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Kybernetika, Vol. 49 (2013), No. 4, 568--589

Persistent URL: http://dml.cz/dmlcz/143446

KYBERNETIKA — VOLUME 49 (2013), NUMBER 4, PAGES 568–589

MEMORYLESS SOLUTION TO THE OPTIMAL CONTROL PROBLEM FOR LINEAR SYSTEMS WITH DELAYED INPUT

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This note investigates the optimal control problem for a time-invariant linear systems with an arbitrary constant time-delay in in the input channel. A state feedback is provided for the infinite horizon case with a quadratic cost function. The solution is memoryless, except at an initial time interval of measure equal to the time-delay. If the initial input is set equal to zero, then the optimal feedback control law is memoryless from the beginning. Stability results are established for the closed loop system, in the scalar case.

Keywords: time-delay systems, optimal control, stability

Classification: 93E12, 62A10

1. INTRODUCTION

The optimal control problem for linear systems with delayed input is a challenging research topic which has received much attention in the literature in both the discrete and the continuous time cases. A time-delay in the input is frequently encountered in many engineering frameworks, such as network control systems and process control, for instance, due to communication of the input signals (see, e. g. [8, 9, 30]). The reader can refer to the recent book [20] for many problems and solutions concerning the control of systems with delay in the input.

For the optimal control problem in the discrete time case, the reader can refer to the pioneering works [10, 29, 32] or to the more recent one [33], where the optimal control problem is set in the general framework of multiple input, multiple time-varying delays.

As far as the continuous time case is concerned, the optimal control problem of linear time-invariant systems with input delays has been treated by [11, 15, 22, 27]. In these papers, the system is rewritten on a suitable Banach or Hilbert space and the solution to the optimal control problem is provided by means of operators on infinite dimensional spaces. Thus, in general, this solution is not directly implementable. Approximation methods are then developed in order to obtain a suboptimal solution (see, for instance, [16, 17] and references therein).

A finite dimensional solution to the finite horizon optimal control of systems with input delays can be found in [1, 2, 3, 4]. Time-varying systems with multiple inputs and multiple time-varying delays are dealt with. The solution is found by means of a suitable Riccati differential equation (see 4.3, 4.4, p. 132, in [1]).

In [33] the finite horizon optimal control of time-varying linear systems with multiple inputs (each channel with a constant time delay), has been treated. The optimal control input is given as the sum of a feedback of the state (euclidean) variable and of an integral in the delay interval of the control input itself (see Theorem 5 in [33]). In [23] the H_2-optimal control of time-invariant linear systems with multiple constant input/output delays is studied. In [19] the same problem is investigated in an H_∞ control setting. In [31] it is shown that a suitable state feedback control which involves the integral of the past control law solves the infinite horizon optimal control problem for linear time-invariant systems with single input time-delay.

As well known, the implementation of control laws involving distributed delays is not an easy task and can arise instability problems (see [24] and references therein). To this aim, a memoryless, asymptotically stabilizing, predictive control law is found in [34], for linear systems, with eigenvalues in the closed left half plane, with single, time-varying, delay in the input channel.

To our knowledge, an exact (i. e. not approximated), memoryless solution to the infinite horizon optimal control problem, for linear time-invariant systems with input delay, is not yet available in the literature.

In this paper we provide the finite dimensional, implementable, solution of the infinite horizon optimal control problem for linear time-invariant systems with a single, arbitrarily large, time-delay in the input channel. The solution is memoryless, thus the easiest possible as far as implementable problems are concerned.

A preliminary version of this paper has been published in [7].

2. PROBLEM SETTING

Let us consider the following linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t-h), \quad t \geq 0, \\ x(0) &= x_0, \\ u(\theta) &= u_0(\theta), \quad \theta \in [-h, 0), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, $u(t) \in \mathbb{R}^p$ is the delayed control input, h is a positive constant and $u_0 \in C([-h,0);\mathbb{R}^p)$ (the space of the continuous and bounded functions mapping $[-h,0)$ into $\mathbb{R}^n$). We assume that the pair (A,B) is stabilizable.

The optimal control problem here investigated is that of minimizing the following infinite horizon cost functional

$$J = \int_0^{+\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t) \right) \mathrm{d}t, \tag{2}$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric nonnegative definite matrix, $R \in \mathbb{R}^{p \times p}$ is a symmetric positive definite matrix. The problem is to find the optimal control $u(t)$, $t \in [0, +\infty)$, such to minimize the functional J. In the next section we will provide the solution, as a memoryless state feedback.

Remark 2.1. Notice that, since $x(t)$, $t \in [0,h]$, depends only on the given initial conditions (i. e. the initial input u_0 and the initial state x_0), the problem of minimizing the functional (2) is equivalent to the problem of minimizing the functional

$$\int_h^{+\infty} \left(x^T(t)Qx(t) + u^T(t-h)Ru(t-h)\right) \mathrm{d}t. \tag{3}$$

3. MEMORYLESS STATE FEEDBACK SOLUTION

Theorem 3.1. The solution to the optimal control problem defined in Section 2 is given by

$$u(t) = \begin{cases} -R^{-1}B^T P e^{(A-BR^{-1}B^T P)t}\left(e^{A(h-t)}x(t) + m(t)\right), & t \in [0,h), \\ -R^{-1}B^T P e^{(A-BR^{-1}B^T P)h} x(t), & t \in [h,+\infty), \end{cases}$$

where $m(t) = \int_t^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau$, $t \in [0,h)$, and $P \in \mathbb{R}^{n\times n}$ is the solution to the Algebraic Riccati Equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{4}$$

Moreover, setting

$$\tilde{x} = e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau,$$

the optimal value of the functional J is equal to

$$\begin{aligned} \tilde{x}^T P\tilde{x} + \int_0^h &\left(e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right)^T Q \\ &\cdot \left(e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right) \mathrm{d}t. \end{aligned} \tag{5}$$

Proof. As a preliminary step of the proof, notice that the control law designed in (4) is well posed, since the stabilizability of the pair (A,B) ensures the existence of the symmetric, positive semidefinite solution P to the Riccati equation (4). Moreover, because of linearity, the closed loop system (1), (4) admits a unique solution defined in the whole positive real set $\mathbb{R}^+$.

The Theorem is proven in the following two steps. First, the following Claim is proven (Step 1), providing the state evolution $x(t)$ of system (1), when the feedback control law (4) is applied:

$$\text{Claim: } x(t) = e^{(A-BR^{-1}B^T P)(t-h)}x(h), \quad t \geq h, \tag{6}$$

where

$$x(h) = e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau = \tilde{x}. \tag{7}$$

Then (Step 2), we exploit Claim (6) to prove that the control law in (4) is the optimal solution to the control problem, and that the optimal value of the functional J is given by (5).

Step 1. Claim (6) is proven by induction, starting to show that it is true for $[h, 2h]$. Indeed, let $t \in [h, 2h]$. We have, by (1) and (4):

$$\begin{aligned} x(t) &= e^{A(t-h)}x(h) + \int_h^t e^{A(t-\tau)}Bu(\tau-h)\,\mathrm{d}\tau \\ &= e^{A(t-h)}x(h) - \int_h^t e^{A(t-\tau)}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)(\tau-h)}\Big(e^{A(2h-\tau)}x(\tau-h) \\ &\quad + \int_{\tau-h}^h e^{A(h-\theta)}Bu_0(\theta-h)\,\mathrm{d}\theta\Big)\,\mathrm{d}\tau. \end{aligned} \tag{8}$$

Since, for $\tau \in [h, t] \subseteq [h, 2h]$,

$$x(\tau-h) = e^{A(\tau-h)}x_0 + \int_0^{\tau-h} e^{A(\tau-h-\theta)}Bu_0(\theta-h)\,\mathrm{d}\theta, \tag{9}$$

we obtain, from (8), (9),

$$\begin{aligned} x(t) &= e^{A(t-h)}\left(e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right) \\ &\quad - \int_h^t e^{A(t-\tau)}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)(\tau-h)} \\ &\quad \cdot\left(e^{Ah}x_0 + \int_0^{\tau-h} e^{A(h-\theta)}Bu_0(\theta-h)\,\mathrm{d}\theta + \int_{\tau-h}^h e^{A(h-\theta)}Bu_0(\theta-h)\,\mathrm{d}\theta\right)\mathrm{d}\tau. \end{aligned} \tag{10}$$

The Claim (6) holds true, for $t \in [h, 2h]$, provided that the right-hand side of (10) is equal to

$$e^{(A-BR^{-1}B^TP)(t-h)}\left(e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right) \tag{11}$$

that is, provided that the following equality holds true:

$$\begin{aligned} &-e^{-At}e^{(A-BR^{-1}B^TP)(t-h)}\left(e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right) \\ &+\left(x_0 + \int_0^h e^{-A\tau}Bu_0(\tau-h)\,\mathrm{d}\tau\right) \\ &= \int_h^t e^{-A\tau}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)(\tau-h)}\left(e^{Ah}x_0 + \int_0^h e^{A(h-\theta)}Bu_0(\theta-h)\,\mathrm{d}\theta\right)\mathrm{d}\tau. \end{aligned}$$

Notice that equality (12) straightforwardly comes by equalling the right-hand side of (10) to (11) and further premultiplying each term by the nonsingular matrix e^{-At}. To prove that equality (12) holds true, define the following function:

$$\Xi(t) = -e^{-At}e^{(A-BR^{-1}B^TP)(t-h)}\left(e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right), \; t \in [h, 2h], \tag{12}$$

so that the left-hand side of (12) becomes $\Xi(t) - \Xi(h)$. Thus, equality (12) is proven by showing that the right-hand side of (12) is:

$$\int_h^t \frac{\mathrm{d}\Xi(\tau)}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{13}$$

Indeed, this is true, since:

$$\begin{aligned}\frac{\mathrm{d}\Xi}{\mathrm{d}t} &= Ae^{-At}e^{(A-BR^{-1}B^TP)(t-h)}\left(e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right)\\ &-e^{-At}\left(A-BR^{-1}B^TP\right)e^{(A-BR^{-1}B^TP)(t-h)}\left(e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right)\\ &= e^{-At}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)(t-h)}\left(e^{Ah}x_0 + \int_0^h e^{A(h-\tau)}Bu_0(\tau-h)\,\mathrm{d}\tau\right)\end{aligned}$$

and thus the Claim is proved for $t \in [h, 2h]$.

Now, by induction, let the Claim (6) holds true for $t \in [h, ih]$, $i \geq 2$ positive integer. We will prove that the Claim holds true for $t \in [h, (i+1)h]$.

Let $t \in [ih, (i+1)h]$. Since, by hypothesis, the Claim (6) holds true for $t \in [h, ih]$, the solution of (1), (4) is given, for $t \in [ih, (i+1)h]$, by

$$\begin{aligned}x(t) &= e^{A(t-ih)}x(ih) - \int_{ih}^t e^{A(t-\tau)}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)h}x(\tau-h)\,\mathrm{d}\tau\\ &= e^{A(t-ih)}e^{(A-BR^{-1}B^TP)(i-1)h}x(h)\\ &\quad - \int_{ih}^t e^{A(t-\tau)}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)h}e^{(A-BR^{-1}B^TP)(\tau-2h)}x(h)\,\mathrm{d}\tau\\ &= e^{A(t-ih)}e^{(A-BR^{-1}B^TP)(i-1)h}x(h)\\ &\quad - \int_{ih}^t e^{A(t-\tau)}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)(\tau-h)}x(h)\,\mathrm{d}\tau.\end{aligned} \tag{14}$$

The Claim (6) holds true, for $t \in [h, (i+1)h]$, if the right-hand side of (14) is equal, for $t \in [ih, (i+1)h]$, to

$$e^{(A-BR^{-1}B^TP)(t-h)}x(h) \tag{15}$$

that is, if the following equality holds true:

$$\begin{aligned}&-e^{-At}e^{(A-BR^{-1}B^TP)(t-h)}x(h) + e^{-A(ih)}e^{(A-BR^{-1}B^TP)(i-1)h}x(h)\\ &= \int_{ih}^t e^{-A\tau}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)(\tau-h)}x(h)\,\mathrm{d}\tau.\end{aligned} \tag{16}$$

Also in this case, equality (16) comes by premultiplying each term by the nonsingular matrix e^{-At}. Again, in order to prove (16), define the following function

$$\Theta(t) = -e^{-At}e^{(A-BR^{-1}B^TP)(t-h)}x(h), \quad t \in [ih,\ (i+1)h], \tag{17}$$

so that the left-hand side of (16) becomes $\Theta(t) - \Theta(ih)$, and equality (16) is proven by showing that the right-hand side is:

$$\int_{ih}^{t} \frac{\mathrm{d}\Theta}{\mathrm{d}\tau}\,\mathrm{d}\tau. \tag{18}$$

Indeed, this is true, since:

$$\begin{aligned}\frac{d\Theta}{dt} &= Ae^{-At}e^{(A-BR^{-1}B^TP)(t-h)}x(h)\\ &\quad -e^{-At}\left(A - BR^{-1}B^TP\right)e^{(A-BR^{-1}B^TP)(t-h)}x(h)\\ &= e^{-At}BR^{-1}B^TPe^{(A-BR^{-1}B^TP)(t-h)}x(h)\end{aligned} \tag{19}$$

and thus the Claim (6) is proved for $t \in [h, (i+1)h]$. By mathematical induction, we conclude that the Claim (6) holds true for $t \in [h, +\infty)$.

Step 2. Notice that the value of $x(h)$ depends of only the given initial conditions of the system (1), i. e. x_0 and u_0. From Claim (6), it follows that $x(t)$ obeys to the equation

$$\begin{aligned}&\dot{x}(t) = Ax(t) + Bv(t), \quad t \geq h,\\ &x(h) = \tilde{x},\end{aligned} \tag{20}$$

where $v(t) = -R^{-1}B^TPx(t)$, that is, $x(t)$, $v(t)$ are the optimal state and optimal input, respectively, to the problem of minimizing the cost functional

$$\int_{h}^{+\infty} \left(x^T(t)Qx(t) + v^T(t)Rv(t)\right)\,\mathrm{d}t, \tag{21}$$

under constraints described by (20). Hence, since Claim (6) holds true, the control law (4) satisfies, for $t \geq h$, the equality $u(t-h) = v(t) = -R^{-1}B^TPx(t)$, thus yielding the optimal solution to the problem stated in Section 2 (see also Remark 2.1). Moreover, the value of the optimal functional is equal to the optimal value of the cost functional (21), under constraints described by (20), i. e. $\tilde{x}^TP\tilde{x}$, plus (see the expression of the functional (2)) the quantity $\int_0^h x^T(t)Qx(t)\,\mathrm{d}t$. From the computation of $x(t)$ in the interval $[0, h]$ (which depends of only x_0 and u_0), the value reported in (5) is obtained. So, the proof of the Theorem is completed. □

Remark 3.2. Notice that, once an h-lengthy time interval has gone by, the optimal control law is the same as in the delay free case, but just a prediction of the current state, a time h ahead, replaces the current state. Such prediction is obtained by the free evolution of the delay free system in closed loop with delay free optimal controller.

Remark 3.3. Notice that, from h on, the optimal control law (4) is a memoryless feedback of the state variable, whichever is the initial input u_0. If $u_0 \equiv 0$, the optimal control law (4) is a memoryless state feedback from 0 on, i. e. from the beginning. If $u_0 \neq 0$, then it is necessary to compute integrals of such initial input law. Indeed the control law involves, for $t \in [0, h]$, $m(t)$, an integral (with suitable kernel) of the initial input u_0.

Remark 3.4. By Theorem 3.1, the optimal control law (4) yields, for the functional reported in (3) (which does not consider the interval where the state evolves according to initial state x_0 and initial input u_0) the value $\tilde{x}^T P \tilde{x}$.

4. STABILITY RESULTS

Denote $\hat{x}(\tau)$, $\tau \in [h, 2h]$, the optimal solution in $[h, 2h]$, obtained by applying the optimal control law (see 4)

$$u(t) = -R^{-1} B^T P e^{(A - BR^{-1}B^T P)t} \left(e^{A(h-t)} x(t) + m(t) \right), \quad t \in [0, h) \tag{22}$$

to system (1). Clearly, $\hat{x}(\tau)$ depends of the state and input initialization, x_0 and u_0, respectively. Then, for $t \geq 2h$, the closed loop system (1)-(4) reveals to be the following system with delay in the state:

$$\begin{array}{ll} \dot{x}(t) = Ax(t) - BR^{-1}B^T P e^{(A - BR^{-1}B^T P)h} x(t-h), & t \geq 2h \\ x(\tau) = \hat{x}(\tau), & \tau \in [h, 2h]. \end{array} \tag{23}$$

Notice that a generic initialization of $x(\tau) \neq \hat{x}(\tau)$, $\tau \in [h, 2h]$ for system (23) does not provide the optimal solution, that occurs only by correctly applying $u(\tau)$ in $[0, h]$ as in (22). In other words, given the initial conditions x_0, u_0 for (1), the optimal control characterizes the initial condition $\hat{x}(\tau)$ for (23) according to which the solution of the system (23) is the optimal solution. As a matter of fact, because of the optimality, the correct initialization of (23) ensures also the convergence to zero of the state of the system. That means: the optimal control law tells us how to set a proper initialization for $\hat{x}(\tau)$ according to which the solution of (23) converges to zero, but it does not ensure the stability of system (23), i. e. it does not ensure the state convergence to zero whichever is the initial condition in $[h, 2h]$. Therefore, it deserves interest to investigate the stability of the delayed system (23), because of unavoidable computation or implementation errors. It may well happen, for instance, that $u(t)$, $t \in [0, h)$, is not computed exactly, since an integral is involved and numerical approximations may be necessary. Also, actuator errors may appear so that the control law (4) may be corrupted by some bounded disturbance (see Proposition 2.5 in [28] for equivalence between asymptotic stability and input-to-state stability). In these cases, if the closed loop system (23) is un-stable, then, besides loosing the optimality, the solution of the closed loop system (1), (4), with the optimal control law not exactly implemented, can well diverge to infinity.

In order to check the stability of the state delay system (23) efficient LMI methodologies available in the literature can be used, providing sufficient conditions to ensure the asymptotic stability (see [5, 12, 13, 14, 25] and references therein).

In this Section we will investigate in details the scalar case, that is $x(t) \in R$, according to which conditions on the length of the delay h will be given to ensure stability. Thus, in the following we will use scalars a, b, q, r, p instead of matrices A, B, Q, R, P so that the DDE closed loop equation (23) becomes

$$\dot{x}(t) = ax(t) - \frac{b^2 p}{r} e^{(a-b^2 p/r))h} x(t-h), \quad t \geq 2h \tag{24}$$

where p is the positive solution to the Riccati equation (4) which, in the scalar case, is easily computed as

$$p = \frac{a+\sqrt{\Delta}}{(b^2/r)}, \quad \Delta = a^2 + b^2 q/r. \tag{25}$$

Then, the closed loop system (24) becomes:

$$\dot{x}(t) = ax(t) - (a+\sqrt{\Delta})e^{-\sqrt{\Delta}h} x(t-h), \quad t \geq 2h. \tag{26}$$

Theorem 4.1. Consider the time-delay system (26). If $a \leq 0$, then, for any given delay $h \geq 0$, the origin is asymptotically stable. If $a > 0$, then the origin is asymptotically stable for $0 \leq h < \overline{h}$, and unstable for $h > \overline{h}$, where

$$\overline{h} = \frac{1}{\sqrt{\Delta}} \ln\left(\frac{a+\sqrt{\Delta}}{a}\right). \tag{27}$$

Proof. In order to investigate the stability of the time-delay, linear system (26), compute the characteristic function:

$$d(\lambda) = \lambda - a + (a+\sqrt{\Delta})e^{-\sqrt{\Delta}h} e^{-\lambda h}. \tag{28}$$

Notice that for $h = 0$, (28) reduces to the first order polynomial

$$d(\lambda) = \lambda - a + (a+\sqrt{\Delta}) = \lambda + \sqrt{\Delta} \tag{29}$$

which admits the unique negative real solution $\lambda = -\sqrt{\Delta}$, that means asymptotic stability is ensured for $h = 0$. Thus, the stability analysis is investigated by taking into account whether by increasing the delay parameter h there appear roots with positive real part. According to the established literature, such a case can only happen for roots with negative real part, crossing the imaginary axis as h increases its value (see e. g. [25, 26]).

To this aim, consider a generic pair of purely imaginary roots for (28), $\lambda = \pm j\omega$, so that:

$$d(j\omega) = j\omega - a + (a+\sqrt{\Delta})e^{-\sqrt{\Delta}h} e^{-j\omega h} = 0 \tag{30}$$

that means:

$$\begin{cases} -a + (a+\sqrt{\Delta})e^{-\sqrt{\Delta}h}\cos(\omega h) = 0 \\ \omega - (a+\sqrt{\Delta})e^{-\sqrt{\Delta}h}\sin(\omega h) = 0 \end{cases} \tag{31}$$

and so:

$$\begin{cases} \cos(\omega h) = \frac{a}{a+\sqrt{\Delta}} e^{\sqrt{\Delta} h} \\ \sin(\omega h) = \frac{\omega}{a+\sqrt{\Delta}} e^{\sqrt{\Delta} h}. \end{cases} \tag{32}$$

Notice that for $a \leq 0$ the trivial solution $\omega = 0$ satisfies only the second equation of (32), therefore it is not a solution of the whole system (32). On the other hand, for positive values of a, the trivial solution $\omega = 0$ satisfies both the equations of (32), provided that $h = \bar{h}$. In any case, no other purely imaginary roots occur. Indeed consider the second equation of (32) and search for nontrivial (i. e. different than $\omega = 0$) positive real roots of the following nonlinear function:

$$\psi(\omega) = \frac{\omega}{a+\sqrt{\Delta}} e^{\sqrt{\Delta} h} - \sin(\omega h) \tag{33}$$

for a given positive h. To this aim, compute the derivative:

$$\psi'(\omega) = \frac{d\psi}{d\omega} = \frac{e^{\sqrt{\Delta} h}}{a+\sqrt{\Delta}} - h\cos(\omega h). \tag{34}$$

It will be shown that $\psi'(\omega) > 0$ for any $\omega \geq 0$, so that $\psi(\omega)$ is monotonically increasing for $\omega \geq 0$, that means $\psi(\omega)$ never vanishes, besides the trivial case $\psi(0) = 0$. Consider the following inequality:

$$\psi'(\omega) \geq \xi = \frac{e^{\sqrt{\Delta} h}}{a+\sqrt{\Delta}} - h \tag{35}$$

and investigate whether there exist values of the delay $h > 0$ according to which $\xi = 0$. To this aim, consider ξ as a function of h, so that:

$$\xi(0) = \frac{1}{a+\sqrt{\Delta}} > 0, \qquad \lim_{h \mapsto +\infty} \xi(h) = +\infty \tag{36}$$

and

$$\xi'(h) = \frac{d\xi}{dh} = \frac{\sqrt{\Delta} e^{\sqrt{\Delta} h}}{a+\sqrt{\Delta}} - 1 \geq 0 \Longleftrightarrow h \geq K = \frac{1}{\sqrt{\Delta}} \ln\left(\frac{a+\sqrt{\Delta}}{\sqrt{\Delta}}\right). \tag{37}$$

Notice that, if $a \leq 0$, then $K \leq 0$ and, therefore, $\xi'(h) \geq 0$ for any positive h. That means $\xi(h) > 0$ for any positive h, and so:

$$\varphi'(\omega) \geq \xi > 0 \quad \forall h \geq 0. \tag{38}$$

On the other hand, consider the case of $a > 0$. Then, $K > 0$ and there exists a minimum for $\xi(h)$ at $h = K$, so that $\xi(h) > 0$ for any $h \geq 0$ if, and only if, $\xi(K) > 0$. After some computations, it comes:

$$\xi(K) = \frac{1}{\sqrt{\Delta}} - \frac{1}{\sqrt{\Delta}} \ln\left(\frac{a+\sqrt{\Delta}}{\sqrt{\Delta}}\right) > 0 \Longleftrightarrow \frac{a+\sqrt{\Delta}}{\sqrt{\Delta}} < e \Longleftrightarrow \Delta > \frac{a^2}{(e-1)^2}. \tag{39}$$

Finally, since $\Delta = a^2 + b^2 q/r$, it is:

$$\xi(K) > 0 \Longleftrightarrow \frac{b^2 q}{a^2 r} \geq \frac{1}{(e-1)^2} - 1. \tag{40}$$

The right-hand side of this last inequality is trivially verified because $1/(e-1)^2 < 1$ and so $\xi(K) > 0$.

In summary, for $a \leq 0$ there is no crossing of the imaginary axis whatever is set the delay h, and so the roots of the characteristic equation never become with positive real part: there is no loss of stability. On the other hand, if $a > 0$, there exists a unique value for the delay h ($h = \overline{h}$, actually) according to which $\lambda = 0$ is a solution of the characteristic equation. In order to investigate whether there is actually a crossing of the imaginary axis, according to the established literature [25, 26], the sign of the derivative of the real part of the eigenvalues with respect to h, when crossing the imaginary axis, has to be computed. To this aim, the following equation [26] will be suitably exploited:

$$\operatorname{sign}\left\{\frac{d(\mathrm{Re}\lambda)}{dh}\right\} = \operatorname{sign}\left\{\mathrm{Re}\left(\frac{d\lambda}{dh}\right)^{-1}\right\}. \tag{41}$$

Rewrite the characteristic function by using the following slightly abuse of notation:

$$\varphi(h) = d\big(\lambda(h), h\big) \tag{42}$$

according to which:

$$\frac{d\varphi}{dh} = \frac{\partial d}{\partial \lambda}\frac{d\lambda}{dh} + \frac{\partial d}{\partial h} = 0 \tag{43}$$

and so:

$$\left(\frac{d\lambda}{dh}\right)^{-1} = \frac{1 - h(a+\sqrt{\Delta})e^{-\sqrt{\Delta}h}\cdot e^{-\lambda h}}{(a+\sqrt{\Delta})(\lambda+\sqrt{\Delta})e^{-\sqrt{\Delta}h}\cdot e^{-\lambda h}}. \tag{44}$$

By computing (44) for $\lambda = j0$ and $h = \overline{h}$ it becomes:

$$\left[\left(\frac{d\lambda}{dh}\right)^{-1}\right]_{\substack{\lambda=j0\\ h=\overline{h}}} = \frac{1-\overline{h}(a+\sqrt{\Delta})e^{-\sqrt{\Delta}\,\overline{h}}}{(a+\sqrt{\Delta})\sqrt{\Delta}\cdot e^{-\sqrt{\Delta}\,\overline{h}}} = \frac{1-a\overline{h}}{a\sqrt{\Delta}} = \frac{1-\frac{a}{\sqrt{\Delta}}\ln\left(1+\frac{\sqrt{\Delta}}{a}\right)}{a\sqrt{\Delta}}. \tag{45}$$

Then,

$$\operatorname{sign}\left\{\left.\frac{d(\mathrm{Re}\lambda)}{dh}\right|_{\substack{\lambda=j0\\ h=\overline{h}}}\right\} > 0 \Longleftrightarrow \frac{\sqrt{\Delta}}{a} > \ln\left(1+\frac{\sqrt{\Delta}}{a}\right). \tag{46}$$

Since $X > \ln(1+X)$ for any $X > 0$, the above inequality holds true for any set of parameters such that $a > 0$: by increasing the delay there will come a value $\overline{h}$ according to which there will be a root crossing the imaginary axis in $j0$ towards the positive real complex half plane. No more imaginary axis crossing is permitted to the roots of the characteristic function, and stability is lost. □

A way to design a control law for system (1) is by disregarding the delay of the control input. In this way, we have the solution of the optimal control law in the delay-free case, that is

$$u(t) = -R^{-1}B^TPx(t). \tag{47}$$

Instead of the optimal control law given by Theorem 3.1, we can try to apply the control law (47) to system (1). Of course, when closing the loop the delay affects the input, so that the closed loop system becomes

$$\dot{x}(t) = Ax(t) - BR^{-1}B^T Px(t-h), \quad t \geq h. \tag{48}$$

In the following it will be shown, for the scalar case, that the stability conditions obtained in Theorem 4.1 are less restrictive than the ones required by the closed loop system (48) that, in the scalar case, becomes:

$$\dot{x}(t) = ax(t) - (a + \sqrt{\Delta})x(t-h), \quad t \geq h. \tag{49}$$

Theorem 4.2. Consider the time-delay system (49). If $a \leq -|b|\sqrt{q/(3r)}$, then, for any given delay $h \geq 0$, the origin is asymptotically stable. If $a > -|b|\sqrt{q/(3r)}$, then the origin is asymptotically stable for $0 \leq h < \tilde{h}$, and unstable for $h > \tilde{h}$, where

$$\tilde{h} = \frac{1}{\tilde{\omega}} \arctan\left(\frac{\tilde{\omega}}{a}\right), \quad \tilde{\omega} = \Delta^{\frac{1}{4}}\sqrt{2a + \sqrt{\Delta}}. \tag{50}$$

P r o o f. Consider the characteristic function of system (49):

$$\tilde{d}(\lambda) = \lambda - a + (a + \sqrt{\Delta})e^{-\lambda h}. \tag{51}$$

Notice that for $h = 0$ it reduces to a first order polynomial

$$\tilde{d}(\lambda) = \lambda + \sqrt{\Delta} \tag{52}$$

which admits the unique negative real solution $\lambda = -\sqrt{\Delta}$, that means asymptotic stability. Again, the stability analysis is investigated by taking into account whether by increasing the delay parameter h there appear roots with positive real part. In this case, we may apply existing theorems investigating the roots placement of transcendental equations like (51): indeed, the Theorem is completed by trivially exploiting Thm. 2.1 in [18]. $\square$

Remark 4.3. It has to be stressed that, besides the loss of optimality, the control law designed in eq.(47) also weakens the stability result, for the scalar case. Indeed, stability is no more ensured for negative values of parameter a, since there is a bound for h also for $-|b|\sqrt{q/(3r)} < a \leq 0$. Furthermore, the bound $\tilde{h}$ is smaller than the bound $\overline{h}$ in eq.(27) whatever are chosen the model parameter with $a > 0$. To prove this last statement, we need to show that, for any (positive) a, Δ it is:

$$\frac{1}{\Delta^{\frac{1}{4}}\sqrt{2a + \sqrt{\Delta}}} \arctan\left(\frac{\Delta^{\frac{1}{4}}\sqrt{2a + \sqrt{\Delta}}}{a}\right) < \frac{1}{\sqrt{\Delta}} \ln\left(\frac{a + \sqrt{\Delta}}{a}\right). \tag{53}$$

To this aim, note that, denoting $\sigma = \sqrt{\Delta}/a > 1$ the left-hand side of (53) can be written as

$$\frac{\arctan(\sqrt{2\sigma + \sigma^2})}{a\sqrt{2\sigma + \sigma^2}}, \tag{54}$$

whilst the right-hand side can be written as

$$\frac{\ln(1+\sigma)}{a\sigma}. \tag{55}$$

Thus the point is to show that:

$$\frac{\arctan(\sqrt{2\sigma+\sigma^2})}{\sqrt{2\sigma+\sigma^2}} < \frac{\ln(1+\sigma)}{\sigma}, \quad \forall\sigma > 1. \tag{56}$$

Indeed, inequality (56) is true, as can be seen by visual inspection from Figure 1.

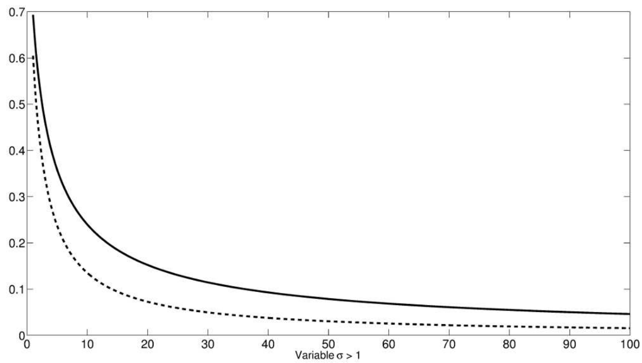

Fig. 1. Plot of the l.h.s. (dotted line) and r.h.s. (full line) of the eq.(56) versus variable $\sigma > 1$.

5. SIMULATION RESULTS

In this section we consider the following system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t-h), \quad x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{57}$$

We have taken the input signal equal to zero in $[-h, 0)$. We have chosen the matrix Q as the identity in $\mathbb{R}^{2\times 2}$ and $R = 1$. We have computed the solution to the Riccati equation up to an error smaller than 10^{-12}, and applied the optimal control law. We have made simulations by means of third order Runge–Kutta method, with fixed integration step $T = 0.0001$secs. Simulations show convergence to 0 of the state variables, up to $h = 0.33$. For $h = 0.34$, a simulation on a time interval of 50 secs starts showing divergence of the state variables. For $h > 0.34$, simulations show clearly that the state variables diverge. This is not in contradiction with the optimality of the control law, since, given that the closed loop system may become unstable, numerical errors deviate the solution from the optimal one and thus the divergence effect appears. For the cases with $h = 0.27$, $h = 0.33$, $h = 0.35$, the plot of the state variables is reported in Figure 2, Figure 4,

Figure 6, and the plot of the optimal input signal is reported in Figure 3, Figure 5, Figure 7, respectively. We have applied to the system also the control input obtained by neglecting the time delay, that is

$$u(t) = -R^{-1}B^T P x(t), \quad t \geq 0. \tag{58}$$

In Figure 8 and in Figure 9 the state variables and the control law signal are reported, respectively. As can be seen, for $h = 0.27$, the state variables diverge to ∞. We observe the convergence of the state to 0, though oscillations appear, by using the control law (58), for $h \leq 0.25$, as shown in Figure 10 and Figure 11, respectively. Thus, simulations show that we can consider a delay h up to 0.25, with the control law (58), and a delay h up to 0.33, with the control law (4). The improvement, as far as the allowed size of the delay is concerned, is evident. For the case $h = 0.33$, we have computed numerically the value of the functional J as 24.9343 (the functional (2) includes the values of the state in free evolution, in the interval $[0, h)$). If we consider the functional (3), we have computed numerically its value as 23.8407. In this case, the computation by Matlab of $\tilde{x} = e^{Ah}x_0$ returns $\tilde{x} = \begin{bmatrix} 0.479 \\ -2.3090 \end{bmatrix}$. The following value is computed for the matrix P,

$$P = \begin{bmatrix} 5.4142 & 0.4142 \\ 0.4142 & 4.4142 \end{bmatrix}. \tag{59}$$

The term $\tilde{x}^T P \tilde{x}$ returns the value 23.8408, thus validating what is stated in remark 3.4 (take into account that the value of the functional computed numerically, 23.8407, besides numerical approximations, is underestimated because of the finite time interval of the simulation). We have also made simulations perturbing the optimal control input. Namely, at any integration step kT, $k = 0, 1, 2, \ldots$, the computed optimal input $u(kT)$ is changed into $u(kT) + 2(rand - 0.5)u(kT)$ (recall that $rand$ is the Matlab command generating a random variable with uniform probability density in $(0, 1)$). We always obtained greater values of the functional, still observing boundedness of the state variables. In Figure 12 and Figure 13, the state variables and the input signal are reported, respectively, when the Matlab $rand$ function is initialized with the command $rand('seed', 0)$. The value of the functional J is in this case computed as 33.1857.

6. CONCLUSIONS

In this paper we have provided a solution to the infinite horizon linear quadratic optimal control problem for linear time-invariant systems with a known constant time-delay in the input channel. Such a solution has the characteristic of being memoryless, except at an initial time interval of measure equal to the time-delay. If the initial input is set equal to zero, then the optimal feedback control law is memoryless from the beginning. Stability results are established for the closed loop system, in the scalar case. Many simulations have been performed, showing the effectiveness of the proposed optimal controller. Moreover, it has been shown in simulations, for a two dimensional system, that the maximum delay range, for which stability is guaranteed, is meaningfully improved with respect to the one obtained with a controller built up by optimal control formulas neglecting the delay.

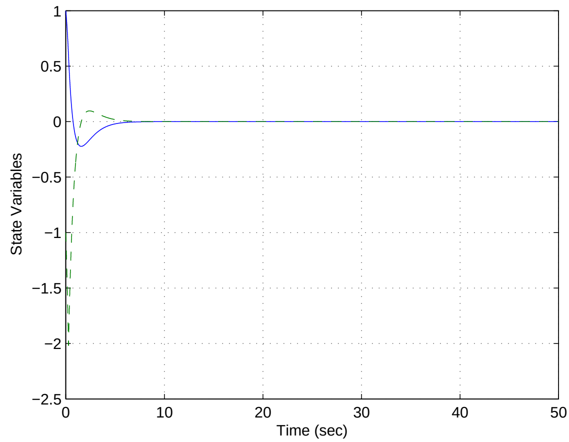

Fig. 2. State Variables, $h = 0.27$.

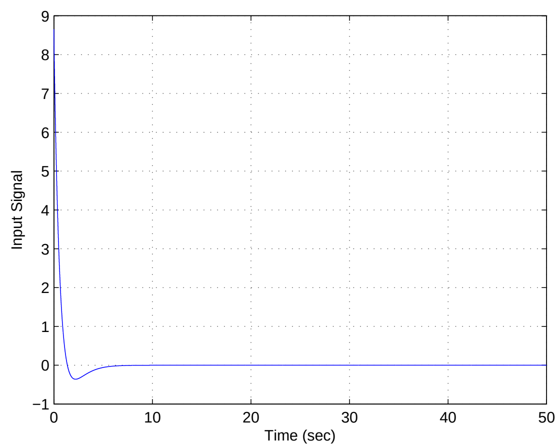

Fig. 3. Optimal Input Signal, $h = 0.27$.

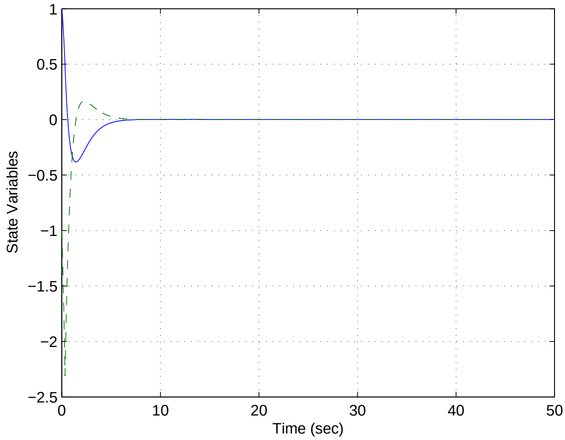

Fig. 4. State Variables, $h = 0.33$.

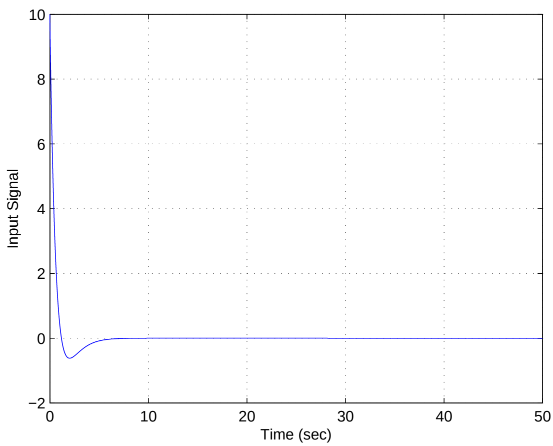

Fig. 5. Optimal Input Signal, $h = 0.33$.

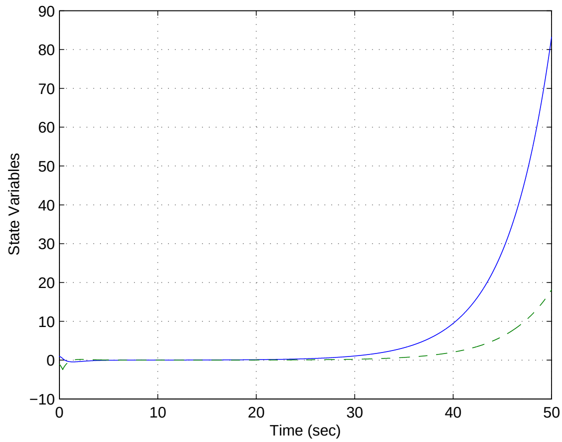

Fig. 6. State Variables, $h = 0.35$.

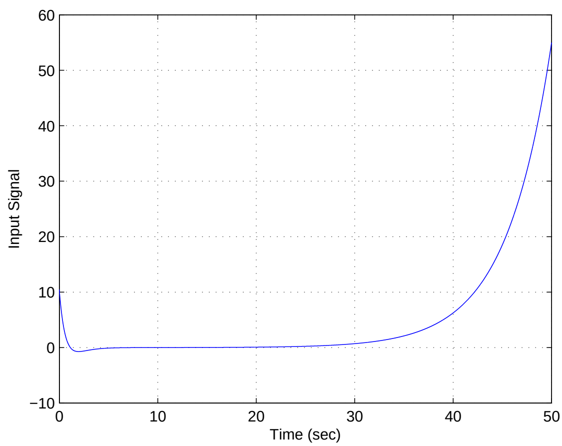

Fig. 7. Optimal Input Signal, $h = 0.35$.

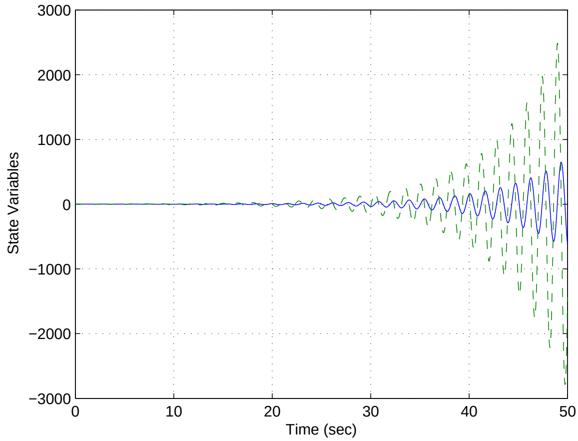

Fig. 8. State Variables, $h = 0.27$, control law (58).

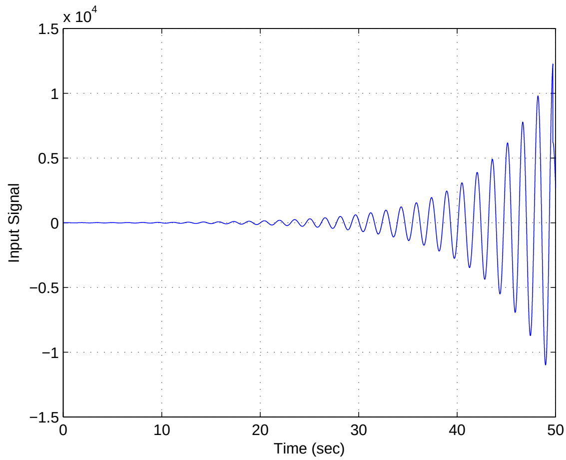

Fig. 9. Input Signal, $h = 0.27$, control law (58).

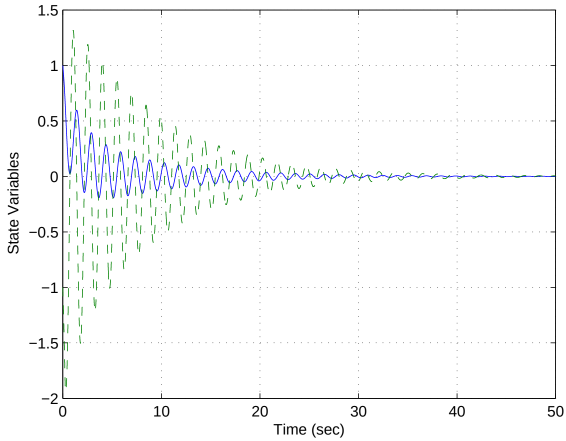

Fig. 10. State Variables, $h = 0.25$, control law (58).

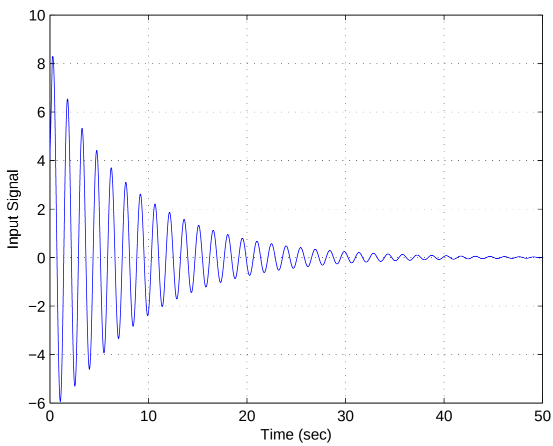

Fig. 11. Input Signal, $h = 0.25$, control law (58).

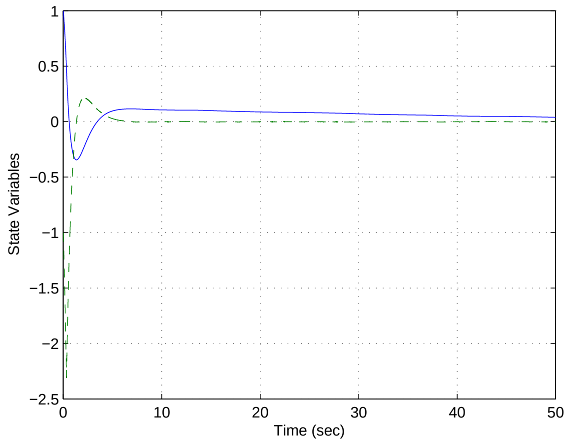

Fig. 12. State Variables, $h = 0.33$, perturbed control law.

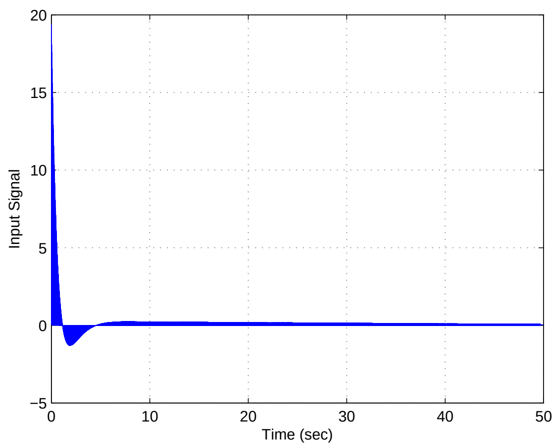

Fig. 13. Input Signal, $h = 0.33$, perturbed control law.

Future work will concern the infinite horizon linear quadratic optimal control problem for time-varying systems with time-varying time-delay in the input channel, on the basis of the results in [6].

ACKNOWLEDGEMENT

This work was partially supported by MIUR PRIN Project 2009, and by the Center of Excellence for Research DEWS. The authors are grateful to the Associate Editor and Reviewers for insightful comments which helped to improve this paper.

(Received November 16, 2012)

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