, we consider the linear map $\alpha_\lambda\colon \operatorname{osp}(1,2) \to \operatorname{osp}(1,2)$ defined by:

$$\alpha_\lambda(X) = \lambda^2 X, \quad \alpha_\lambda(Y) = \frac{1}{\lambda^2}Y, \quad \alpha_\lambda(H) = H, \quad \alpha_\lambda(F) = \frac{1}{\lambda}F, \quad \alpha_\lambda(G) = \lambda G.$$

We define a superalgebra bracket $[\cdot,\cdot]_\lambda$ with respect to the basis, for $\lambda \neq 0$, by

$$[H,X]_\lambda = 2\lambda^2 X, \quad [H,Y]_\lambda = -\frac{2}{\lambda^2}Y, \quad [X,Y]_\lambda = H, \quad [Y,G]_\lambda = \frac{1}{\lambda}F,$$
$$[X,F]_\lambda = \lambda G, \quad [H,F]_\lambda = -\frac{1}{\lambda}F, \quad [H,G]_\lambda = \lambda G, \quad [G,F]_\lambda = H, \quad [G,X] = 0,$$
$$[Y,F] = 0, \quad [G,G]_\lambda = -2\lambda^2 X, \quad [F,F]_\lambda = \frac{2}{\lambda^2}Y.$$

Then $\operatorname{osp}(1,2)_\lambda = (\operatorname{osp}(1,2), [\cdot,\cdot]_\lambda, \alpha_\lambda)$ is a Hom-Lie superalgebra.

Let $f \in C^1_{\alpha,\mathrm{Id}_\mathbb{C}}(\operatorname{osp}(1,2),\mathbb{C})$. The scalar 2-coboundary operator is defined according to (3.4) by

$$\begin{aligned}\delta^2(f)(x_0,x_1,x_2) = {} & -f([x_0,x_1],\alpha(x_2)) \\ & + (-1)^{|x_2||x_1|} f([x_0,x_2],\alpha(x_1)) + f(\alpha(x_0),[x_1,x_2]).\end{aligned} \tag{3.26}$$

Now, we suppose that f is a 2-cocycle of $\operatorname{osp}(1,2)_\lambda$. Then f satisfies

$$-f([x_0,x_1],\alpha(x_2)) + (-1)^{|x_2||x_1|} f([x_0,x_2],\alpha(x_1)) + f(\alpha(x_0),[x_1,x_2]) = 0. \tag{3.27}$$

By plugging the triples

$$(H,X,F),\ (H,X,Y),\ (H,X,G),\ (H,Y,G),\ (X,Y,F),\ (X,F,G), (Y,F,G),$$
$$(H,Y,F),\ (X,Y,G),\ (H,F,G),\ (H,F,F),\ (H,G,G),\ (X,G,G)$$

respectively, in (3.27) we obtain

$$f(H,G) = f(X,F),\ f(G,X) = 0,\ f(H,F) = f(G,Y),\ f(G,G) = f(X,H),$$
$$f(F,F) = f(Y,H)\ f(F,Y) = 0,\ f(X,Y) = f(G,F).$$

So, if we consider the map $g\colon \operatorname{osp}(1,2) \to \mathbb{R}$ defined by

$$g(X) = \frac{1}{2\lambda^2} f(H,X), \quad g(Y) = -\frac{\lambda^2}{2} f(H,X), \quad g(F) = -\lambda f(H,F),$$
$$g(G) = \frac{1}{\lambda} f(H,G), \quad g(H) = f(X,Y),$$

we obtain

$$\begin{aligned}& f(a_1H + a_2X + a_3Y + a_4F + a_5G, b_1H + b_2X + b_3Y + b_4F + b_5G) \\ & \quad = \delta(g)(a_1H + a_2X + a_3Y + a_4F + a_5G, b_1H + b_2X + b_3Y + b_4F + b_5G).\end{aligned}$$

Therefore $H^2(\mathrm{osp}(1,2)_\lambda, \mathbb{C}) = \{0\}$.

Notice that this result is the same for any $r \geqslant 1$.

4. Extensions of Hom-Lie superalgebras

The extension theory of Hom-Lie algebras was presented first in [5], [7].

An extension of a Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$ by a Hom-module (V, α_V) is an exact sequence

$$0 \longrightarrow (V, \alpha_V) \xrightarrow{\ i\ } (\widetilde{\mathcal{G}}, \widetilde{\alpha}) \xrightarrow{\ \pi\ } (\mathcal{G}, \alpha) \longrightarrow 0$$

satisfying $\tilde{\alpha} \circ i = i \circ \alpha_V$ and $\alpha \circ \pi = \pi \circ \tilde{\alpha}$.

We say that the extension is central if $[\widetilde{\mathcal{G}}, i(V)]_{\widetilde{\mathcal{G}}} = 0$.

Two extensions

$$0 \longrightarrow (V, \alpha_V) \xrightarrow{\ i_k\ } (\mathcal{G}_k, \alpha_k) \xrightarrow{\ \pi_k\ } (\mathcal{G}, \alpha) \longrightarrow 0 \ (k = 1, 2)$$

are equivalent if there is an isomorpism $\varphi\colon (\mathcal{G}_1, \alpha_1) \to (\mathcal{G}_2, \alpha_2)$ such that $\varphi \circ i_1 = i_2$ and $\pi_2 \circ \varphi = \pi_1$.

4.1. Trivial representation of Hom-Lie superalgebras. Let $V = \mathbb{C}$ (or $\mathbb{R}$) and $[\cdot,\cdot]_V = 0$. Obviously, $\forall \beta \in \mathrm{End}(\mathbb{C})$, $(\mathcal{G}, [\cdot,\cdot]_V, \beta)$ is a representation of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$. This representation is called the trivial representation of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$.

In the following we consider central extensions of a Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$. We will see that it is controlled by the second cohomology group $H^2(\mathcal{G}, V)$. Let $\theta \in C^2_\alpha(\mathcal{G}, V)$, we consider the direct sum $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}_0} \oplus \widetilde{\mathcal{G}_1}$ where $\widetilde{\mathcal{G}_0} = \mathcal{G}_0 \oplus \mathbb{C}$ and $\widetilde{\mathcal{G}_1} = \mathcal{G}_1$ with the bracket

$$[(x, s), (y, t)]_\theta = ([x, y], \theta(x, y)) \quad \forall x, y \in \mathcal{G},\ \ s, t \in \mathbb{C}.$$

Define $\widetilde{\alpha}\colon \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}$ by $\widetilde{\alpha}(x, s) = (\alpha(x), s)$.

Theorem 4.1. *The triple $(\widetilde{\mathcal{G}}, [\cdot,\cdot]_\theta, \widetilde{\alpha})$ is a Hom-Lie superalgebra if and only if θ is a 2-cocycle (i.e. $\delta^2(\theta) = 0$).*

We call the Hom-Lie superalgebra $(\widetilde{\mathcal{G}}, [\cdot,\cdot]_\theta, \tilde{\alpha})$ the central extension of $(\mathcal{G}, [\cdot,\cdot], \alpha)$ by $\mathbb{C}$.

P r o o f. The map $\tilde{\alpha}$ is an algebra morphism with respect to the bracket $[\cdot,\cdot]_\theta$ as follows from the fact that $\theta \circ \alpha = \theta$. More precisely, we have

$$\tilde{\alpha}[(x, s), (y, t)]_\theta = (\alpha[x, y], \theta(x, y)).$$

On the other hand, we have

$$[\tilde{\alpha}(x,s),\tilde{\alpha}(y,t)]_\theta = [(\alpha(x),s),(\alpha(y),t)]_\theta = ([\alpha(x),\alpha(y)],\theta(\alpha(x),\alpha(y))).$$

Since α is an algebra morphism and $\theta(\alpha(x),\alpha(y)) = \theta(x,y)$, we deduce that $\tilde{\alpha}$ is an algebra morphism.

By direct computation, we have

$$\begin{aligned}
&\circlearrowleft_{(x,s),(y,t),(z,m)} (-1)^{|(x,s)||(z,m)|}[\tilde{\alpha}(x,s),[(y,t),(z,m)]_\theta]_\theta \\
&\quad= \circlearrowleft_{(x,s),(y,t),(z,m)} (-1)^{|x||z|}[(\alpha(x),s),([y,z],\theta(y,z))]_\theta \\
&\quad= \circlearrowleft_{x,y,z} (-1)^{|x||z|}([\alpha(x),[y,z]],\theta(\alpha(x),[y,z])) \\
&\quad= \circlearrowleft_{x,y,z} (-1)^{|x||z|}(0,\theta(\alpha(x),\theta(\alpha(x),[y,z])).
\end{aligned}$$

Thus, by the Hom-Jacobi identity of $\mathcal{G}$, the bracket $[\cdot,\cdot]_\theta$ satisfies the Hom-Jacobi identity if and only if

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}\theta(\alpha(x),[y,z]) = 0.$$

This means that $\delta^2\theta = 0$. □

4.2. Cohomology space $H^2(\mathcal{G},V)$ and central extensions.

Proposition 4.2. *Let $(\mathcal{G},[\cdot,\cdot],\alpha)$ be a multiplicative Hom-Lie superalgebra and V a $\mathcal{G}$-Hom-module. The second cohomology space $H^2(\mathcal{G},V) = Z^2(\mathcal{G},V)/B^2(\mathcal{G},V)$ is in one-to-one correspondence with the set of the equivalence classes of central extensions of $(\mathcal{G},[\cdot,\cdot],\alpha)$ by (V,β).*

Proof. Let

$$0 \longrightarrow (V,\beta) \xrightarrow{i} (\widetilde{\mathcal{G}},\tilde{\alpha}) \xrightarrow{\pi} (\mathcal{G},\alpha) \longrightarrow 0$$

be a central extension of the Hom-Lie superalgebra $(\mathcal{G},\alpha)$ by (V,β), so there is a space H such that $\widetilde{\mathcal{G}} = H \oplus i(V)$.

The maps $\pi_{/H}\colon H \to \mathcal{G}$ and $k\colon V \to i(V)$ defined, respectively, by $\pi_{/H}(x) = \pi(x)$ and $k(v) = i(v)$ are bijective, their inverses are denoted by s and l. Considering the map $\varphi\colon \mathcal{G}\times V \to \widetilde{\mathcal{G}}$ defined by $\varphi(x,v) = s(x)+i(v)$, it is easy to verify that φ is a bijective.

Since π is a homomorphism of Hom-Lie superalgebras hence $\pi([s(x),s(y)]_{\widetilde{\mathcal{G}}} - s([x,y])) = 0$. So $[s(x),s(y)]_{\widetilde{\mathcal{G}}} - s([x,y]) \in i(V)$.

We set $[s(x),s(y)] - s([x,y]) = G(x,y) \in i(V)$. Then $F(x,y) = l\circ G(x,y) \in V$ and it is easy to see that $F(x,x) = 0$ and then $F \in C^2(\mathcal{G},V)$ is a 2-cochain that defines a bracket on $\widetilde{\mathcal{G}}$. In fact, we can identify the superspace $L \times V$ and $\widetilde{\mathcal{G}}$ by $\varphi\colon (x,v) \to s(x)+i(v)$ where the bracket is

$$[s(x)+i(v), s(y)+i(w)]_{\widetilde{\mathcal{G}}} = [s(x), s(y)]_{\widetilde{\mathcal{G}}} = s([x,y]) + F(x,y).$$

Viewed as elements of $\mathcal{G} \times V$ we have $[(x,v),(y,w)] = ([x,y], F(x,y))$ and the homogeneous elements (x,v) of $\mathcal{G} \times V$ are such that $|x| = |v|$ and we have in this case $|(x,v)| = |x|$.

We deduce that we can assign a 2-cocycle $F \in Z^2(\mathcal{G}, V)$ to every central extension

$$0 \longrightarrow (V,\beta) \xrightarrow{\ i\ } (\widetilde{\mathcal{G}}, \tilde{\alpha}) \xrightarrow{\ \pi\ } (\mathcal{G},\alpha) \longrightarrow 0.$$

Indeed, for $x, y \in \mathcal{G}$, if we set

$$F(x,y) = l([s(x), s(y)] - s([x,y])) \in V,$$

then we have $F(x,y) \in V$ and F satisfies the 2-cocycle conditions.

Conversely, for each $f \in Z^2(\mathcal{G}, V)$ one can define a central extension

$$0 \longrightarrow (V,\beta) \longrightarrow (\mathcal{G}_f, \alpha_f) \longrightarrow (\mathcal{G},\alpha) \longrightarrow 0$$

by

$$[(x,v),(y,w)]_f = ([x,y], f(x,y)),$$

where $x, y \in \mathcal{G}$ and $v, w \in V$.

Let f and g be two elements of $Z^2(\mathcal{G}, V)$ such that $f - g \in B^2(\mathcal{G}, V)$, i.e. $(f-g)(x,y) = h([x,y])$, where $h\colon \mathcal{G} \to V$ is a linear map satisfying $h \circ \alpha = \beta \circ h$. Now we prove that the extensions defined by f and g are equivalent. Let us define $\Phi\colon \mathcal{G}_f \to \mathcal{G}_g$ by

$$\Phi(x,v) = (x, v - h(x)).$$

It is clear that Φ is bijective. Let us check that Φ is a homomorphism of Hom-Lie superalgebras. We have

$$\begin{aligned}[]
[\Phi((x,v)), \Phi((y,w))]_g &= [(x, v-h(x)), (y, w-h(y))]_g = ([x,y], g(x,y)) \\
&= ([x,y], f(x,y) - h([x,y])) = \Phi(([x,y], f(x,y))) = \Phi([(x,v),(y,w)]_f)
\end{aligned}$$

and

$$\begin{aligned}
\Phi \circ \tilde{\alpha}((x,v)) &= \Phi(\alpha(x), \beta(v)) = (\alpha(x), \beta(v) - h(\alpha(x))) \\
&= (\alpha(x), \beta(v) - \beta \circ h(x)) = (\alpha(x), \beta(v - h(x))) = \tilde{\alpha} \circ \Phi(x,v).
\end{aligned}$$

Next, we show that for $f, g \in Z^2(\mathcal{G}, V)$ such that the central extensions $0 \to (V, \beta) \to (\mathcal{G}_f, \tilde{\alpha}) \to (\mathcal{G}, \alpha) \to 0$ and $0 \to (V, \beta) \to (\mathcal{G}_g, \tilde{\alpha}) \to (\mathcal{G}, \alpha) \to 0$ are equivalent, we have $f - g \in B^2(\mathcal{G}, V)$. Let Φ be a homomorphism of Hom-Lie superalgebras such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V,\beta) & \xrightarrow{i_1} & (\mathcal{G}_f, \widetilde{\alpha}) & \xrightarrow{\pi_1} & (\mathcal{G}, \alpha) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}_V} & & \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{G}}} & & \\ 0 & \longrightarrow & (V,\beta) & \xrightarrow{i_2} & (\mathcal{G}_g, \tilde{\alpha}) & \xrightarrow{\pi_2} & (\mathcal{G}, \alpha) & \longrightarrow & 0 \end{array}$$

commutes. We can express $\Phi(x, v) = (x, v - h(x))$ for some linear map $h\colon \mathcal{G} \to V$. Then we have

$$\begin{aligned} \Phi([(x,v),(y,w)]_f) &= \Phi(([x,y], f(x,y))) = ([x,y], f(x,y) - h([x,y])), \\ [\Phi((x,v)), \Phi((y,w))]_g &= [(x, v - h(x)), (y, w - h(y))]_g = ([x,y], g(x,y)), \end{aligned}$$

and thus $(f - g)(x, y) = h([x, y])$ (i.e. $f - g \in B^2(\mathcal{G}, V)$), so we have completed the proof. □

4.3. The adjoint representation of Hom-Lie superalgebras.

In this section we generalize some results of [19].

Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. We consider $\mathcal{G}$ as a representation on itself via the bracket and with respect to the morphism α.

Definition 4.3. The α^s-adjoint representation of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$, which we denote by ad_s, is defined by

$$\mathrm{ad}_s(a)(x) = [\alpha^s(a), x], \ \forall a, \ x \in \mathcal{G}.$$

Lemma 4.4. *With the above notation, we have that $(\mathcal{G}, \mathrm{ad}_s(\cdot)(\cdot), \alpha)$ is a representation of the Hom-Lie superalgebra $\mathcal{G}$.*

P r o o f. The result follows from

$$\mathrm{ad}_s(\alpha(a))(\alpha(x)) = [\alpha^{s+1}(a), \alpha(x)] = \alpha([\alpha^s(a), x]) = \alpha \circ \mathrm{ad}_s(a)(x),$$

and

$$\begin{aligned} \mathrm{ad}_s([x,y])(\alpha(z)) &= [\alpha^s([x,y]), \alpha(z)] = [[\alpha^s(x), \alpha^s(y)], \alpha(z)] \\ &= -(-1)^{|z||[x,y]|}[\alpha(z), [\alpha^s(x), \alpha^s(y)]] \\ &= (-1)^{|z||x|}(-1)^{|z||x|}[\alpha^{s+1}(x), [\alpha^s(y), z]] \\ &\quad + (-1)^{|z||x|}(-1)^{|y||x|}[\alpha^{s+1}(y), [z, \alpha^s(x)]] \\ &= [\alpha^{s+1}(x), [\alpha^s(y), z]] - (-1)^{|x||y|}[\alpha^{s+1}(y), [\alpha^s(x), z]]. \end{aligned}$$

□

The set of k-hom-cochains on $\mathcal{G}$ with coefficients in $\mathcal{G}$, which we denote by $C^k_\alpha(\mathcal{G};\mathcal{G})$, is given by

$$C^k_\alpha(\mathcal{G};\mathcal{G}) = \{f \in C^k(\mathcal{G};\mathcal{G}) \colon\ f\circ\alpha = \alpha\circ f\}.$$

In particular, the set of 0-Hom-cochains is given by:

$$C^0_\alpha(\mathcal{G};\mathcal{G}) = \{x \in \mathcal{G} \colon\ \alpha(x) = x\}.$$

Proposition 4.5. *With respect to the α^s-adjoint representation* ad_s, *of the Hom-Lie superalgebra $(\mathcal{G},[\cdot,\cdot],\alpha)$, $D \in C^1_{\alpha,\mathrm{ad}_s}$ is a 1-cocycle if and only if D is an α^{s+1}-derivation of the Hom-Lie superalgebra $(\mathcal{G},[\cdot,\cdot],\alpha)$, i.e. $D \in \mathrm{Der}_{\alpha^{s+1}}(\mathcal{G})$.*

Proof. The conclusion follows directly from the definition of the coboundary operator δ. D is closed if and only if

$$\begin{aligned}\delta(D)(x,y) = &-D([x,y]) + (-1)^{|x||D|}[\alpha^{s+1}(x), D(y)]\\ &+ (-1)^{1+|y|(|D|+|x|)}[\alpha^{s+1}(y), D(x)] = 0,\end{aligned}$$

so

$$D([x,y]) = [D(x),\alpha^{s+1}(y)] + (-1)^{|x||D|}[\alpha^{s+1}(x), D(y)],$$

which implies that D is an α^{s+1}-derivation. □

4.3.1. The α^{-1}-adjoint representation ad_{-1}.

Proposition 4.6. *With respect to the α^{-1}-adjoint representation* ad_{-1}, *we have*

$$\begin{aligned}&H^0(\mathcal{G},\mathcal{G}) = C^0_\alpha(\mathcal{G};\mathcal{G}) = \{x\in\mathcal{G}\colon\ \alpha(x) = x\};\\ &H^1(\mathcal{G},\mathcal{G}) = \mathrm{Der}_{\alpha^0}(\mathcal{G}).\end{aligned}$$

Proof. For any 0-hom-cochain $x \in C^0_\alpha(\mathcal{G};\mathcal{G})$ we have $\delta(x)(y) = (-1)^{|y||x|}[\alpha^{-1}(y), x] = 0$ for all $y \in \mathcal{G}$.

Therefore, any 0-hom-cochain is closed. Thus, we have $H^0(\mathcal{G},\mathcal{G}) = C^0_\alpha(\mathcal{G};\mathcal{G}) = \{x \in \mathcal{G}\colon\ \alpha(x) = x\}$. Since there is no exact 1-hom-cochain, by Proposition 4.5 we have $H^1(\mathcal{G},\mathcal{G}) = \mathrm{Der}_{\alpha^0}(\mathcal{G})$. □

Let $\omega \in C^2_\alpha(\mathcal{G};\mathcal{G})$ be an even super-skew-symmetric bilinear operator commuting with α. Consider a t-parametrized family of bilinear operations

$$[x,y]_t = [x,y] + t\omega(x,y).$$

Since ω commutes with α, α is a morphism with respect to the bracket $[\cdot,\cdot]_t$ for every t. If $(\mathcal{G}[[t]], [\cdot,\cdot]_t, \alpha)$ is a Hom-Lie superalgebra, we say that ω generates a deformation of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$. The super Hom-Jacobi identity of $[\cdot,\cdot]_t$, is equivalent to

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}(\omega(\alpha(x),[y,z]) + [\alpha(x),[y,z]]) = 0, \tag{4.1}$$

$$\circlearrowleft_{x,y,z} (-1)^{|x||z|}\omega(\alpha(x),\omega(y,z)) = 0. \tag{4.2}$$

Obviously, (4.1) means that ω is an even 2-cycle with respect to the α^{-1}-adjoint representation ad_{-1}. Furthermore, (4.2) means that ω must itself define a Hom-Lie superalgebra structure on $\mathcal{G}$.

4.3.2. The α^0-adjoint representation ad_0.

Proposition 4.7. *With respect to the α^0-adjoint representation ad_0, we have*

$$H^0(\mathcal{G};\mathcal{G}) = \{x \in \mathcal{G}\colon\ \alpha(x) = x, [x,y] = 0\ \forall y \in \mathcal{G}\},$$
$$H^1(\mathcal{G};\mathcal{G}) = \mathrm{Der}_\alpha(\mathcal{G})/\mathrm{Inn}_\alpha(\mathcal{G}).$$

Proof. For any 0-hom-cochain we have $d_0x(y) = [\alpha^0(y), x] = [x,y]$.

Therefore, the set of 0-cycles $Z^0(\mathcal{G},\mathcal{G})$ is given by $Z^0(\mathcal{G},\mathcal{G}) = \{x \in C^0_\alpha(\mathcal{G},\mathcal{G})\colon [x,y] = 0\ \forall y \in \mathcal{G}\}$. Since $B^0(\mathcal{G},\mathcal{G}) = \{0\}$, we deduce that $H^0(\mathcal{G};\mathcal{G}) = \{x \in \mathcal{G}\colon \alpha(x) = x, [x,y] = 0\ \forall y \in \mathcal{G}\}$.

For any $f \in C^1_\alpha(\mathcal{G},\mathcal{G})$ we have

$$\delta(f)(x,y) = -f([x,y]) + (-1)^{|x||f|}[\alpha(x), f(y)] + (-1)^{1+|y|(|f|+|x|)}[\alpha(y), f(x)],$$

so,

$$\delta(f)(x,y) = -f([x,y]) + [f(x),\alpha(y)] + (-1)^{|x||f|}[\alpha(x), f(y)].$$

Therefore, the set of 1-cocycles $Z^1(\mathcal{G},\mathcal{G})$ is exactly the set of α-derivation Der_α.

Furthermore, it is obvious that any exact 1-coboundary is of the form of $[x,\cdot]$ for some $x \in C^0_\alpha(\mathcal{G};\mathcal{G})$. Therefore, we have $B^1(\mathcal{G},\mathcal{G}) = \mathrm{Inn}_\alpha(\mathcal{G})$. This implies that $H^1(\mathcal{G};\mathcal{G}) = \mathrm{Der}_\alpha(\mathcal{G})/\mathrm{Inn}_\alpha(\mathcal{G})$. □

4.3.3. The coadjoint representation ad^*.

Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra and $(\mathcal{G}, [\cdot,\cdot]_V, \beta)$ a representation of $\mathcal{G}$. Let V^* be the dual vector space of V. We define an even bilinear map $[\cdot,\cdot]_{V^*}\colon \mathcal{G} \times V^* \to V^*$ by

$$[x,f]_{V^*}(v) = -f([x,v]_V),\ \forall x \in \mathcal{G},\ f \in V^*,\ \text{and } v \in V.$$

Let $f \in V^*$, $x, y \in \mathcal{G}$ and $v \in V$. We compute the right hand side of the identity (4.2):

$$\begin{aligned}
&[\alpha(x), [y, f]_{V^*}]_{V^*}(v) - (-1)^{|x||y|}[\alpha(y), [x, v]_{V^*}]_{V^*} \\
&\quad = -[y, f]_{V^*}([\alpha(x), v]_V) + (-1)^{|x||y|}[x, f]_{V^*}([\alpha(y), v]_V) \\
&\quad = f([y, [\alpha(x), v]_V]_V) - (-1)^{|x||y|} f([x, [\alpha(y), v]_V]_V).
\end{aligned}$$

On the other hand, since the twisted map for $[\cdot,\cdot]_{V^*}$ is $\beta^* = {}^t\beta$, the left hand side of the identity (4.2) reads

$$\begin{aligned}
[[x, y], \beta^*(f)]_{V^*}(v) &= -\beta^*(f)([[x, y], v]_V) = -{}^t\beta(f)([[x, y], v]_V) \\
&= -f \circ \beta([[x, y], v]_V).
\end{aligned}$$

Therefore, we have the following proposition:

Proposition 4.8. *Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra and $(V, [\cdot,\cdot]_V, \beta)$ a representation of $\mathcal{G}$. The triple $(V^*, [\cdot,\cdot]_{V^*}, \beta^*)$, where $[x, f]_{V^*}(v) = -f([x, v]_V)$, $\forall x \in \mathcal{G}$, $f \in V^*$, $v \in V$, defines a representation of the Hom-Lie superalgebra $(\mathcal{G}, [\cdot,\cdot], \alpha)$ if and only if*

$$[[x, y], \beta(v)]_V = (-1)^{|x||y|}[x, [\alpha(y), v]_V]_V - [y, [\alpha(x), v]_V]_V.$$

We obtain the following characterization in the case of adjoint representation.

Corollary 4.9. *Let $(\mathcal{G}, [\cdot,\cdot], \alpha)$ be a Hom-Lie superalgebra and $(\mathcal{G}, \mathrm{ad}, \alpha)$ the adjoint representation of $\mathcal{G}$, where $\mathrm{ad}\colon \mathcal{G} \to \mathrm{End}(\mathcal{G})$. We set $\mathrm{ad}^*\colon \mathcal{G} \to \mathrm{End}(\mathcal{G}^*)$ and $\mathrm{ad}^*(x)(f) = -f \circ \mathrm{ad}(x)$.*

Then $(\mathcal{G}^*, \mathrm{ad}^*, \alpha^*)$ is a representation of $\mathcal{G}$ if and only if

$$[[x, y], \alpha(z)] = (-1)^{|x||y|}[x, [\alpha(y), z]] - [y, [\alpha(x), z]], \quad \forall x, y, z \in \mathcal{G}.$$

5. Cohomology of q-Witt superalgebra

In the following, we describe the q-Witt Hom-Lie superalgebra obtained in [3] and compute its derivations and the second cohomology group.

Let $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ be an associative superalgebra. We assume that $\mathcal{A}$ is supercommutative, that is, for homogeneous elements a, b the identity $ab = (-1)^{|a||b|}ba$ holds. For example, $\mathcal{A}_0 = \mathbb{C}[t, t^{-1}]$ and $\mathcal{A}_1 = \theta\mathcal{A}_0$ where θ is the Grassman variable ($\theta^2 = 0$). Let $q \in \mathbb{C} \setminus \{0, 1\}$ and $n \in \mathbb{N}$, we set $\{n\} = \frac{1-q^n}{1-q}$, a q-number. Let σ be the algebra endomorphism on $\mathcal{A}$ defined by

$$\sigma(t^n) = q^n t^n \quad \text{and} \quad \sigma(\theta) = q\theta.$$

Let ∂_t and ∂_θ be two linear maps on $\mathcal{A}$ defined by

$$\partial_t(t^n) = \{n\}t^n, \quad \partial_t(\theta t^n) = \{n\}\theta t^n,$$
$$\partial_\theta(t^n) = 0, \quad \partial_\theta(\theta t^n) = q^n t^n.$$

Definition 5.1. Let $i \in \mathbb{Z}_2$. A σ-derivation D_i on $\mathcal{A}$ is an endomorphism satisfying:

$$D_i(ab) = D_i(a)b + (-1)^{i|a|}\sigma(a)D_i(b)$$

where $a, b \in \mathcal{A}$ are homogeneous elements and $|a|$ is the parity of a.

A σ-derivation D_0 is called an even σ-derivation and D_1 is called an odd σ-derivation. The set of all σ-derivations is denoted by $\mathrm{Der}_\sigma(\mathcal{A})$. Therefore, $\mathrm{Der}_\sigma(\mathcal{A}) = \mathrm{Der}_\sigma(\mathcal{A})_0 \oplus \mathrm{Der}_\sigma(\mathcal{A})_1$, where $\mathrm{Der}_\sigma(\mathcal{A})_0$ and $\mathrm{Der}_\sigma(\mathcal{A})_1$ are the spaces of even and odd σ-derivations, respectively.

Lemma 5.2. *The linear map $\Delta = \partial_t + \theta\partial_\theta$ on $\mathcal{A}$ is an even σ-derivation. Hence,*

$$\Delta(t^n) = \{n\}t^n,$$
$$\Delta(\theta t^n) = \{n+1\}\theta t^n.$$

Let $\mathcal{W}^q = \mathcal{A} \cdot \Delta$, be a superspace generated by the elements $L_n = t^n \cdot \Delta$ of parity 0 and the elements $G_n = \theta t^n \cdot \Delta$ of parity 1.

Let $[-,-]_\sigma$ be the bracket on the superspace $\mathcal{W}^q$ defined by

$$[L_n, L_m]_\sigma = (\{m\} - \{n\})L_{n+m}, \tag{5.1}$$
$$[L_n, G_m]_\sigma = (\{m+1\} - \{n\})G_{n+m}. \tag{5.2}$$

The other brackets are obtained by supersymmetry or are 0.

It is easy to see that $\mathcal{W}^q$ is a $\mathbb{Z}$-graded algebra

$$\mathcal{W}^q = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}^q_n,$$

where

$$\mathcal{W}_n^q = \operatorname{span}_{\mathbb{C}}\{L_n, G_n\}.$$

Let α be an even linear map on $\mathcal{W}^q$ defined on the generators by

$$\alpha(L_n) = (1+q^n)L_n,$$
$$\alpha(G_n) = (1+q^{n+1})G_n.$$

Proposition 5.3 ([3])**.** *The triple $(\mathcal{W}^q, [-,-]_\sigma, \alpha)$ is a Hom-Lie superalgebra.*

5.1. Derivations of the Hom-Lie superalgebra $\mathcal{W}^q$.

A homogeneous α^k-derivation is said to be of degree s if there exists $s \in \mathbb{Z}$ such that for all $n \in \mathbb{Z}$ we have $D(\langle L_n \rangle) \subset \langle L_{n+s} \rangle$. The corresponding subspace of homogeneous α^k-derivations of degree s is denoted by $\operatorname{Der}^s_{\alpha^k,i}$ $(i \in \mathbb{Z}_2)$.

It is easy to check that $\operatorname{Der}_{\alpha^k}(\mathcal{W}^q) = \bigoplus\limits_{s\in\mathbb{Z}} (\operatorname{Der}^s_{\alpha,0}(\mathcal{W}^q) \oplus \operatorname{Der}^s_{\alpha,1}(\mathcal{W}^q))$.

Let D be a homogeneous α^k-derivation

$$D([x,y]) = [D(x), \alpha^k(y)] + (-1)^{|x||D|}[\alpha^k(x), D(y)] \quad \text{for all homogeneous } x, y \in \mathcal{W}^q.$$

We deduce that

$$(5.3) \qquad (\{m\} - \{n\})D(L_{n+m}) = (1+q^m)^k[D(L_n), L_m] + (1+q^n)^k[L_n, D(L_m)]$$

and

$$(5.4) \qquad (\{m+1\} - \{n\})D(G_{n+m}) = (1+q^{m+1})^k[D(L_n), G_m] + (1+q^n)^k[L_n, D(L_m)] \quad \forall n, m \in \mathbb{Z}.$$

5.1.1. The α^0-derivation of the Hom-Lie superalgebra $\mathcal{W}^q$.

Proposition 5.4. *The set of α^0-derivations of the Hom-Lie superalgebra $\mathcal{W}^q$ is*

$$\operatorname{Der}_{\alpha^0}(\mathcal{W}^q) = \langle D_1 \rangle \oplus \langle D_2 \rangle$$

where D_1 and D_2 are defined with respect to the basis as

$$D_1(L_n) = nL_n, \quad D_1(G_n) = G_n,$$
$$D_2(L_n) = nG_{n-1}, \quad D_2(G_n) = L_{n-1}.$$

P r o o f. We consider two cases $|D| = 0$ and $|D| = 1$.

Case 1: $|D| = 0$.

Let D be an even derivation of degree s, $D(L_n) = a_{s,n}L_{s+n}$ and $D(G_n) = b_{s,n}G_{s+n}$. By (5.3) we have

$$(\{m\} - \{n\})a_{s,n+m} = (\{m\} - \{n+s\})a_{s,n} + (\{m+s\} - \{n\})a_{s,m}.$$

We deduce that

$$(q^n - q^m)a_{s,n+m} = (q^{n+s} - q^m)a_{s,n} + (q^n - q^{m+s})a_{s,m}.$$

If $m = 0$, we have

$$q^n(1 - q^s)a_{s,n} = (q^n - q^s)a_{s,0}.$$

If $s \neq 0$ we have

$$a_{s,n} = \frac{1 - q^{s-n}}{1 - q^s}\, a_{s,0}.$$

We deduce that

$$(5.5)\quad (q^n - q^m)\frac{1 - q^{s-n}}{1 - q^s}a_{s,0} = (q^{n+s} - q^m)\frac{1 - q^{s-n}}{1 - q^s}a_{s,0} + (q^n - q^{m+s})\frac{1 - q^{s-m}}{1 - q^s}a_{s,0}.$$

Taking $n = 2s$, $m = s$ in (5.5) we obtain $a_{s,0} = 0$, so $a_{s,n} = 0$.

If $s = 0$ and $n \neq m$ we have $a_{s,n} = na_{s,1}$.

By (5.4) and $D(G_n) = b_{s,n}G_{n+s}$ we have

$$(\{m+1\} - \{n\})b_{s,n+m} = (\{m+s+1\} - \{n\})b_{s,m}.$$

So

$$(q^n - q^{m+1})b_{s,n+m} = (q^n - q^{m+s+1})b_{s,m}.$$

Taking $n = 0$, we have $(q^{m+1} - q^{m+s+1})b_{s,m} = 0$, hence if $s \neq 0$ we have $b_{s,m} = 0$. If $s = 0$ and $n \neq m+1$ we have $b_{s,n+m} = b_{s,m}$, so $b_{s,n} = b_{s,0}$. Finally, it follows that the set of even α^0-derivations is $\mathrm{Der}_{\alpha^0,0}(\mathcal{W}^q) = \mathrm{Der}^0_{\alpha,0}(\mathcal{W}^q) = \langle D_1 \rangle$ with $D_1(L_n) = nL_n$ and $D_1(G_n) = G_n$.

Case 2: $|D| = 1$.

Let D be an odd derivation of degree s, $D(L_n) = a_{s,n}G_{s+n}$ and $D(G_n) = b_{s,n}L_{s+n}$. By (5.3) we have

$$(\{m\} - \{n\})a_{s,n+m} = (\{m\} - \{n+s+1\})a_{s,n} + (\{m+s+1\} - \{n\})a_{s,m}.$$

We deduce that

$$(q^n - q^m)a_{s,n+m} = (q^{n+s+1} - q^m)a_{s,n} + (q^n - q^{m+s+1})a_{s,m}.$$

If $m = 0$, we have

$$q^n(1 - q^{s+1})a_{s,n} = (q^n - q^{s+1})a_{s,0}.$$

If $s \neq -1$ we have $a_{s,n} = ((1 - q^{s+1-n})/(1 - q^{s+1}))a_{s,0}$.
Then

$$(5.6) \qquad (q^n - q^m)\frac{1 - q^{s+1-n}}{1 - q^{s+1}}a_{s,0} = (q^{n+s+1} - q^m)\frac{1 - q^{s+1-n}}{1 - q^{s+1}}a_{s,0} + (q^n - q^{m+s+1})\frac{1 - q^{s+1-m}}{1 - q^{s+1}}a_{s,0}.$$

Taking $n = 2s + 2$ and $m = s + 1$ in (5.6) we obtain $a_{s,0} = 0$, so $a_{s,n} = 0$.
If $s = -1$ and $n \neq m$, then $a_{s,n} = na_{s,1}$.
By (5.4) and $D(G_n) = b_{s,n}L_{n+s}$ we have

$$(\{m + 1\} - \{n\})b_{s,n+m} = (\{m + s + 1\} - \{n\})b_{s,m}.$$

Taking $n = 0$, we have $(q^{m+1} - q^{m+s+1})b_{s,m} = 0$, hence if $s \neq -1$ we have $b_{s,m} = 0$.
If $s = -1$ and $n \neq m + 1$, we obtain $b_{s,n+m} = b_{s,m}$. So $b_{s,n} = b_{s,0}$.
Finally, it follows that the set of odd α^0-derivations is

$$\mathrm{Der}_{\alpha^0,1}(\mathcal{W}^q) = \mathrm{Der}^{-1}_{\alpha,0}(\mathcal{W}^q) = \langle D_2\rangle \quad \text{with } D_2(L_n) = nG_{n-1} \text{ and } D_2(G_n) = L_{n-1}.$$

□

5.1.2. The α^1-derivations of the Hom-Lie superalgebra $\mathcal{W}^q$.

Proposition 5.5. *If D is an α-derivation then $D = 0$.*

Proof. *Case 1*: $|D| = 0$.
Let D be an even derivation of degree s, $D(L_n) = a_{s,n}L_{s+n}$ and $D(G_n) = b_{s,n}G_{s+n}$.
By (5.3) we have

$$(\{m\} - \{n\})a_{s,n+m} = (1 + q^m)(\{m\} - \{n + s\})a_{s,n} + (1 + q^n)(\{m + s\} - \{n\})a_{s,m}.$$

We deduce that

$$a_{s,n+m} = \frac{(1 + q^m)(q^{n+s} - q^m)}{q^n - q^m}a_{s,n} + \frac{(1 + q^n)(q^n - q^{m+s})}{q^n - q^m}a_{s,m}.$$

If $m = 0$, we have $a_{s,n} = (((1+q^n)(q^n - q^s))/(1+q^n-2q^{n+s}))a_{s,0}$. So

$$a_{s,n+m} = \frac{(1+q^{n+m})(q^{n+m}-q^s)}{1+q^{n+m}-2q^{n+m+s}}a_{s,0}.$$

Then

$$\frac{(1+q^{n+m})(q^{n+m}-q^s)}{1+q^{n+m}-2q^{n+m+s}}a_{s,0} = \frac{(1+q^n)(q^n-q^{m+s})(1+q^m)(q^m-q^s)}{(q^n-q^m)(1+q^m-2q^{m+s})}a_{s,0} - \frac{(1+q^m)(q^m-q^{n+s})(1+q^n)(q^n-q^s)}{(q^n-q^m)(1+q^n-2q^{n+s})}a_{s,0}.$$

If $q \in [0,1[$, then letting $n, m \to +\infty$ we obtain $a_{s,0} = 0$. If $q > 1$ and for a fixed $m = s$, then when n goes to infinity we obtain $a_{s,0} = 0$. We deduce that $D(L_n) = 0$.

By (5.4) and $D(G_n) = b_{s,n}G_{n+s}$ we have

$$(\{m+1\} - \{n\})b_{s,n+m} = (1+q^n)(\{m+s+1\} - \{n\})b_{s,m}.$$

So

$$(q^n - q^{m+1})b_{s,n+m} = (1+q^n)(q^n - q^{m+s+1})b_{s,m}.$$

Taking $n = 0$, we have $(1+q^{m+1}-2q^{m+s+1})b_{s,m} = 0$. Then $b_{s,m} = 0$, so $f(G_n) = 0$. Hence $D \equiv 0$.

Case 2: $|D| = 1$. Let D be an odd derivation of degree s, $D(L_n) = a_{s,n}G_{s+n}$ and $D(G_n) = b_{s,n}L_{s+n}$. By (5.3) we have

$$(\{m\}-\{n\})a_{s,n+m} = (1+q^m)(\{m\}-\{n+s+1\})a_{s,n} + (1+q^n)(\{m+s+1\}-\{n\})a_{s,m}.$$

Then

$$a_{s,n+m} = \frac{(1+q^n)(q^n-q^{m+s+1})}{(q^n-q^m)}a_{s,m} - \frac{(1+q^m)(q^m-q^{n+s+1})}{(q^n-q^m)}a_{s,n}.$$

If $m = 0$, we have

$$a_{s,n} = \frac{(1+q^n)(q^n-q^{s+1})}{1+q^n-2q^{n+s+1}}a_{s,0}.$$

So

$$a_{s,n+m} = \frac{(1+q^{n+m})(q^{n+m}-q^{s+1})}{1+q^{n+m}-2q^{n+m+s+1}}a_{s,0}.$$

Then

$$\frac{(1+q^{n+m})(q^{n+m}-q^{s+1})}{1+q^{n+m}-2q^{n+m+s+1}}a_{s,0} = \frac{(1+q^n)(q^n-q^{m+s+1})(1+q^m)(q^m-q^{s+1})}{(q^n-q^m)(1+q^m-2q^{m+s+1})}a_{s,0} - \frac{(1+q^m)(q^m-q^{n+s+1})(1+q^n)(q^n-q^{s+1})}{(q^n-q^m)(1+q^n-2q^{n+s+1})}a_{s,0}.$$

If $q \in [0,1[$, then letting $n, m \to +\infty$, we obtain $a_{s,0} = 0$. If $q > 1$ and setting $m = s$, then if n goes to infinity we obtain $a_{s,0} = 0$. We deduce that $D(L_n) = 0$. By (5.4) and $D(G_n) = b_{s,n} L_{n+s}$ we obtain

$$(\{m+1\} - \{n\}) b_{s,n+m} L_{m+n+s} = (1+q^n)(\{m+s\} - \{n\}) b_{s,m} L_{m+s+n}.$$

So

$$(q^n - q^{m+1}) b_{s,n+m} = (1+q^n)(q^n - q^{m+s}) b_{s,m}.$$

Taking $n = 0$ leads to $(1 + q^{m+1} - 2q^{m+s}) b_{s,m} = 0$.

It turns out that $b_{s,m} = 0$, so $D(G_n) = 0$. Hence $D \equiv 0$. □

5.1.3. The q-derivations of the Hom-Lie superalgebra $\mathcal{W}^q$. In this section we study the q-derivations of $\mathcal{W}^q$. The derivation algebra of $\mathcal{W}^q$ is denoted by $\operatorname{Der} \mathcal{W}^q$. Since $\mathcal{W}^q$ is a $\mathbb{Z}_2$-graded Hom-Lie superalgebra, we have

$$\operatorname{Der} \mathcal{W}^q = (\operatorname{Der} \mathcal{W}^q)_0 \oplus (\operatorname{Der} \mathcal{W}^q)_1,$$

where $(\operatorname{Der} \mathcal{W}^q)_0 = \{D \in \operatorname{Der} \mathcal{W}^q \colon D((\mathcal{W}^q)_i) \subset (\mathcal{W}^q)_i, i \in \mathbb{Z}_2\}$ denotes the set of even derivations of $\mathcal{W}^q$, and $(\operatorname{Der} \mathcal{W}^q)_1 = \{D \in \operatorname{Der} \mathcal{W}^q \colon D((\mathcal{W}^q)_i) \subset (\mathcal{W}^q)_{i+1}, i \in \mathbb{Z}_2\}$ denotes the set of odd derivations of $\mathcal{W}^q$.

The space $\mathcal{W}^q$ may be viewed also as a $\mathbb{Z}$-graded space. Define

$$(\operatorname{Der} \mathcal{W}^q)_s = \{D \in \operatorname{Der} \mathcal{W}^q \colon D(\mathcal{W}^q_n) \subset \mathcal{W}^q_{n+s}\}.$$

Then we have $\operatorname{Der} \mathcal{W}^q = \bigoplus_{s \in \mathbb{Z}} (\operatorname{Der} \mathcal{W}^q)_s$. Obviously, the $\mathbb{Z}$-graded and $\mathbb{Z}_2$-graded structures are compatible.

Moreover, let $\operatorname{Der}_q \mathcal{W}^q_0 = \bigoplus_{s \in \mathbb{Z}} (\operatorname{Der} \mathcal{W}^q)'_s$, $\operatorname{Der}_q \mathcal{W}^q_1 = \bigoplus_{s \in \mathbb{Z}} (\operatorname{Der} \mathcal{W}^q)''_s$, where $(\operatorname{Der} \mathcal{W}^q)'_s \oplus (\operatorname{Der} \mathcal{W}^q)''_s = (\operatorname{Der} \mathcal{W}^q)_s$.

Definition 5.6. An element $\varphi \in (\operatorname{Der} \mathcal{W}^q)_0 \cap (\operatorname{Der} \mathcal{W}^q)_s$ or $\varphi \in (\operatorname{Der} \mathcal{W}^q)_1 \cap (\operatorname{Der} \mathcal{W}^q)_s$ is a q-derivation if, respectively,

$$\varphi([x,y]) = \frac{1}{1+q^s}([\varphi(x), \alpha(y)] + [\alpha(x), \varphi(y)]) \tag{5.7}$$

or

$$\varphi([x,y]) = \frac{1}{1+q^{s+1}}([\varphi(x), \alpha(y)] + (-1)^{|x|}[\alpha(x), \varphi(y)]) \tag{5.8}$$

where $x, y \in \mathcal{W}^q$ are homogeneous elements.

For a fixed $a \in (\mathcal{W}^q)_i$, we obtain the q-derivation

$$\begin{aligned} \varphi_a \colon \mathcal{W}^q &\longrightarrow \mathcal{W}^q, \\ x &\longmapsto [a, x]. \end{aligned}$$

The map is denoted by ad_a and is called the inner q-derivation.

Proposition 5.7. *If φ is an odd q-derivation of degree s then it is an inner q-derivation, more precisely:*

$$(\operatorname{Der}\mathcal{W}^q)_1 = \bigoplus_{s\in\mathbb{Z}} \langle \operatorname{ad}_{G_s}\rangle.$$

P r o o f. Let φ be an odd q-derivation of degree s:

$$\varphi(L_n) = a_{s,n}G_{n+s} \text{ and } \varphi(G_n) = b_{s,n}L_{n+s}. \tag{5.9}$$

Case 1: $s \neq -1$.

By (5.1) and (5.9), we have

$$\begin{aligned}\{n\}\varphi(L_n) = \varphi([L_0, L_n]) &= \frac{1}{1+q^{s+1}}([\varphi(L_0), \alpha(L_n)] + [\alpha(L_0), \varphi(L_n)])\\ &= \frac{1}{1+q^{s+1}}([a_{s,0}G_s, (1+q^n)L_n] + [2L_0, a_{s,n}G_{s+n}])\\ &= \frac{1+q^n}{1+q^{s+1}}(\{n\} - \{s+1\})a_{s,0}G_{n+s} + 2a_{s,n}\frac{1}{1+q^{s+1}}\{n+s+1\}G_{n+s}.\end{aligned}$$

We deduce that, when $s \neq -1$ then $a_{s,n} = ((q^{s+1} - q^n)/(q^{s+1}-1))a_{s,0}$. On the other hand,

$$\begin{aligned}-\frac{a_{s,0}}{\{s+1\}}\operatorname{ad}_{G_s}(L_n) = -\frac{a_{s,0}}{\{s+1\}}[G_s, L_n] &= \frac{a_{s,0}}{\{s+1\}}(\{s+1\} - \{n\})G_{n+s}\\ &= \frac{q^n - q^{s+1}}{1-q^{s+1}}a_{s,0}G_{n+s} = a_{s,n}G_{n+s}.\end{aligned}$$

So $\varphi(L_n) = -(a_{s,0}/\{s+1\})\operatorname{ad}_{G_s}(L_n)$.

By (5.2) and (5.9), we have

$$\begin{aligned}\{n+1\}\varphi(G_n) = \varphi([L_0, G_n]) &= \frac{1}{1+q^{s+1}}([\varphi(L_0), \alpha(G_n)] + [\alpha(L_0), \varphi(G_n)])\\ &= \frac{1}{1+q^{s+1}}([a_{s,0}G_s, (1+q^{n+1})G_n] + [2L_0, b_{s,n}L_{s+n}])\\ &= 2b_{s,n}\frac{1}{1+q^{s+1}}\{n+s\}L_{n+s}.\end{aligned}$$

We deduce that $\{n+1\}b_{s,n} = 2b_{s,n}(1/(1+q^{s+1}))\{n+s\}$, so $b_{s,n} = 0$. Moreover,

$$-\frac{a_{s,0}}{\{s+1\}}\operatorname{ad}_{G_s}(G_n) = -\frac{a_{s,0}}{\{s+1\}}[G_s, G_n] = 0 = b_{s,n}G_{n+s} = \varphi(G_n),$$

which implies in this case $\varphi = -(a_{s,0}/\{s+1\})\operatorname{ad}_{G_s}$.

Case 2: $s=-1$.

By (5.1) and (5.9), we have

$$\begin{aligned}(\{m\}-\{n\})\varphi(L_{m+n}) &= \varphi([L_n,L_m]) = \frac{1}{2}([\varphi(L_n),\alpha(L_m)]+[\alpha(L_n),\varphi(L_m)])\\ &= \frac{1}{2}([a_{-1,n}G_{n-1},(1+q^m)L_m]+[(1+q^n)L_n,a_{-1,m}G_{m-1}])\\ &= -\frac{1+q^m}{2}a_{-1,n}(\{n\}-\{m\})G_{m+n-1}+\frac{1+q^n}{2}a_{-1,m}(\{m\}-\{n\})G_{m+n-1}.\end{aligned}$$

Then

$$(\{m\}-\{n\})a_{-1,n+m} = -\frac{1+q^m}{2}a_{-1,n}(\{n\}-\{m\})+\frac{1+q^n}{2}a_{-1,m}(\{m\}-\{n\}).$$

So for $m\neq n$ we have

$$a_{-1,n+m}=\frac{1+q^m}{2}a_{-1,n}+\frac{1+q^n}{2}a_{-1,m}. \tag{5.10}$$

Setting $m=0$ in (5.10), we obtain $a_{-1,0}=0$.

Setting $m=1$, $n=4$ in (5.10), then

$$a_{-1,5}=\frac{1+q}{2}a_{-1,4}+\frac{1+q^4}{2}a_{-1,1}. \tag{5.11}$$

Setting $m=1$, $n=3$ in (5.10), we obtain

$$a_{-1,4}=\frac{1+q}{2}a_{-1,3}+\frac{1+q^3}{2}a_{-1,1}. \tag{5.12}$$

Setting $m=1$, $n=2$ in (5.10), we obtain

$$a_{-1,3}=\frac{1+q}{2}a_{-1,2}+\frac{1+q^2}{2}a_{-1,1}. \tag{5.13}$$

We deduce that

$$\begin{aligned}a_{-1,5} &= \Big(\frac{1+q^4}{2}+\frac{1+q}{2}\,\frac{1+q^3}{2}+\Big(\frac{1+q}{2}\Big)^2\frac{1+q^2}{2}\Big)a_{-1,1}\\ &\quad+\Big(\frac{1+q}{2}\Big)^3a_{-1,2}.\end{aligned} \tag{5.14}$$

Now, setting $m=2$, $n=3$ in (5.10), we obtain

$$a_{-1,5}=\frac{1+q^2}{2}a_{-1,3}+\frac{1+q^3}{2}a_{-1,2}. \tag{5.15}$$

By (5.13) and (5.15), we deduce that

$$a_{-1,5} = \Big(\frac{1+q^2}{2}\Big)^2 a_{-1,1} + \Big(\frac{1+q}{2}\frac{1+q^2}{2} + \frac{1+q^3}{2}\Big)a_{-1,2}. \tag{5.16}$$

Then, we deduce (by (5.14) and (5.16)) that $a_{-1,2} = (1+q)a_{-1,1} = \{2\}a_{-1,1}$.

Setting $m = 1$ in (5.10), we obtain $a_{-1,n+1} = \frac{1}{2}(1+q^1)a_{-1,n} + \frac{1}{2}(1+q^n)a_{-1,1}$. By induction, we can show that $a_{-1,n} = \{n\}a_{-1,1}$.

So, $\varphi(L_n) = \{n\}a_{-1,1}G_{n-1} = a_{-1,1}[G_{-1}, L_n]$, therefore

$$\varphi(L_n) = a_{-1,1}\mathrm{ad}_{G_{-1}}(L_n). \tag{5.17}$$

Now, we calculate $\varphi(G_n)$: by (5.2) and (5.9) we have

$$\begin{aligned}(\{m+1\} - \{n\})\varphi(G_{n+m}) &= \frac{1}{2}([\varphi(L_n), \alpha(G_m)] + [\alpha(L_n), \varphi(G_m)])\\ &= \frac{1}{2}([a_{-1,n}G_{n-1}, (1+q^{m+1})G_m] + [(1+q^n)L_n, b_{-1,m}L_{m-1}])\\ &= b_{-1,m}\frac{1+q^n}{2}(\{m-1\} - \{n\})L_{m+n-1}.\end{aligned}$$

We deduce that

$$(\{m+1\} - \{n\})b_{-1,m+n} = b_{-1,m}\frac{1+q^n}{2}(\{m-1\} - \{n\}).$$

So for $m + 1 \neq n$ we have

$$b_{-1,n+m} = \frac{1+q^n}{2}\frac{q^n - q^{m-1}}{q^n - q^{m+1}}b_{-1,m}. \tag{5.18}$$

Setting $m = 0$ in (5.18) (so $n \neq 1$), we obtain

$$b_{-1,n} = \frac{1+q^n}{2}\frac{q^n - q^{-1}}{q^n - q}b_{-1,0}. \tag{5.19}$$

So

$$b_{-1,n+m} = \frac{1+q^{n+m}}{2}\frac{q^{n+m} - q^{-1}}{q^{n+m} - q}b_{-1,0} \tag{5.20}$$

and

$$b_{-1,m} = \frac{1+q^m}{2}\frac{q^m - q^{-1}}{q^m - q}b_{-1,0}. \tag{5.21}$$

By (5.18) and (5.20), we have

$$\frac{1+q^{n+m}}{2}\frac{q^{n+m}-q^{-1}}{q^{n+m}-q}b_{-1,0}=\frac{1+q^n}{2}\frac{q^n-q^{m-1}}{q^n-q^{m+1}}b_{-1,m}.$$

If we replace $b_{-1,m}$ by its value given in (5.21), we obtain

$$\frac{1+q^{n+m}}{2}\frac{q^{n+m}-q^{-1}}{q^{n+m}-q}b_{-1,0}=\frac{1+q^n}{2}\frac{q^n-q^{m-1}}{q^n-q^{m+1}}\frac{1+q^m}{2}\frac{q^m-q^{-1}}{q^m-q}b_{-1,0}. \tag{5.22}$$

Setting $n=2$, $m=-3$ in (5.22), we obtain $b_{-1,0}=0$. By (5.19) we deduce that $b_{-1,n}=0$ for all $n\neq 1$.

Setting $m=1$ in (5.18), we obtain $b_{-1,n+1}=\frac{1}{2}((1+q^n)(q^n-1)/(q^n-q^2))b_{-1,1}$. We deduce that

$$b_{-1,4}=\frac{1+q^3}{2}\frac{q^3-1}{q^3-q^2}b_{-1,1}. \tag{5.23}$$

So $b_{-1,1}=0$.

Since $b_{-1,n}=0$ for all $n\neq 1$ and $b_{-1,1}=0$, we have $\varphi(G_n)=0$, for all $n\in\mathbb{Z}$.

Since $\varphi(G_n)=0=a_{-1,1}[G_{-1},G_n]$, we have

$$\varphi(G_n)=a_{-1,1}\mathrm{ad}_{G_{-1}}(G_n). \tag{5.24}$$

By (5.24) and (5.17), we deduce that $\varphi=a_{-1,1}\ \mathrm{ad}_{G_{-1}}$. □

Proposition 5.8. *If φ is an even q-derivation of degree s then it is an inner derivation, more precisely:*

$$(\mathrm{Der}\,\mathcal{W}^q)_0=\bigoplus_{s\in\mathbb{Z}}\langle \mathrm{ad}_{L_s}\rangle.$$

Proof. Let φ be an even q-derivation of degree s:

$$\varphi(L_n)=a_{s,n}L_{n+s},\ \text{ and } \varphi(G_n)=b_{s,n}G_{n+s}. \tag{5.25}$$

Case 1: $s\neq 0$.

By (5.1) and (5.25), we have

$$\begin{aligned}\{n\}\varphi(L_n)=\varphi([L_0,L_n])&=\frac{1}{1+q^s}([\varphi(L_0),\alpha(L_n)]+[\alpha(L_0),\varphi(L_n)])\\&=\frac{1}{1+q^s}([a_{s,0}L_s,(1+q^n)L_n]+[2L_0,a_{s,n}L_{s+n}])\\&=\frac{1+q^n}{1+q^s}(\{n\}-\{s\})a_{s,0}L_{n+s}+2a_{s,n}\frac{1}{1+q^s}\{n+s\}L_{n+s}.\end{aligned}$$

We deduce that, when $s \neq 0$, then $a_{s,n} = \big((q^s - q^n)/(q^s - 1)\big)a_{s,0}$. Moreover,

$$
\begin{aligned}
-\frac{a_{s,0}}{\{s\}}\mathrm{ad}_{L_s}(L_n) &= -\frac{a_{s,0}}{\{s\}}[L_s, L_n] = -\frac{a_{s,0}}{\{s\}}(\{n\} - \{s\})L_{n+s} \\
&= -\frac{q^s - q^n}{1 - q^s}a_{s,0}L_{n+s} = a_{s,n}L_{n+s}.
\end{aligned}
$$

So

$$
\varphi(L_n) = -\frac{a_{s,0}}{\{s\}}\mathrm{ad}_{L_s}(L_n). \tag{5.26}
$$

Applying the same relations (5.1) and (5.25), we obtain

$$
\begin{aligned}
\{n+1\}\varphi(G_n) = \varphi([L_0, G_n]) &= \frac{1}{1+q^s}([\varphi(L_0), \alpha(G_n)] + [\alpha(L_0), \varphi(G_n)]) \\
&= \frac{1}{1+q^s}([a_{s,0}L_s, (1+q^{n+1})G_n] + [2L_0, b_{s,n}G_{s+n}]) \\
&= \frac{1+q^{n+1}}{1+q^s}a_{s,0}(\{n+1\} - \{s\})L_{n+s} + 2b_{s,n}\frac{1}{1+q^s}\{n+s+1\}L_{n+s}.
\end{aligned}
$$

We deduce that $b_{s,n} = a_{s,0}(q^s - q^{n+1})/(q^s - 1)$. On the other hand,

$$
\begin{aligned}
-\frac{a_{s,0}}{\{s\}}\mathrm{ad}_{L_s}(G_n) &= -\frac{a_{s,0}}{\{s\}}[L_s, G_n] = -\frac{a_{s,0}}{\{s\}}(\{n+1\} - \{s\})G_{n+s} \\
&= a_{s,0}\frac{q^s - q^{n+1}}{q^s - 1}G_{n+s} = b_{s,n}G_{n+s}.
\end{aligned}
$$

So

$$
\varphi(G_n) = -\frac{a_{s,0}}{\{s\}}\mathrm{ad}_{L_s}(G_n). \tag{5.27}
$$

Using (5.26) and (5.27), we deduce that $\varphi = -(a_{s,0}/\{s\})\mathrm{ad}_{L_s}$.

Case 2: $s = 0$.

By (5.1) and (5.25), we have

$$
\begin{aligned}
(\{m\} - \{n\})\varphi(L_{m+n}) = \varphi([L_n, L_m]) &= \frac{1}{2}([\varphi(L_n), \alpha(L_m)] + [\alpha(L_n), \varphi(L_m)]) \\
&= \frac{1}{2}([a_{0,n}L_n, (1+q^m)L_m] + [(1+q^n)L_n, a_{0,m}L_m]) \\
&= a_{0,n}\frac{1+q^m}{2}(\{m\} - \{n\})L_{m+n} + a_{0,m}\frac{1+q^n}{2}(\{m\} - \{n\})L_{m+n}.
\end{aligned}
$$

This implies that

$$
a_{0,m+n}(\{m\} - \{n\}) = \frac{1}{2}a_{0,n}(1+q^m)(\{m\} - \{n\}) + \frac{1}{2}a_{0,m}(1+q^n)(\{m\} - \{n\}).
$$

So for $m \neq n$, we have

$$a_{0,n+m} = \frac{1+q^m}{2} a_{0,n} + \frac{1+q^n}{2} a_{0,m}. \tag{5.28}$$

Setting $m = 0$ in (5.28), we obtain $a_{0,0} = 0$.

Setting $m = 1$ in (5.28), we obtain $a_{0,n+1} = \frac{1}{2}(1+q)a_{0,n} + \frac{1}{2}(1+q^n)a_{0,1}$.

By induction, we prove that $a_{0,n} = \{n\}a_{0,1}$. So $\varphi(L_n) = \{n\}a_{0,1}L_n$, that is

$$\varphi(L_n) = \{n\}a_{0,1}L_n = a_{0,1}[L_0, L_n],$$

which leads to

$$\varphi(L_n) = a_{0,1}\mathrm{ad}_{L_0}(L_n). \tag{5.29}$$

By (5.2) and (5.25), we have

$$\begin{aligned}
(\{m+1\} - \{n\})\varphi(G_{m+n}) &= \varphi([L_n, G_m]) = \frac{1}{2}([\varphi(L_n), \alpha(G_m)] + [\alpha(L_n), \varphi(G_m)]) \\
&= \frac{1}{2}([a_{0,n}L_n, (1+q^{m+1})G_m] + [(1+q^n)L_n, b_{0,m}G_m]) \\
&= a_{0,n}(\{m+1\} - \{n\})\frac{1+q^{m+1}}{2}G_{m+n} \\
&\quad + b_{0,m}\frac{1+q^n}{2}(\{m+1\} - \{n\})G_{m+n}.
\end{aligned}$$

We deduce that $b_{0,m+n} = \frac{1}{2}a_{0,n}(\{m+1\}-\{n\})(1+q^{m+1}) + \frac{1}{2}b_{0,m}(1+q^n)(\{m+1\} - \{n\})$. So, for $m+1 \neq n$, it follows that

$$b_{0,n+m} = a_{0,n}\frac{1+q^{m+1}}{2} + b_{0,m}\frac{1+q^n}{2}. \tag{5.30}$$

Taking $m = 0$ in (5.30), we obtain $b_{0,n} = \frac{1}{2}a_{0,n}(1+q) + \frac{1}{2}b_{0,0}(1+q^n)$. Since $a_{0,n} = a_{0,1}\{n\}$, we have

$$b_{0,n} = a_{0,1}\{n\}\frac{1+q}{2} + b_{0,0}\frac{1+q^n}{2}. \tag{5.31}$$

Taking $m = 1$, $n = -1$ in (5.30) and $n = 1$ in (5.31), we obtain $b_{0,0} = a_{0,1}$. So using (5.31), we get

$$\begin{aligned}
b_{0,n} &= a_{0,1}\{n\}\frac{1+q}{2} + b_{0,0}\frac{1+q^n}{2} \\
&= a_{0,1}\frac{1-q^n}{1-q}\frac{1+q}{2} + a_{0,1}\frac{1+q^n}{2} \\
&= a_{0,1}\{n+1\}.
\end{aligned}$$

Then $\varphi(G_n) = b_{0,n}G_n = a_{0,1}\{n+1\}G_n = a_{0,1}[L_0, G_n]$. Therefore

$$\varphi(G_n) = a_{0,1}\mathrm{ad}_{L_0}(G_n). \tag{5.32}$$

By (5.29) and (5.32), we deduce that $\varphi = a_{0,1}\ \mathrm{ad}_{L_0}$. □

5.2. Cohomology space $H^2_{r,0}(\mathcal{W}^q)$ of $\mathcal{W}^q$.

Now we describe the cohomology space $H^2_0(\mathcal{W}^q, \mathbb{C})$. We denote by $[f]$ the cohomology class of an element f.

Theorem 5.9.

$$H^2_{r,0}(\mathcal{W}^q) = \mathbb{C}[\varphi_1] \oplus \mathbb{C}[\varphi_2],$$

where

$$\begin{aligned}\varphi_1(xL_n + yG_m, zL_p + tG_k) &= xzb_n\delta_{n+p,0},\\ \varphi_2(xL_n + yG_m, zL_p + tG_k) &= xtb_n\delta_{n+k,-1} - yzb_p\delta_{p+m,-1},\end{aligned}$$

with

$$b_n = \begin{cases} \dfrac{1}{q^{n-2}}\,\dfrac{1+q^2}{1+q^n}\,\dfrac{(1-q^{n+1})(1-q^n)(1-q^{n-1})}{(1-q^3)(1-q^2)(1-q)}, & \textit{if } n \geqslant 0,\\ -b_{-n} & \textit{if } n < 0.\end{cases} \tag{5.33}$$

Proof. For all $f \in C^2(\mathcal{W}^q, \mathbb{C})$, we have (see (3.4))

$$\begin{aligned}\delta(f)(x_0, x_1, x_2) = &- f([x_0, x_1], \alpha(x_2)) + (-1)^{|x_2||x_1|} f([x_0, x_2], \alpha(x_1))\\ &+ f(\alpha(x_0), [x_1, x_2]).\end{aligned} \tag{5.34}$$

Now, suppose that f is a q-deformed 2-cocycle on $\mathcal{W}^q$. From (5.34) we obtain

$$-f([x_0, x_1], \alpha(x_2)) + (-1)^{|x_2||x_1|} f([x_0, x_2], \alpha(x_1)) + f(\alpha(x_0), [x_1, x_2]) = 0. \tag{5.35}$$

By (5.35) and taking the triple (x, y, z) to be (L_n, L_m, L_p), (L_n, L_m, G_p), and (L_n, G_m, G_p), respectively, we obtain $f(L_n, L_p)$, $f(L_n, G_p)$ and $f(G_n, G_p)$ which define f.

Case 1: $X = (L_n, L_m, L_p)$.

Using (5.35), we have

$$-f([L_n, L_m], \alpha(L_p)) + f([L_n, L_p], \alpha(L_m)) + f(\alpha(L_n), [L_m, L_p]) = 0.$$

Since $[L_m, L_p] = (\{p\} - \{m\})L_{m+p}$ and $\alpha(L_n) = (1 + q^n)L_n$, then

$$(5.36)\qquad \begin{aligned} -(1+q^p)(\{m\}-\{n\})f(L_{n+m},L_p) + (1+q^m)(\{p\}-\{n\})f(L_{n+p},L_m) \\ + (1+q^n)(\{p\}-\{m\})f(L_n,L_{m+p}) = 0. \end{aligned}$$

Setting $m = 0$ in (5.36), we obtain $f(L_n, L_p) = ((q^n - q^p)/(1 - q^{n+p}))f(L_0, L_{n+p})$ $(n + p \neq 0)$.

Setting $m = 0$, $n = -p$ in (5.36), we obtain $f(L_0, L_0) = 0$.

Setting $m = -n - p$ in (5.36), we obtain

$$(5.37)\qquad \begin{aligned} -(1+q^p)(q^n - q^{-n-p})f(L_{-p},L_p) + (1+q^{-n-p})(q^n - q^p)f(L_{n+p},L_{-n-p}) \\ + (1+q^n)(q^{-n-p} - q^p)f(L_n,L_{-n}) = 0. \end{aligned}$$

Setting $p = 1$ (5.37), we obtain

$$(5.38)\qquad \begin{aligned} -(1+q)(q^{2n+1}-1)f(L_{-1},L_1) + q(1+q^{n+1})(q^{n-1}-1)f(L_{n+1},L_{-n-1}) \\ + (1+q^n)(1-q^{n+2})f(L_n,L_{-n}) = 0. \end{aligned}$$

Hence,

$$(5.39)\qquad \begin{aligned} f(L_{n+1},L_{-n-1}) &= \frac{1}{q}\frac{1+q^n}{1+q^{n+1}}\frac{1-q^{n+2}}{1-q^{n-1}}f(L_n,L_{-n}) \\ &\quad - \frac{1}{q}\frac{1+q}{1+q^{n+1}}\frac{1-q^{2n+1}}{1-q^{n-1}}f(L_1,L_{-1}), \quad \text{for } n \neq 1, \end{aligned}$$

$$(5.40)\qquad \begin{aligned} f(L_n,L_{-n}) &= q\frac{1+q^{n+1}}{1+q^n}\frac{1-q^{n-1}}{1-q^{n+2}}f(L_{n+1},L_{-n-1}) \\ &\quad + \frac{1+q}{1+q^n}\frac{1-q^{2n+1}}{1-q^{n+2}}f(L_1,L_{-1}), \quad \text{for } n \neq -2. \end{aligned}$$

Setting

$$\alpha_n = \frac{1}{q}\frac{1+q^{n-1}}{1+q^n}\frac{1-q^{n+1}}{1-q^{n-2}} \quad \text{and} \quad \beta_n = -\frac{1}{q}\frac{1+q}{1+q^n}\frac{1-q^{2n-1}}{1-q^{n-2}},$$

from the formula (5.39) we get $f(L_n, L_{-n}) = a_n f(L_1, L_{-1}) + b_n f(L_2, L_{-2})$, for $n > 2$, where

$$a_n = \beta_n + \alpha_n\beta_{n-1} + \alpha_n\alpha_{n-1}\beta_{n-2} + \ldots + \alpha_n\alpha_{n-1}\ldots\alpha_4\beta_3,$$

and

$$b_n = \frac{1}{q^{n-2}}\frac{1+q^2}{1+q^n}\frac{(1-q^{n+1})(1-q^n)(1-q^{n-1})}{(1-q^3)(1-q^2)(1-q)}.$$

Setting

$$\alpha'_n = q\frac{1+q^{n+1}}{1+q^n}\,\frac{1-q^{n-1}}{1-q^{n+2}} \quad \text{and} \quad \beta'_n = q\frac{1+q}{1+q^n}\,\frac{1-q^{2n+1}}{1-q^{n+2}},$$

from the formula (5.40) we get $f(L_n, L_{-n}) = a'_n f(L_{-1}, L_1) + b'_n f(L_{-2}, L_2)$ for $n < -2$, where

$$\begin{aligned} a'_n &= \beta'_n + \alpha_n\beta'_{n+1} + \alpha'_n\alpha'_{n+1}\beta'_{n+2} + \ldots + \alpha'_n\alpha'_{n+1}\ldots\alpha'_{-4}\beta'_{-3},\\ b'_n &= \frac{1}{q^{n+2}}\,\frac{1+q^{-2}}{1+q^n}\,\frac{(1-q^{n-1})(1-q^n)(1-q^{n+1})}{(1-q^{-3})(1-q^{-2})(1-q^{-1})} = -b_{-n}. \end{aligned}$$

Case 2: $X = (L_n, L_m, G_p)$.

By (5.35) we have

$$-f([L_n, L_m], \alpha(G_p)) + f([L_n, G_p], \alpha(L_m)) + f(\alpha(L_n), [L_m, G_p]) = 0.$$

Since $[L_n, G_p] = (\{p+1\} - \{n\})G_{n+p}$ and $\alpha(G_n) = (1+q^{n+1})G_n$, we have

$$\begin{aligned} (5.41) \qquad -(1+q^{p+1})(\{m\}-\{n\})f(L_{n+m}, G_p) + (1+q^m)(\{p+1\}-\{n\})\\ \times f(G_{n+p}, L_m) + (1+q^n)(\{p+1\}-\{m\})f(L_n, G_{m+p}) = 0. \end{aligned}$$

Taking $m = 0$ in (5.41), we obtain

$$(5.42) \qquad (1-q^{n+p+1})f(L_n, G_p) = (q^n - q^{p+1})f(L_0, G_{n+p}).$$

Then

$$f(L_n, G_p) = \frac{q^n - q^{p+1}}{1-q^{n+p+1}} f(L_0, G_{n+p}) \quad \text{for } n+p+1 \neq 0.$$

Taking $n = 1$, $p = -2$ in (5.42), we obtain $f(L_0, G_{-1}) = 0$.

Taking $m = -n$, $p = -1$ in (5.41), we obtain (with $f(L_0, G_{-1}) = 0$)

$$f(L_n, G_{-n-1}) = -f(L_{-n}, G_{n-1}).$$

Then $f(L_1, G_{-2}) = -f(L_{-1}, G_0)$, $f(L_2, G_{-3}) = -f(L_{-2}, G_1)$.

Taking $m = -1$, $p = -n$ in (5.41), we obtain

$$\begin{aligned} -(1+q^{n-1})(q^{n+1}-1)f(L_{n-1}, G_{-n}) + (1+q)(q^{2n-1}-1)f(G_0, L_{-1})\\ + q(1+q^n)(q^{n-2}-1)f(L_n, G_{-n-1}) = 0. \end{aligned}$$

Hence

$$(5.43)\qquad f(L_n,G_{-n-1})=\frac{1}{q}\frac{1+q^{n-1}}{1+q^n}\frac{1-q^{n+1}}{1-q^{n-2}}f(L_{n-1},G_{-n}) - \frac{1}{q}\frac{1+q}{1+q^n}\frac{1-q^{2n-1}}{1-q^{n-2}}f(L_1,G_{-2})\quad \text{for } n\neq 2,$$

$$(5.44)\qquad f(L_{n-1},G_{-n})=q\frac{1+q^n}{1+q^{n-1}}\frac{1-q^{n-2}}{1-q^{n+1}}f(L_n,G_{-n-1}) + \frac{1+q}{1+q^{n-1}}\frac{1-q^{2n-1}}{1-q^{n+1}}f(L_{-1},G_0)\quad \text{for } n\neq -1.$$

Comparing (5.39) and (5.43), we deduce that

$$f(L_n,G_{-n-1})=a_nf(L_1,G_{-2})+b_nf(L_2,G_{-3})\quad \text{for } n>2.$$

Comparing (5.40) and (5.44), we deduce that

$$f(L_n,G_{-n-1})=a'_nf(L_{-1},G_0)+b'_nf(L_{-2},G_1)\quad \text{for } n<-2,$$

where a_n, b_n, a'_n and b'_n are defined as in the previous case.

Case 3: $X=(L_n,G_m,G_p)$.

By (5.35) we have

$$-f([L_n,G_m],\alpha(G_p))-f([L_n,G_p],\alpha(G_m))+f(\alpha(L_n),[G_m,G_p])=0.$$

So

$$(5.45)\qquad -(1+q^{p+1})(\{m+1\}-\{n\})f(G_{m+n},G_p) - (1+q^{m+1})(\{p+1\}-\{n\})f(G_{p+n},G_m)=0.$$

Taking $m=0$ in (5.45), we obtain

$$(5.46)\quad (1+q^{p+1})(\{1\}-\{n\})f(G_n,G_p)+(1+q)(\{p+1\}-\{n\})f(G_{p+n},G_0)=0.$$

Taking $n=1$ and replacing $p+1$ by k in (5.46), we obtain

$$f(G_k,G_0)=0\ \text{ for }\ k\neq 1.$$

Hence,

$$f(G_n,G_p)=0\ \text{ for }\ n\neq 1,\ p+n\neq 1.$$

Taking $p = 1 - n$ in (5.46), we obtain

$$f(G_n, G_{1-n}) = -\frac{1+q}{1+q^{2-n}}(1+q^{1-n})f(G_1, G_0) \quad (n \neq 1). \tag{5.47}$$

Replacing n by $1-n$ and p by n in (5.46), we obtain

$$f(G_{1-n}, G_n) = -\frac{1+q}{1+q^{n+1}}(1+q^{n})f(G_1, G_0) \quad (n \neq 0). \tag{5.48}$$

Then using the super skew-symmetry of f, we get $f(G_1, G_0) = 0$.

We deduce that $f(G_n, G_m) = 0$ for all n, $m \in \mathbb{Z}$.

We denote by g the linear map defined on $\mathcal{W}^q$ by

$$\begin{aligned} g(L_n) &= -\frac{1}{\{n\}}f(L_0, L_n) \ \text{ if } \ n \neq 0, \ g(L_0) = -\frac{q}{q+1}f(L_1, L_{-1}), \\ g(G_n) &= \frac{1}{\{n+1\}}f(L_0, G_n) \ \text{ if } \ n \neq -1, \ g(G_{-1}) = -\frac{q}{q+1}f(L_1, G_{-2}). \end{aligned}$$

It is easy to verify that $\delta(g)(L_n, L_p) = ((q^p - q^n)/(1-q^{n+p}))f(L_0, L_{n+p})$ $(p \neq -n)$, $\delta(g)(L_n, L_{-n}) = 0$, $\delta(g)(L_n, G_p) = ((q^{p+1} - q^n)/(1-q^{p+n+1}))f(L_0, G_{n+p})$ $(p + n \neq -1)$ and $\delta(g)(G_n, G_p) = 0$.

Let $h = f - \delta^1 g$. Then we have

$$\begin{aligned} h(L_1, L_{-1}) &= h(L_1, G_{-2}) = 0, \\ h(L_n, L_p) &= 0 \quad \text{for } n+p \neq 0, \\ h(L_n, G_p) &= 0 \quad \text{for } n+p \neq -1, \\ h(G_n, G_p) &= 0 \quad \text{for all } n, p \in \mathbb{Z}. \end{aligned}$$

Since h is a 2-cocycle we deduce that:

$$\begin{aligned} h(L_n, L_{-n}) &= a_n h(L_1, L_{-1}) + b_n h(L_2, L_{-2}) = b_n h(L_2, L_{-2}), \\ h(L_n, G_{-n-1}) &= a_n h(L_1, G_{-2}) + b_n f(L_2, G_{-3}) = b_n h(L_2, G_{-3}), \\ h(G_n, G_m) &= 0. \end{aligned}$$

Using the above equalities, we deduce that

$$\begin{aligned} h(xL_n + yG_m, zL_p + tG_k) &= xz\delta_{n+p,0} b_n h(L_2, L_{-2}) + xt\delta_{n+k,-1} b_n h(L_2, G_{-3}) \\ &\quad - yz\delta_{p+m,-1} b_p h(L_2, G_{-3}) \\ &= h(L_2, L_{-2})\varphi_1(xL_n + yG_m, zL_p + tG_k) \\ &\quad + h(L_2, G_{-3})\varphi_2(xL_n + yG_m, zL_p + tG_k), \end{aligned}$$

which completes the proof. □

Corollary 5.10. *Let V be a trivial representation of $\mathcal{W}^q$ and $f \in C^2(\mathcal{W}^q, V)$. Define a bracket and a morphism on $\widetilde{\mathcal{W}^q} = \mathcal{W}^q \oplus V$ by*

$$[(x,a),(y,b)]_{\widetilde{\mathcal{W}^q}} = ([x,y], f(x,y)),$$
$$\tilde{\alpha}(x,a) = (\alpha(x), a) \quad \forall x,y \in \mathcal{W}^q, a,b \in V.$$

The triple $(\widetilde{\mathcal{W}^q}, [\cdot,\cdot]_{\widetilde{\mathcal{W}^q}}, \tilde{\alpha})$ is a Hom-Lie superalgebra if and only if f is in $\mathbb{C}[\varphi_1] \oplus \mathbb{C}[\varphi_2]$.

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