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REGULARITY RESULTS FOR A CLASS OF OBSTACLE PROBLEMS IN HEISENBERG GROUPS

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Abstract. We study regularity results for solutions $u \in HW^{1,p}(\Omega)$ to the obstacle problem

$$\int_\Omega \mathcal{A}(x, \nabla_{\mathbb{H}} u)\nabla_{\mathbb{H}}(v-u)\,\mathrm{d}x \geqslant 0 \quad \forall v \in \mathcal{K}_{\psi,u}(\Omega)$$

such that $u \geqslant \psi$ a.e. in Ω, where $\mathcal{K}_{\psi,u}(\Omega) = \{v \in HW^{1,p}(\Omega)\colon\ v - u \in HW_0^{1,p}(\Omega) v \geqslant \psi$ a.e. in $\Omega\}$, in Heisenberg groups $\mathbb{H}^n$. In particular, we obtain weak differentiability in the T-direction and horizontal estimates of Calderon-Zygmund type, i.e.

$$T\psi \in HW^{1,p}_{\rm loc}(\Omega) \Rightarrow Tu \in L^p_{\rm loc}(\Omega),$$
$$|\nabla_{\mathbb{H}}\psi|^p \in L^q_{\rm loc}(\Omega) \Rightarrow |\nabla_{\mathbb{H}} u|^p \in L^q_{\rm loc}(\Omega),$$

where $2 < p < 4$, $q > 1$.

Keywords: obstacle problem, weak solution, regularity, Heisenberg group

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1. Introduction

The aim of this paper is the study of some regularity results for solutions of one-side obstacle problems in the Heisenberg group. More precisely, let Ω be an open and bounded domain in the Heisenberg group $\mathbb{H}^n$. We will consider the weak solution $u \in HW^{1,p}(\Omega)$ of the obstacle problem

$$\int_\Omega \mathcal{A}(x, \nabla_{\mathbb{H}} u)\nabla_{\mathbb{H}}(v-u)\,\mathrm{d}x \geqslant 0 \quad \forall v \in \mathcal{K}_{\psi,u}(\Omega) \tag{1.1}$$

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such that $u \geqslant \psi$ a.e. in Ω, where $\nabla_{\mathbb{H}}$ and $HW^{1,p}(\Omega)$ are respectively the *horizontal gradient* and the *horizontal Sobolev space* introduced in (2.5), $\psi \in HW^{1,p}(\Omega)$ is a given obstacle function and

$$\mathcal{K}_{\psi,u}(\Omega) = \{v \in HW^{1,p}(\Omega)\colon\ v - u \in HW_0^{1,p}(\Omega),\ v \geqslant \psi \text{ a.e. in } \Omega\}, \tag{1.2}$$

where $HW_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $HW^{1,p}(\Omega)$.

We need the following assumptions, with positive constants α and β, to hold for the operator $\mathcal{A}\colon\ \Omega \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$:

$$x \mapsto \mathcal{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^{2n}; \tag{1.3}$$

$$\xi \mapsto \mathcal{A}(x,\xi) \text{ is continuous for almost all } x \in \Omega; \tag{1.4}$$

$$\mathcal{A}(x,\xi)\cdot\xi \geqslant \alpha|\xi|^p \text{ for almost all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{2n}; \tag{1.5}$$

$$|\mathcal{A}(x,\xi)| \leqslant \beta(|\xi|^{p-1}+1) \text{ for almost all } x \in \Omega \text{ and } \xi \in \mathbb{R}^{2n}; \tag{1.6}$$

$$\langle(\mathcal{A}(x,\eta)-\mathcal{A}(x,\xi)),(\eta-\xi)\rangle \geqslant c^*(\alpha)(\mu^2+|\eta|^2+|\xi|^2)^{(p-2)/2}|\eta-\xi|^2 \tag{1.7}$$
$$\text{for almost all } x \in \Omega \text{ and } \xi \neq \eta \in \mathbb{R}^{2n}.$$

We may assume that $\alpha \leqslant \beta$, by choosing β larger, if necessary. We will refer to this set of conditions as the *structure conditions* of $\mathcal{A}$.

It is worth noticing that the first four structural conditions are not strong enough to give a unique solution to the $\mathcal{K}_{\psi,u}$-obstacle problem. However, if $\mathcal{A}$ satisfies the monotonicity condition

$$(\mathcal{A}(x,\xi_1)-\mathcal{A}(x,\xi_2))\cdot(\xi_1-\xi_2) > 0, \quad \xi_1 \neq \xi_2 \tag{1.8}$$

for a.e. $x \in \mathbb{R}^n$, then it can be shown, working as in the Euclidean case, that the $\mathcal{K}_{\psi,u}$-obstacle problem admits a unique solution provided that $\mathcal{K}_{\psi,u} \neq \emptyset$ (see for instance [23], Chapters 3 and 7).

The study of the classical obstacle problem, which started in the sixties with the pioneering work of Stampacchia, Lewy and Lions [26], [27], [35] has led in the last decades to deep developments in the calculus of variations and partial differential equations; among other, some fundamental results have been achieved by Caffarelli ([3], [4]) concerning the theory of free boundaries for the obstacle problem. From that moment onwards many authors have contributed, also following different points of view bringing regularity results for single and double obstacle problem (see among others [8], [12], [18], [19], [20], [33], [34] together with the references therein).

As already mentioned, the aim of this paper is to prove some basic regularity results for the solution to the obstacle problem (1.1) in the Heisenberg group. Beside his mathematical importance as a model of the metric space, the interest in the

Heisenberg group has grown in the last years due to its many applications. The former has been in the modellizations of nonholonomic mechanic (see [7] and reference therein), other ones have been in control theory and in engineering (for instance the motion of robot arms) [37] and neurobiology (models of perceptual completion) [9]. The study of regularity properties of solutions to sub-elliptic equations in Heisenberg groups and in more general Carnot groups started with Hormander [24] and has been developed more recently by the works of Capogna, Garofalo, Danielli, Manfredi, Mingione, Goldstein-Zatorska, and Domokos [5], [6], [12], [14], [15], [16], [17], [21], [28], [29], [32]. We quote the recent and important papers of Mingione and coworkers [32] and Domokos [14], [15], which are fundamental in the techniques of proofs of our results.

As we said we obtain integrability estimates on Tu and $\nabla_{\mathbb{H}}u$, where u is the weak solution of the obstacle problem (1.1). The regularity result in the vertical direction T is obtained under the assumption $\mathcal{A}(x, \nabla_{\mathbb{H}}u) = \mathcal{A}(\nabla_{\mathbb{H}}u)$. We implement iteration methods on fractional difference quotients, using the techniques of Domokos in [14], [15]. In particular, we consider as test functions in the weak form of (1.1), the fractional difference quotients of the weak solution multiplied by a corresponding cut-off function. Notice that this method has been applied in the Euclidean setting to regularity problems of nonlinear second order equations ([32]). The results we prove (see Theorems 3.4, 3.5) can be summarized as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $2 \leqslant p < 4$ and let $u \in HW^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of the obstacle problem* (1.1) *with $T\psi \in W^{1,p}_{\mathbb{H},\text{loc}}(\Omega)$, where $\mathcal{A}(x, \nabla_{\mathbb{H}}u) = \mathcal{A}(\nabla_{\mathbb{H}}u)$. Then $Tu \in L^p_{\text{loc}}(\Omega)$.*

The second result we achieve goes along the lines of the nonlinear Calderón-Zygmund theory; indeed, due to the recent result provided by Mingione and coworkers [32] we are able to obtain a Calderón-Zygmund type estimate for the solution u to the obstacle problem in the following sense: provided that the obstacle function ψ belongs to $HW^{1,q}(\Omega)$ with some $q > p$, p being the natural growth exponent appearing in the structure conditions for $\mathcal{A}$, then also $u \in HW^{1,q}(\Omega)$. The study of nonlinear Calderón-Zygmund type estimates goes back to the fundamental paper of Iwaniec [25] in the case of elliptic equations with constant p growth, and to the paper of Di Benedetto and Manfredi [13] in the case of elliptic systems. Recently, Acerbi and Mingione proved estimates of this kind for parabolic systems in [1] and Bögelein, Duzaar and Mingione proved similar results in the elliptic and in the parabolic case in [2], using the technique introduced by [1]. The result of [2] has been subsequently extended by Eleuteri and Habermann to the variable exponent case, see [19]. Furthermore, Mingione [30], [31] developed a natural extension of the Calderón Zygmund theory for problems with measure data, showing appropriate fractional differentiability of the solution. The result we prove extends to the subelliptic case the original result of [2] and can be summarized as follows.

Theorem 1.2. *Let $u \in HW^{1,p}(\Omega)$ be a solution to the obstacle problem* (1.1) *under the assumptions* (1.3)–(1.8) *and* $2 < p < 4$. *If* $|\nabla_{\mathbb{H}}\psi|^p \in L^q_{\mathrm{loc}}(\Omega)$ *for some* $q > 1$, *then* $|\nabla_{\mathbb{H}}u|^p \in L^q_{\mathrm{loc}}(\Omega)$.

The proof of this result goes through several steps. As in the Euclidean case, the key point to the proof of a quantified higher integrability of the gradient of the solution u to the obstacle problem (1.1) is a decay estimate of the level sets of the maximal function of $|\nabla_{\mathbb{H}}u|^p$ to increasing levels, as we can see in (4.24) (recall also the definitions of μ_1 and μ_2 in (4.20)). Iteration of (4.24) in combination with the well known L^p estimates for the maximal function then directly provides the desired integrability result. To prove (4.24), we make use of Lemma 4.2 which is a direct consequence of a Calderón-Zygmund type covering argument. To apply this lemma on super level sets of the maximal function (see the definitions of E and G in (4.22) and (4.23)), it turns out to be crucial to show that assumption (ii) in Lemma 4.2 is fulfilled. This is the statement of Lemma 4.3. In order to prove Lemma 4.3, the strategy consists in a comparison of the solution to the original obstacle problem to the solution to the Dirichlet problem

$$\begin{cases} \mathrm{div}_{\mathbb{H}}\mathcal{A}(x_0, \nabla_{\mathbb{H}}z) = 0 & \text{in } B, \\ z = u & \text{on } \partial B. \end{cases} \tag{1.9}$$

The structure conditions of this problem—a nonlinear degenerate elliptic equation with constant growth exponent—guarantee an L^∞ estimate for the gradient of z, namely the following theorem which is the novelty brought by Mingione and coworkers in [32]:

$$\sup_{B_{R/2}} |\nabla_{\mathbb{H}}u| \leqslant c\left(\fint_{B_R} (\mu + |\nabla_{\mathbb{H}}u|^p)\,\mathrm{d}h \right)^{1/p}. \tag{1.10}$$

To compare the solution to the original obstacle problem to the solution to (1.9), it turns out to be necessary to include further two comparison processes, in order to be able, through the different comparison estimates, to pass the *sup* estimate on the solution u to the original obstacle problem.

The structure of the paper is the following: in Section 2 we recall some preliminary results and definitions in the Heisenberg group, Section 3 is devoted to the study of the vertical derivative Tu and Section 4 to the Calderón-Zygmund type estimates of the horizontal gradient $\nabla_{\mathbb{H}}u$.

The Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} = \mathbb{R}^{2n+1}$ is the simplest example of the Carnot group, endowed with a left-invariant metric d_∞, which is not equivalent to the Euclidean metric.

We shall denote the points of $\mathbb{H}^n$ by $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}, t)$. If $x = (x', t)$, $y = (y', s) \in \mathbb{H}^n$, we define the group operation

$$x \cdot y := \Big(x' + y', t + s - \frac{1}{2}\sum_i [y_i x_{i+n} - x_i y_{i+n}]\Big) \tag{2.1}$$

and the family of non isotropic dilations $\delta_r(x) := (Rx', R^2 t)$, for $R > 0$. The Heisenberg Lie algebra $\mathfrak{h}$ is (linearly) generated by

$$X_j = \frac{\partial}{\partial x_j} - \frac{x_{j+n}}{2}\frac{\partial}{\partial t}, \quad X_{j+n} = \frac{\partial}{\partial x_{j+n}} + \frac{x_j}{2}\frac{\partial}{\partial t} \quad \text{for } j = 1, \ldots, n; \quad T = \frac{\partial}{\partial t}, \tag{2.2}$$

the only non-trivial commutator relations are $[X_j, X_{j+n}] = T$ for $j = 1, \ldots, n$. Let us define $\|x\|_\infty := \max\{|x'|, |t|^{1/2}\}$ and the distance d_∞, defined as $d_\infty(x, y) := \|x^{-1} \cdot y\|_\infty$.

Proposition 2.1. *For any bounded subset $\Omega \in \mathbb{H}^n$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that*

$$c_1(\Omega)|x - y|_{\mathbb{R}^{2n+1}} \leqslant d_\infty(x, y) \leqslant c_2(\Omega)|x - y|^{1/2}_{\mathbb{R}^{2n+1}} \quad \textit{for } x, y \in \Omega. \tag{2.3}$$

Hence, the topologies defined by d_∞ and by the Euclidean distance coincide on $\mathbb{H}^n$, therefore the topological dimension of $\mathbb{H}^n$ is $2n + 1$. On the contrary, the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $\mathcal{Q} = 2n + 2$. $\mathcal{Q}$ is called the homogeneous dimension of $\mathbb{H}^n$.

We will indicate the ball with center $x_0 \in \mathbb{H}^n$ and radius R with respect to the distance d_∞ by $B(x_0, R) := \{x \in \mathbb{H}^n \colon d_\infty(x, x_0) \leqslant R\}$. The ball $B(x_0, R)$ has a doubling property, i.e. there exists a constant C, depending only on the homogeneous dimension $\mathcal{Q}$ such that

$$\mathcal{L}^{2n+1}(B(x_0, 2R)) \leqslant C\mathcal{L}^{2n+1}(B(x_0, R)). \tag{2.4}$$

When the center of the ball is not important, we shall use the notation $B_R = B(x_0, R)$ and when no ambiguity may arise, we shall also denote $\lambda B = B(x_0, \lambda R)$ for $\lambda > 0$.

There is a natural measure $\mathrm{d}x$ on $\mathbb{H}^n$ which is given by the Lebesgue measure $\mathrm{d}\mathcal{L}^{2n+1} = \mathrm{d}x$ on $\mathbb{R}^{2n+1}$. The measure $\mathrm{d}x$ is left (and right) invariant and it is the Haar measure of the group.

Definition 2.2. Let $B_R \subset \mathbb{H}^n$ be a ball and $f\colon B_R \to \mathbb{R}^k$ an intagrable function. Let us define the average of f over B_R as

$$(f)_R := \fint_{B_R} f(x)\,\mathrm{d}x = \frac{1}{\mathcal{L}^{2n+1}(B_R)} \int_{B_R} f(x)\,\mathrm{d}x.$$

We shall identify vector fields and associated first order differential operators; thus the vector fields $X_1, \dots, X_n$, $X_{n+1}, \dots, X_{2n}$ generate a vector bundle on $\mathbb{H}^n$, the so called *horizontal* vector bundle $\mathrm{H}\mathbb{H}^n$ according to the notation of Gromov (see [22]), that is a vector subbundle of $\mathrm{T}\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$.

Let $\Omega \subset \mathbb{H}^n$ be an open set and $u \in C^0(\Omega)$. We will define in the sense of distributions as the *horizontal gradient* of u the vector

$$\nabla_{\mathbb{H}} u := (X_1 u, \dots, X_n u, X_{n+1} u, \dots, X_{2n} u).$$

It is well-known that $\nabla_{\mathbb{H}}$ acts as a gradient operator in $\mathbb{H}^n$. Let us denote by $C^1_{\mathbb{H}}(\Omega)$ the set of continuous real functions in Ω such that $\nabla_{\mathbb{H}} u$ is continuous in Ω. The notion of $C^k_{\mathbb{H}}(\Omega)$ is given analogously. Finally, let us define the horizontal Sobolev space

$$HW^{1,p}(\Omega) := \{u \in C^0(\Omega)\colon \ \nabla_{\mathbb{H}} u \in L^p(\Omega; \mathbb{R}^{2n})\}; \tag{2.5}$$

$HW^{1,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{HW^{1,p}(\Omega)} := \|u\|_{L^p_{\mathbb{H}}(\Omega)} + \|\nabla_{\mathbb{H}} u\|_{L^p_{\mathbb{H}}(\Omega;\mathbb{R}^{2n})}.$$

As already mentioned, $HW^{1,p}_0(\Omega)$ is defined in the usual way, as the closure of $C^\infty_0(\Omega)$ in $HW^{1,p}(\Omega)$. We will write $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ if $u \in HW^{1,p}(K)$ for every compact set $K \subset \Omega$.

To conclude this section, let us recall that if Z is an invariant vector field, then for some $P = (x_1, \dots x_{2n}, t)$ we can write

$$Z = \sum_{i=1}^{2n} x_i X_i + tT.$$

The exponential mapping in canonical coordinates is defined by $\mathrm{e}^Z = P$. Let us finally recall the Baker-Campbell-Hausdorff formula for the invariant vector fields Z, V

$$\mathrm{e}^Z \mathrm{e}^V = \mathrm{e}^{Z+V+\frac{1}{2}[Z,V]}.$$

3. A regularity result for the vertical derivative

In this section we consider the obstacle problem (1.1) in the case $\mathcal{A}(x,\nabla_{\mathbb{H}}u)=\mathcal{A}(\nabla_{\mathbb{H}}u)$, i.e.

$$\int_\Omega \mathcal{A}(\nabla_{\mathbb{H}}u)\nabla_{\mathbb{H}}(v-u)\,\mathrm{d}x\geqslant 0\quad \forall v\in\mathcal{K}_{\psi,u}(\Omega) \tag{3.1}$$

under the assumptions (1.3)–(1.8). Let us recall some preliminary definitions and results about fractional difference quotients, following the notation of [14].

Definition 3.1. Let $\Omega\subset\mathbb{H}^n$ be a bounded open set. Let $x\in\Omega$, let Z be a left invariant vector field, $s\in\mathbb{R}$, $0<\alpha,\theta\leqslant 1$ and let $u\colon\ \Omega\to\mathbb{R}$. We define

$$\begin{aligned}
D_{Z,s,\theta}u(x)&:=\frac{u(x\cdot \mathrm{e}^{sZ})-u(x)}{|s|^\theta},\\
D_{Z,-s,\theta}u(x)&:=\frac{u(x\cdot \mathrm{e}^{-sZ})-u(x)}{-|s|^\theta},\\
\Delta_{Z,s}u(x)&:=u(x\cdot \mathrm{e}^{sZ})-u(x),\\
\Delta^2_{Z,s}u(x)&:=u(x\cdot \mathrm{e}^{sZ})+u(x\cdot \mathrm{e}^{-sZ})-2u(x).
\end{aligned}$$

Let us notice that

$$\begin{aligned}
D_{Z,-s,\alpha}D_{Z,s,\theta}u(x)&=D_{Z,s,\theta}D_{Z,-s,\alpha}u(x)\\
&=\frac{u(x\cdot \mathrm{e}^{sZ})+u(x\cdot \mathrm{e}^{-sZ})-2u(x)}{|s|^{\alpha+\theta}}=\frac{\Delta^2_{Z,s}u(x)}{|s|^{\alpha+\theta}}.
\end{aligned}$$

If $\theta=1$, we will denote $D_{Z,s,1}u\equiv D_{Z,s}u$. We will use the following results from [5], [14], [24].

Proposition 3.2. *Let $\Omega\subset\mathbb{H}^n$ be an open set, $K\subset\Omega$ a compact set, Z a left invariant vector field and $u\in L^p_{\mathrm{loc}}(\Omega)$. If there exist constants $\sigma,C>0$ such that*

$$\sup_{0<|s|<\sigma}\int_K |D_{Z,s,1}u(x)|^p\,\mathrm{d}x\leqslant C^p,$$

then $Zu\in L^p(K)$ and $\|Zu\|_{L^p(K)}\leqslant C$. Conversly, if $Zu\in L^p(K)$ then for some $\sigma>0$

$$\sup_{0<|s|<\sigma}\int_K |D_{Z,s,1}u(x)|^p\,\mathrm{d}x\leqslant (2\|Zu\|_{L^p(K)})^p.$$

Proposition 3.3. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $1 < p < \infty$, let $u \in HW^{1,p}_{\rm loc}(\Omega)$, $x_0 \in \Omega$, and $R > 0$ be such that $B_{3R} = B(x_o, 3R) \subset \Omega$. Then there exists a positive constant c independent of u such that*

$$\int_{B_R} |D_{T,s,\frac{1}{2}} u(x)|^p \,\mathrm{d}x \leqslant c \int_{B_{2R}} (|u|^p + |\nabla_{\mathbb{H}} u|^p) \,\mathrm{d}x.$$

We are now able to show our result.

Theorem 3.4. *Let $\Omega \subset \mathbb{H}^n$ be an open set, let $u \in HW^{1,p}_{\rm loc}(\Omega)$ be a weak solution of the obstacle problem (3.1) with $T\psi \in HW^{1,p}_{\rm loc}(\Omega)$, $x_0 \in \Omega$, $R > 0$ such that $B_R = B(x_0, R) \subset \Omega$. Let us suppose that there exist $c > 0$, $\sigma > 0$, and $\alpha \in [0, 1/2)$ such that*

$$\sup_{0 \neq |s| \leqslant \sigma} \int_{B_R} |D_{T,s,\frac{1}{2}+\alpha} u(x)|^p \,\mathrm{d}x \leqslant c \int_{B_{2R}} (\mu^2 + |\nabla_{\mathbb{H}} u(x)|^2)^{p/2} + |u(x)|^p \,\mathrm{d}x. \tag{3.2}$$

If $(1 + 2\alpha)/p < 1/2$ then with possibly different $c > 0$ and $\sigma > 0$ we have

$$\begin{aligned} \sup_{0 \neq |s| \leqslant \sigma} \int_{B_{R/2}} & |D_{T,s,1/2+1/p+(2/p)\alpha} u(x)|^p \,\mathrm{d}x \\ & \leqslant c \int_{B_{2R}} (\mu^2 + |\nabla_{\mathbb{H}} u(x)|^2)^{p/2} + |u(x)|^p + |T\psi(x)|^p + |\nabla_{\mathbb{H}} T\psi(x)|^p \,\mathrm{d}x. \end{aligned} \tag{3.3}$$

If $(1 + 2\alpha)/p \geqslant 1/2$ then

$$\begin{aligned} \int_{B_{R/4}} & |Tu(x)|^p \,\mathrm{d}x \\ & \leqslant c \int_{B_{2R}} (\mu^2 + |\nabla_{\mathbb{H}} u(x)|^2)^{p/2} + |u(x)|^p + |T\psi(x)|^p + |\nabla_{\mathbb{H}} T\psi(x)|^p \,\mathrm{d}x. \end{aligned} \tag{3.4}$$

Proof. Let $u \in HW^{1,p}(\Omega)$ be a weak solution of the problem (3.1) and let η be a cut-off function between $B_{R/2}$ and B_R such that there exists $C_\eta > 0$ such that $|\nabla_{\mathbb{H}} \eta| \leqslant C_\eta / R$. Let us define the function

$$\varphi(x) := u(x) + D_{T,-s,\gamma}(\eta^2 D_{T,s,\gamma}[u - \psi]). \tag{3.5}$$

Let us verify that φ is a good test function. Indeed,

$$\begin{aligned}
\varphi(x) &:= u(x) + D_{T,-s,\gamma}(\eta^2 D_{T,s,\gamma}[u-\psi]) = u(x) + D_{T,-s,\gamma}(\eta^2 D_{T,s,\gamma}u) \\
&\quad - D_{T,-s,\gamma}(\eta^2 D_{T,s,\gamma}\psi) = u(x) + D_{T,-s,\gamma}\Big(\eta^2(x)\frac{u(x\cdot \mathrm{e}^{sT}) - u(x)}{s^\gamma}\Big) \\
&\quad - D_{T,-s,\gamma}\Big(\eta^2(x)\frac{\psi(x\cdot \mathrm{e}^{sT}) - \psi(x)}{s^\gamma}\Big) = u(x) \\
&\quad + \frac{1}{s^{2\gamma}}[-\eta^2(x\cdot \mathrm{e}^{-sT})u(x) + \eta^2(x)u(x\cdot \mathrm{e}^{sT}) + \eta^2(x\cdot \mathrm{e}^{-sT})u(x\cdot \mathrm{e}^{-sT}) - \eta^2(x)u(x)] \\
&\quad + \frac{1}{s^{2\gamma}}[\eta^2(x\cdot \mathrm{e}^{-sT})\psi(x) - \eta^2(x)\psi(x\cdot \mathrm{e}^{sT}) - \eta^2(x\cdot \mathrm{e}^{-sT})\psi(x\cdot \mathrm{e}^{-sT}) - \eta^2(x)\psi(x)] \\
&= u(x)\Big[1 - \frac{1}{s^{2\gamma}}\eta^2(x\cdot \mathrm{e}^{-sT}) - \frac{1}{s^{2\gamma}}\eta^2(x)\Big] + \frac{1}{s^{2\gamma}}\eta^2(x)u(x\cdot \mathrm{e}^{-sT}) \\
&\quad + \frac{1}{s^{2\gamma}}\eta^2(x\cdot \mathrm{e}^{-sT})u(x\cdot \mathrm{e}^{-sT}) + \frac{1}{s^{2\gamma}}[\eta^2(x\cdot \mathrm{e}^{-sT}) + \eta^2(x)]\psi(x) \\
&\quad - \frac{1}{s^{2\gamma}}\eta^2(x)\psi(x\cdot \mathrm{e}^{-sT}) - \frac{1}{s^{2\gamma}}\eta^2(x\cdot \mathrm{e}^{-sT})\psi(x\cdot \mathrm{e}^{-sT}) \\
&\geqslant \psi(x)\Big[1 - \frac{1}{s^{2\gamma}}\eta^2(x\cdot \mathrm{e}^{-sT}) - \frac{1}{s^{2\gamma}}\eta^2(x)\Big] + \frac{1}{s^{2\gamma}}\eta^2(x)\psi(x\cdot \mathrm{e}^{-sT}) \\
&\quad + \frac{1}{s^{2\gamma}}\eta^2(x\cdot \mathrm{e}^{-sT})\psi(x\cdot \mathrm{e}^{-sT}) + \frac{1}{s^{2\gamma}}[\eta^2(x\cdot \mathrm{e}^{-sT}) + \eta^2(x)]\psi(x) \\
&\quad - \frac{1}{s^{2\gamma}}\eta^2(x)\psi(x\cdot \mathrm{e}^{-sT}) - \frac{1}{s^{2\gamma}}\eta^2(x\cdot \mathrm{e}^{-sT})\psi(x\cdot \mathrm{e}^{-sT}) = \psi(x).
\end{aligned}$$

Let us consider now the equation

$$\int_{B_R} \mathcal{A}(\nabla_{\mathbb{H}} u(x))(\nabla_{\mathbb{H}}(D_{T,-s,\gamma}(\eta^2 D_{T,s,\gamma}[u-\psi])))\,\mathrm{d}x \geqslant 0.$$

Since $D_{T,s,\gamma}$, $D_{T,-s,\gamma}$ and X_i are commutative, we have

$$\begin{aligned}
&\int_{B_R} D_{T,s,\gamma}(\mathcal{A}(\nabla_{\mathbb{H}} u(x)))\nabla_{\mathbb{H}}(\eta^2 D_{T,s,\gamma}[u-\psi])\,\mathrm{d}x \leqslant 0, \\
&\int_{B_R} D_{T,s,\gamma}(\mathcal{A}(\nabla_{\mathbb{H}} u(x)))\eta^2\nabla_{\mathbb{H}}(D_{T,s,\gamma}[u-\psi])\,\mathrm{d}x \\
&\qquad + \int_{B_R} D_{T,s,\gamma}(\mathcal{A}(\nabla_{\mathbb{H}} u(x)))2\eta\nabla_{\mathbb{H}}\eta D_{T,s,\gamma}[u-\psi]\,\mathrm{d}x \leqslant 0.
\end{aligned}$$

Then

$$\int_{B_R} D_{T,s,\gamma}(\mathcal{A}(\nabla_{\mathbb{H}} u(x)))\eta^2\nabla_{\mathbb{H}}(D_{T,s,\gamma}u)\,\mathrm{d}x \leqslant A - B + C, \tag{3.6}$$

where

$$\begin{aligned}
A &= \int_{B_R} D_{T,s,\gamma}(\mathcal{A}(\nabla_{\mathbb{H}} u(x)))\eta^2 \nabla_{\mathbb{H}}(D_{T,s,\gamma}\psi)\,\mathrm{d}x,\\
B &= \int_{B_R} D_{T,s,\gamma}(\mathcal{A}(\nabla_{\mathbb{H}} u(x)))2\eta \nabla_{\mathbb{H}}\eta D_{T,s,\gamma}u\,\mathrm{d}x,\\
C &= \int_{B_R} D_{T,s,\gamma}(\mathcal{A}(\nabla_{\mathbb{H}} u(x)))2\eta \nabla_{\mathbb{H}}\eta D_{T,s,\gamma}\psi\,\mathrm{d}x.
\end{aligned}$$

Using the same estimates of equation (3.9) as in Lemma 3.1 of [14] and denoting

$$A(x) := \mu^2 + |\nabla_{\mathbb{H}} u(x)|^2 + |\nabla_{\mathbb{H}} u(x\cdot \mathrm{e}^{sT})|^2,$$

we obtain

$$\begin{aligned}
&\int_{B_R} D_{T,s,\gamma}(\mathcal{A}(\nabla_{\mathbb{H}} u(x)))\eta^2 \nabla_{\mathbb{H}}(D_{T,s,\gamma}u)\,\mathrm{d}x\\
&\qquad\leqslant c\underbrace{\int_{B_R} \eta^2 A(x)^{(p-2)/2}|D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|\,|D_{T,s,\gamma}\nabla_{\mathbb{H}}\psi(x)|\,\mathrm{d}x}_{A'}\\
&\qquad\quad+2c\underbrace{\int_{B_R} \eta|\nabla_{\mathbb{H}}\eta| A(x)^{(p-2)/2}|D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|\,|D_{T,s,\gamma}u(x)|\,\mathrm{d}x}_{B'}\\
&\qquad\quad+2c\underbrace{\int_{B_R} \eta|\nabla_{\mathbb{H}}\eta| A(x)^{(p-2)/2}|D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|\,|D_{T,s,\gamma}\nabla_{\mathbb{H}}\psi(x)|\,\mathrm{d}x}_{C'}.
\end{aligned}$$

Applying the ε-Young inequality to A', B' and C', we obtain with a possible different constant $c > 0$

$$\begin{aligned}
A' + B' + C' \leqslant c&\int_{B_R} \varepsilon\eta^2 A(x)^{(p-2)/2}|D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^2\,\mathrm{d}x\\
&+\frac{c}{\varepsilon}\int_{B_R} \eta^2 A(x)^{(p-2)/2}|D_{T,s,\gamma}\nabla_{\mathbb{H}}\psi(x)|^2\,\mathrm{d}x\\
&+\frac{c}{\varepsilon}\int_{B_R} |\nabla_{\mathbb{H}}\eta|^2 A(x)^{(p-2)/2}|D_{T,s,\gamma}u(x)|^2\,\mathrm{d}x\\
&+\frac{c}{\varepsilon}\int_{B_R} |\nabla_{\mathbb{H}}\eta|^2 A(x)^{(p-2)/2}|D_{T,s,\gamma}\psi(x)|^2\,\mathrm{d}x.
\end{aligned}\tag{3.7}$$

Hence, we have that

$$\int_{B_R} \eta^2 A(x)^{(p-2)/2}|D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^2\,\mathrm{d}x \leqslant A' + B' + C'$$

by the Young inequality and for all sufficiently small $\varepsilon > 0$ we obtain

$$\begin{aligned}\int_{B_R} &\eta^2 A(x)^{(p-2)/2} |D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^2 \,\mathrm{d}x \\ &\leqslant C \int_{B_{2R}} (\mu^2 + |\nabla_{\mathbb{H}} u(x)|^2)^{(p)/2} + |u(x)|^p \,\mathrm{d}x + C \int_{B_R} |\nabla_{\mathbb{H}}\eta|^p |D_{T,s,\gamma}\psi|^p \,\mathrm{d}x \\ &\quad + C \int_{B_R} \eta^p |D_{T,s,\gamma}\nabla_{\mathbb{H}}\psi|^p \,\mathrm{d}x.\end{aligned}$$

Since for every $|s| < 1$ the quantity $D_{T,s,\gamma}\psi$ is monotone increasing with respect to γ and since $\gamma < 1$, we have

$$\begin{aligned}\int_{B_R} &\eta^2 A(x)^{(p-2)/2} |D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^2 \,\mathrm{d}x \\ &\leqslant C \int_{B_{2R}} (\mu^2 + |\nabla_{\mathbb{H}} u(x)|^2)^{p/2} + |u(x)|^p \,\mathrm{d}x + C \int_{B_R} |\nabla_{\mathbb{H}}\eta|^p |D_{T,s,1}\psi|^p \,\mathrm{d}x \\ &\quad + C \int_{B_R} \eta^p |D_{T,s,1}\nabla_{\mathbb{H}}\psi|^p \,\mathrm{d}x.\end{aligned} \tag{3.8}$$

Finally, applying Proposition 3.2 we deduce that for all $s > 0$ sufficiently small

$$\begin{aligned}\int_{B_R} &\eta^2 A(x)^{(p-2)/2} |D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^2 \,\mathrm{d}x \\ &\leqslant C \int_{B_{2R}} (\mu^2 + |\nabla_{\mathbb{H}} u(x)|^2)^{p/2} + |u(x)|^p \,\mathrm{d}x \\ &\quad + \frac{C}{R}(2\|T\psi\|_{L^p(B_R)})^p + C(2\|\nabla_{\mathbb{H}} T\psi\|_{L^p(B_R)})^p.\end{aligned} \tag{3.9}$$

By virtue of

$$|D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^p = |D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^{p-2} \cdot |D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^2$$

and the inequality

$$|s^\gamma D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)| \leqslant \sqrt{2}(\mu^2 + |\nabla_{\mathbb{H}} u(x)|^2 + |\nabla_{\mathbb{H}} u(x \cdot \mathrm{e}^{sT})|)^{1/2},$$

formula (3.8) gives

$$\begin{aligned}\int_{B_R} &\eta^2 s^{(p-2)\gamma} |D_{T,s,\gamma}\nabla_{\mathbb{H}} u(x)|^p \,\mathrm{d}x \\ &\leqslant C \int_{B_{2R}} (\mu^2 + |\nabla_{\mathbb{H}} u(x)|^2)^{p/2} + |u(x)|^p \,\mathrm{d}x + C \int_{B_R} |\nabla_{\mathbb{H}}\eta|^p |D_{T,s,1}\psi|^p \,\mathrm{d}x \\ &\quad + C \int_{B_R} \eta^p |D_{T,s,1}\nabla_{\mathbb{H}}\psi|^p \,\mathrm{d}x.\end{aligned} \tag{3.10}$$

Since

$$D_{T,s,\gamma}\nabla_{\mathbb{H}}(\eta^2 u)(x) = D_{T,s,\gamma}\nabla_{\mathbb{H}}(\eta^2)(x)u(x\cdot \mathrm{e}^{sT}) + \nabla_{\mathbb{H}}(\eta^2)(x)D_{T,s,\gamma}u(x) + D_{T,s,\gamma}\eta^2(x)\nabla_{\mathbb{H}}u(x\cdot \mathrm{e}^{sT}) + \eta^2(x)D_{T,s,\gamma}\nabla_{\mathbb{H}}u(x),$$

we have that

$$\int_{B_R} |D_{T,s,2\gamma/p}\nabla_{\mathbb{H}}u(x)|^p\,\mathrm{d}x \leqslant C\int_{B_{2R}} (\mu^2 + |\nabla_{\mathbb{H}}u(x)|^2)^{p/2} + |u(x)|^p\,\mathrm{d}x + C\int_{B_R} |\nabla_{\mathbb{H}}\eta|^p |D_{T,s,1}\psi|^p\,\mathrm{d}x + C\int_{B_R} \eta^p |D_{T,s,1}\nabla_{\mathbb{H}}\psi|^p\,\mathrm{d}x. \tag{3.11}$$

We denote the right-hand side of (3.11) by M^p. Using Proposition 3.3, we obtain

$$\int_{B_R} \Big|D_{T,-s,1/2}D_{T,s,2\gamma/p}(\eta^2 u)(x)\Big|^p\,\mathrm{d}x \leqslant M^p. \tag{3.12}$$

Therefore, for all s small enough we find that

$$\frac{\|\Delta^2_{T,s}(\eta^2 u)\|_{L^p(\Omega)}}{s^{1/2+(1+2\alpha)/p}} \leqslant M. \tag{3.13}$$

If $(1+2\alpha)/p < 1/2$ then by Theorem 1.1 of [14] we get (3.3). If $(1+2\alpha)/p > 1/2$ then, by Remark 2.2 of [14], we have $Tu \in L^p_{\mathrm{loc}}(\Omega)$ and estimate (3.4) is valid.

If $(1+2\alpha)/p = 1/2$, since $\alpha \in [0,(1/2))$ we get $0 \leqslant (p-2)/4 < 1/2$ which gives $2 \leqslant p \leqslant 4$. By Theorem 1.1 of [14] it follows that we can use α' arbitrarily close to $1/2$, in particular $\alpha' > (p-2)/4$, and the following form of (3.2):

$$\sup_{0\neq|s|\leqslant\sigma}\int_{B_{R/2}} \Big|D_{T,s,1/2+\alpha'}u(x)\Big|^p\,\mathrm{d}x \leqslant c\int_{B_{2R}} \Big(\mu^2 + (|\nabla_{\mathbb{H}}u(x)|^2)^{p/2} + |u(x)|^p\Big)\,\mathrm{d}x.$$

Using a cut-off function η between $B_{R/4}$ and $B_{R/2}$ we get (3.13) with $(1+\alpha')/p > \frac{1}{2}$ and then the previous case. □

Following the proof of Theorem 1.2 in [14], we obtain now this result:

Theorem 3.5. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $2 \leqslant p < 4$, and let $u \in HW^{1,p}_{\mathrm{loc}}(\Omega)$ be a weak solution of the obstacle problem (3.1) with $T\psi \in HW^{1,p}_{\mathrm{loc}}(\Omega)$. Consider $x_0 \in \Omega$, and $R > 0$ such that $B(x_0,3R) \subset \Omega$. Then there exist a number $k \in \mathbb{N}$ depending only on p and a constant $c > 0$ such that*

$$\int_{B(x_0,R/2^{k+1})} |Tu(x)|^p\,\mathrm{d}x \leqslant c\int_{B(x_0,2R)} ((\mu^2 + |\nabla_{\mathbb{H}}u(x)|^2)^{p/2} + |u(x)|^p + |T\psi(x)|^p + |\nabla_{\mathbb{H}}T\psi(x)|^p)\,\mathrm{d}x, \tag{3.14}$$

and hence $Tu \in L^p_{\mathrm{loc}}(\Omega)$.

4. Horizontal Calderón-Zygmund estimates

At the beginning of this section let us recall some preliminary material. The following lemma can be found in [10].

Lemma 4.1. *Let $p \in [\gamma_1, \gamma_2]$ and $\mu \in (0,1]$. There exists a constant $c \equiv c(k, \gamma_1, \gamma_2)$ such that, if $v, w \in \mathbb{R}^k$, then*

$$(\mu^2 + |v|^2)^{p/2} \leqslant c(\mu^2 + |w|^2)^{p/2} + c(\mu^2 + |v|^2 + |w|^2)^{(p-2)/2}|v - w|^2.$$

The following lemma is a direct consequence of a Calderón-Zygmund type covering argument and can be inferred from [19], [21], [32].

Lemma 4.2. *Let $B_{R_0} \in \mathbb{H}^n$ be a ball with radius R_0. Assume that $E, G \subset B_{R_0}$ are measurable sets satisfying the following conditions:*

(i) *there exists $\delta \in (0,1)$ such that $|E| \leqslant \delta |B_{R_0}|$;*
(ii) *for any ball $B(x_0, R)$ centered in B_{R_0}, with radius $R \leqslant 2R_0$ and such that $|E \cap B(x_0, 5R)| > \delta |B_{R_0} \cap B(x_0, R)|$, we have $E \cap B(x_0, 5R) \subset G$.*

Then it follows that $|E| \leqslant \delta |G|$.

Let $B_{R_0} \subset \mathbb{R}^n$ be a ball. We will consider the *Restricted Maximal Function Operator relative to B_{R_0}*, which is defined as

$$M^*_{B_{R_0}}(f)(x) := \sup_{\substack{B \subset B_{R_0} \\ x \in B}} \fint_B |f(x)| \, \mathrm{d}x \tag{4.1}$$

whenever $f \in L^1(B_{R_0})$, where B denotes any ball contained in B_{R_0}, not necessarily with the same center, as long as it contains the point (x, y, t). In the same way, for $s > 1$ we define

$$M^*_{s, B_{R_0}}(f)(x) := \sup_{\substack{B \subset B_{R_0} \\ x \in B}} \left(\fint_B |f(x)|^s \, \mathrm{d}x \right)^{1/s} \tag{4.2}$$

whenever $f \in L^s(B_{R_0})$. We recall the following estimate for $M^*_{1, B_{R_0}} \equiv M^*_{B_{R_0}}$:

$$|\{x \in B_{R_0} \colon |M^*_{B_{R_0}}(f)(x)| \geqslant \lambda\}| \leqslant \frac{c_W}{\lambda^\gamma} \int_{B_{R_0}} |f(x)|^\gamma \, \mathrm{d}x \quad \forall \lambda > 0, \gamma \geqslant 1, \tag{4.3}$$

which is valid for any $f \in L^1(B_{R_0})$; the constant c_W depends only on $\mathcal{Q}$; for this and related issues we refer to [36]. A standard consequence of the previous inequality is then

$$\int_{B_{R_0}} |M^*_{B_{R_0}}(f)(x)|^\gamma \, \mathrm{d}x \leqslant \frac{c(\mathcal{Q},\gamma)}{\gamma-1} \int_{B_{R_0}} |f(x)|^\gamma \, \mathrm{d}x, \quad \gamma > 1. \tag{4.4}$$

A similar estimate for the $M^*_{s,B_{R_0}}$ operator is

$$\int_{B_{R_0}} |M^*_{s,B_{R_0}}(f)(x)|^\gamma \, \mathrm{d}x \leqslant \frac{c(\mathcal{Q},\gamma)}{s(\gamma-s)} \int_{B_{R_0}} |f(x)|^\gamma \, \mathrm{d}x, \quad \gamma > s, \tag{4.5}$$

which can be deduced from (4.4), compare [25], Section 7.

Now let us fix an arbitrarily fixed open subset $\Omega' \Subset \Omega$; for the rest of the section all balls B considered will be such that $B \Subset \Omega'$ unless otherwise specified, and in the sequel all the regularity results we are going to prove are in Ω'. Since the choice of Ω' is arbitrary, the corresponding local regularity of $\nabla_{\mathbb{H}} u$ in Ω will also follow.

In the following we shall concentrate on a ball B_{R_0} such that $B_{2R_0} \subset \Omega'$. The symbol M^* will denote the restricted maximal operator relative to the ball B_{2R_0} in the sense of (4.1): $M^* \equiv M^*_{B_{2R_0}}$; accordingly we shall denote by $M^*_{q/p}$ the restricted maximal operator in the sense of (4.2), again relative to B_{2R_0}, that is, $M^*_{q/p} \equiv M^*_{q/p,B_{2R_0}}$.

We can now prove the following important lemma.

Lemma 4.3. *Let $u \in HW^{1,p}(\Omega)$ be a solution to the $\mathcal{K}_{\psi,w}(\Omega)$-obstacle problem under assumptions (1.3)–(1.8) with $2 \leqslant p < 4$. Then there exist numbers $\varepsilon \equiv \varepsilon(\alpha,\beta,q,n,p) \in (0,1)$ and $A \equiv A(n,p,q,\alpha,\beta) \geqslant 1$ such that the following holds:*

If B is a CC-ball centered in B_{R_0} and with radius less than $2R_0$ satisfying

$$|E \cap 5B| > \delta |B \cap B_{R_0}| \tag{4.6}$$

then

$$5B \cap B_{R_0} \subset G, \tag{4.7}$$

where we set

$$E := \{x \in B_{R_0} \colon M^*(\mu^p + |\nabla_{\mathbb{H}} u|^p|)(x) > A\lambda \text{ and } M^*_{q/p}(|\nabla_{\mathbb{H}}\psi|^p + 1)(x) \leqslant \varepsilon\lambda\},$$

and

$$G := \{x \in B_{R_0} \colon M^*(\mu^p + |\nabla_{\mathbb{H}} u|^p)(x) > \lambda\}.$$

P r o o f. We proceed by contradiction, therefore we assume that (4.7) fails and we thus show that, if we operate a suitable choice of ε and A, also (4.6) fails (but with the dependence on the constants as in the statement of the lemma).

Step 1: beginning

Indeed, assuming that (4.7) fails but (4.6) still holds true, we can infer that there exists $z_1 \in 5B \cap B_{R_0}$ such that $M^*(\mu^p + |\nabla_{\mathbb{H}} u|^p)(z_1) \leqslant \lambda$. On the other hand, $E \cap 5B$ is nonempty and therefore there exists $z_2 \in 5B \cap B_{R_0}$ such that $M^*_{q/p}(|\nabla_{\mathbb{H}} \psi|^p)(z_2)\,\mathrm{d}x \leqslant (\varepsilon\lambda)$. This means that we have

$$\fint_{40B} (\mu^p + |\nabla_{\mathbb{H}} u|^p)\,\mathrm{d}x \leqslant \lambda \quad \text{and} \quad \fint_{40B} (|\nabla_{\mathbb{H}} \psi|^q + 1)\,\mathrm{d}x \leqslant (\varepsilon\lambda)^{q/p}. \tag{4.8}$$

Step 2: comparison to some reference problems

We define $v \in u + HW^{1,p}_0(20B)$ as the solution to the obstacle problem

$$\int_B \mathcal{A}(x_0, \nabla_{\mathbb{H}} v)(\nabla_{\mathbb{H}} v - \nabla_{\mathbb{H}} \varphi)\,\mathrm{d}x \leqslant 0 \tag{4.9}$$

for all $\varphi \in K_{\psi,f}(\Omega)$, where x_0 is the center of $B_R \equiv 20B$.

Now we introduce $w \in u + HW^{1,p}(B_R)$ as the unique solution to the Dirichlet problem

$$\begin{cases} \mathrm{div}_{\mathbb{H}} \mathcal{A}(x_0, \nabla_{\mathbb{H}} w) = \mathrm{div}_{\mathbb{H}} \mathcal{A}(x_0, \nabla_{\mathbb{H}} \varphi) & \text{in } B_R, \\ w = u & \text{on } \partial B_R. \end{cases} \tag{4.10}$$

Let us notice that by the maximum principle (see for instance Theorem 2.5 in [11]) we have $w \geqslant \psi$ on B, since $w \geqslant \psi$ on ∂B.

Finally, let $z \in u + HW^{1,p}_0(B_R)$ be the unique solution to the Dirichlet problem

$$\begin{cases} \mathrm{div}_{\mathbb{H}} \mathcal{A}(x_0, \nabla_{\mathbb{H}} z) = 0 & \text{in } B_R, \\ z = u & \text{on } \partial B_R. \end{cases} \tag{4.11}$$

By the recent results for degenerate elliptic equations in the Heisenberg group, for z the following estimate holds true (for more details we refer to [32]):

$$\sup_{B_{R/2}} |\nabla_{\mathbb{H}} z| \leqslant c\biggl(\fint_{B_R} |\nabla_{\mathbb{H}} z|^p\,\mathrm{d}x\biggr)^{1/p}, \tag{4.12}$$

where c is a constant depending only on n, p, α, β.

Step 3: comparison estimates—part I

We now establish the comparison estimates. First of all, we test (4.11) using $z-u$ as an admissible test function. We have

$$\begin{aligned}\alpha\int_{B_R}|\nabla_{\mathbb{H}}z|^p\,\mathrm{d}x &\overset{(1.5)}{\leqslant}\int_{B_R}\langle\mathcal{A}(x_0,\nabla_{\mathbb{H}}z),\nabla_{\mathbb{H}}z\rangle\,\mathrm{d}x\\ &\overset{(4.11)}{=}\int_{B_R}\langle\mathcal{A}(x_0,\nabla_{\mathbb{H}}z),\nabla_{\mathbb{H}}u\rangle\,\mathrm{d}x\\ &\overset{(1.6)}{\leqslant}\beta\int_{B_R}(|\nabla_{\mathbb{H}}z|^{p-1}+1)|\nabla_{\mathbb{H}}u|\,\mathrm{d}x.\end{aligned}$$

By averaging and applying Young's inequality, we have that

$$\fint_{B_R}|\nabla_{\mathbb{H}}z|^p\,\mathrm{d}x\leqslant c\fint_{B_R}|\nabla_{\mathbb{H}}u|^p+1\,\mathrm{d}x, \tag{4.13}$$

with a constant c only dependent on n, p, α, β.

On the other hand, (4.8), (4.12) together with (4.13) yield

$$\sup_{B_{R/2}}(\mu^2+|\nabla_{\mathbb{H}}z|^2)^{p/2}\leqslant c\lambda^{1/p}, \tag{4.14}$$

where the constant c only depends on n,p,α,β.

On the other hand, if we test (4.10) by the admissible function $w-u$, again using the structure conditions for the operator a and Young's inequality, we immediately deduce

$$\begin{aligned}\alpha\int_{B_R}|\nabla_{\mathbb{H}}w|^p\,\mathrm{d}x &\overset{(1.5)}{\leqslant}\int_{B_R}\langle\mathcal{A}(x_0,\nabla_{\mathbb{H}}w),\nabla_{\mathbb{H}}w\rangle\,\mathrm{d}x\\ &\overset{(4.10)}{=}\int_{B_R}\langle\mathcal{A}(x_0,\nabla_{\mathbb{H}}w),\nabla_{\mathbb{H}}u\rangle\,\mathrm{d}x+\int_{B_R}\langle\mathcal{A}(x_0,\nabla_{\mathbb{H}}\psi),\nabla_{\mathbb{H}}w-\nabla_{\mathbb{H}}u\rangle\,\mathrm{d}x\\ &\overset{(1.6)}{\leqslant}\frac{\alpha}{2}\int_{B_R}|\nabla_{\mathbb{H}}w|^p\,\mathrm{d}x+c\int_{B_R}|\nabla_{\mathbb{H}}u|^p\,\mathrm{d}x+c\int_{B_R}|\nabla_{\mathbb{H}}\psi|^p+1\,\mathrm{d}x,\end{aligned}$$

which gives after averaging

$$\fint_{B_R}|\nabla_{\mathbb{H}}w|^p\,\mathrm{d}x\leqslant c\fint_{B_R}|\nabla_{\mathbb{H}}u|^p\,\mathrm{d}x+c\fint_{B_R}|\nabla_{\mathbb{H}}\psi|^p+1\,\mathrm{d}x, \tag{4.15}$$

with a constant c which depends only on n,p,α,β.

Finally we deduce the last comparison estimate for this part, which concerns v and u; we exploit (4.9) in the following way:

$$\begin{aligned}
\alpha \int_{B_R} |\nabla_{\mathbb{H}} v|^p \,\mathrm{d}x &\overset{(1.5)}{\leqslant} \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} v), \nabla_{\mathbb{H}} v\rangle \,\mathrm{d}x \\
&= \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} v), \nabla_{\mathbb{H}} v - \nabla_{\mathbb{H}} u\rangle \,\mathrm{d}x + \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} v), \nabla_{\mathbb{H}} u\rangle \,\mathrm{d}x \\
&\overset{(1.6),(4.9)}{\leqslant} \frac{\alpha}{2} \int_{B_R} |\nabla_{\mathbb{H}} v|^p \,\mathrm{d}x + c\int_{B_R} |\nabla_{\mathbb{H}} u|^p + 1 \,\mathrm{d}x,
\end{aligned}$$

which gives, once more after averaging,

$$\fint_{B_R} |\nabla_{\mathbb{H}} v|^p \,\mathrm{d}x \leqslant c \fint_{B_R} |\nabla_{\mathbb{H}} u|^p + 1 \,\mathrm{d}x \tag{4.16}$$

with a constant c only dependent on n, p, α, β.

Step 4: comparison estimates—part II

We now establish the following three comparison estimates:

$$I := \int_{B_R} (\mu^2 + |\nabla_{\mathbb{H}} w|^2 + |\nabla_{\mathbb{H}} z|^2)^{(p-2)/2} |\nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} z|^2 \,\mathrm{d}x \leqslant c\varepsilon^{(p-1)/p} R^n \lambda, \tag{4.17}$$

$$II := \int_{B_R} (\mu^2 + |\nabla_{\mathbb{H}} v|^2 + |\nabla_{\mathbb{H}} w|^2)^{(p-2)/2} |\nabla_{\mathbb{H}} v - \nabla_{\mathbb{H}} w|^2 \,\mathrm{d}x \leqslant c\varepsilon^{(p-1)/p} R^n \lambda, \tag{4.18}$$

$$III := \int_{B_R} (\mu^2 + |\nabla_{\mathbb{H}} u|^2 + |\nabla_{\mathbb{H}} v|^2)^{(p-2)/2} |\nabla_{\mathbb{H}} u - \nabla_{\mathbb{H}} v|^2 \,\mathrm{d}x \leqslant c\varepsilon^{(p-1)/p} R^n \lambda \tag{4.19}$$

with constants $c \equiv c(n, p, \alpha, \beta)$. First of all, exploiting the structure conditions on the field $\mathcal{A}$—notice that $p \geqslant 2$—the comparison problems (4.10) and (4.11) and Hölder's inequality, we have

$$\begin{aligned}
c^*(\alpha) I &\overset{(1.7)}{\leqslant} \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} w) - \mathcal{A}(x_0, \nabla_{\mathbb{H}} z), \nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} z\rangle \,\mathrm{d}x \\
&\overset{(4.11)}{=} \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} w), \nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} z\rangle \,\mathrm{d}x \\
&\overset{(4.10)}{=} \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} \psi), \nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} z\rangle \,\mathrm{d}x \\
&\overset{(1.6)}{\leqslant} \beta \int_{B_R} (|\nabla_{\mathbb{H}} \psi|^{p-1} + 1) |\nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} z| \,\mathrm{d}x \\
&\leqslant cR^n \left(\fint_{B_R} (|\nabla_{\mathbb{H}} \psi|^{p-1} + 1)^{p/(p-1)} \,\mathrm{d}x \right)^{(p-1)/p} \left(\fint_{B_R} |\nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} z|^p \right)^{1/p} \\
&\leqslant cR^n \left(\fint_{B_R} |\nabla_{\mathbb{H}} \psi|^p + 1 \,\mathrm{d}x \right)^{(p-1)/p} \left(\fint_{B_R} |\nabla_{\mathbb{H}} w|^p + |\nabla_{\mathbb{H}} z|^p \,\mathrm{d}x \right)^{1/p}.
\end{aligned}$$

Using the comparison estimates established at Step 3, namely (4.15) and (4.13), we can immediately estimate the second integral as

$$\fint_{B_R} (|\nabla_{\mathbb{H}} w|^p + |\nabla_{\mathbb{H}} z|^p)\,\mathrm{d}x \leqslant c \fint_{B_R} |\nabla_{\mathbb{H}} u|^p\,\mathrm{d}x + c \fint_{B_R} |\nabla_{\mathbb{H}}\psi|^p\,\mathrm{d}x + 1.$$

Putting together the last two estimates, we obtain by means of Hölder's inequality

$$\begin{aligned}
c^*(\alpha) I &\leqslant cR^n \Big(\fint_{B_R} |\nabla_{\mathbb{H}}\psi|^p + 1\,\mathrm{d}x \Big)^{(p-1)/p} \Big(\fint_{B_R} |\nabla_{\mathbb{H}} u|^p + \fint_{B_R} |\nabla_{\mathbb{H}}\psi|^p + 1\,\mathrm{d}x \Big)^{1/p} \\
&= cR^n \Big(\fint_{B_R} |\nabla_{\mathbb{H}}\psi|^p + 1\,\mathrm{d}x \Big) \\
&\quad + cR^n \Big(\fint_{B_R} |\nabla_{\mathbb{H}}\psi|^p + 1\,\mathrm{d}x \Big)^{(p-1)/p} \Big(\fint_{B_R} |\nabla_{\mathbb{H}} u|^p + 1\,\mathrm{d}x \Big)^{1/p} \\
&\overset{(4.8)}{\leqslant} cR^n \Big(\fint_{B_R} |\nabla_{\mathbb{H}}\psi|^q + 1\,\mathrm{d}x \Big)^{p/q} \\
&\quad + cR^n \Big(\fint_{B_R} |\nabla_{\mathbb{H}}\psi|^q + 1\,\mathrm{d}x \Big)^{(p-1)/q} \Big(\fint_{B_R} |\nabla_{\mathbb{H}} u|^p + 1\,\mathrm{d}x \Big)^{1/p} \\
&\leqslant cR^n(\varepsilon\lambda) + cR^n(\varepsilon\lambda)^{(p-1)/p}\lambda^{1/p} = cR^n(\varepsilon\lambda) + cR^n\varepsilon^{(p-1)/p}\lambda \\
&\leqslant c(n,p,q,\alpha,\beta)\varepsilon^{(p-1)/p} R^n \lambda,
\end{aligned}$$

where $q > 1$ appears in the assumption on the horizontal gradient of the obstacle function.

Concerning the second comparison estimate, we again exploit the structure conditions for the operator $\mathcal{A}$ but this time we also use the obstacle problem (4.9) together with the comparison estimates established in Step 3, namely (4.15) and (4.16). We thus deduce

$$\begin{aligned}
c^*(\alpha) II &\overset{(1.7)}{\leqslant} \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} v) - \mathcal{A}(x_0, \nabla_{\mathbb{H}} w), \nabla_{\mathbb{H}} v - \nabla_{\mathbb{H}} w \rangle\,\mathrm{d}x \\
&\overset{(4.9)}{\leqslant} \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} w), \nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} v \rangle\,\mathrm{d}x \\
&\overset{(4.10)}{=} \int_{B_R} \langle \mathcal{A}(x_0, \nabla_{\mathbb{H}} \psi), \nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} v \rangle\,\mathrm{d}x \\
&\overset{(1.6)}{\leqslant} \beta \int_{B_R} (|\nabla_{\mathbb{H}}\psi|^{p-1} + 1)|\nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} v|\,\mathrm{d}x \\
&\leqslant cR^n \Big(\fint_{B_R} (|\nabla_{\mathbb{H}}\psi|^{p-1} + 1)^{p/(p-1)}\,\mathrm{d}x \Big)^{(p-1)/p} \Big(\fint_{B_R} |\nabla_{\mathbb{H}} w - \nabla_{\mathbb{H}} v|^p \Big)^{1/p} \\
&\leqslant cR^n \Big(\fint_{B_R} |\nabla_{\mathbb{H}}\psi|^p + 1\,\mathrm{d}x \Big)^{(p-1)/p} \Big(\fint_{B_R} |\nabla_{\mathbb{H}} w|^p + |\nabla_{\mathbb{H}} v|^p\,\mathrm{d}x \Big)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\overset{(4.15),(4.16)}{\leqslant} cR^n\left(\fint_{B_R}|\nabla_{\mathbb{H}}\psi|^p+1\,\mathrm{d}x\right)^{(p-1)/p}\left(\fint_{B_R}|\nabla_{\mathbb{H}}u|^p+\fint_{B}|\nabla_{\mathbb{H}}\psi|^p+1\,\mathrm{d}x\right)^{1/p}\\
&\leqslant\ cR^n\left(\fint_{B_R}|\nabla_{\mathbb{H}}\psi|^p+1\,\mathrm{d}x\right)\\
&\quad+cR^n\left(\fint_{B_R}|\nabla_{\mathbb{H}}\psi|^p+1\,\mathrm{d}x\right)^{(p-1)/p}\left(\fint_{B_R}|\nabla_{\mathbb{H}}u|^p+1\,\mathrm{d}x\right)^{1/p}\\
&\leqslant\ c(n,p,\alpha,\beta)\varepsilon^{(p-1)/p}R^n\lambda,
\end{aligned}
$$

where the conclusion came working exactly as in the previous estimate of I.

We finally conclude with the estimate of III; we have

$$
\begin{aligned}
c^*(\alpha)III\ &\leqslant\ \int_{B_R}\langle\mathcal{A}(x_0,\nabla_{\mathbb{H}}u)-\mathcal{A}(x_0,\nabla_{\mathbb{H}}v),\nabla_{\mathbb{H}}u-\nabla_{\mathbb{H}}v\rangle\,\mathrm{d}x\\
&\overset{(1.5)}{\leqslant}\beta\int_{B_R}2(1+|\nabla_{\mathbb{H}}u|^{p-1})|\nabla_{\mathbb{H}}u-\nabla_{\mathbb{H}}v|\,\mathrm{d}x\\
&\leqslant\ cR^n\left(\fint_{B_R}(|\nabla_{\mathbb{H}}u|^{p-1}+1)^{p/(p-1)}\,\mathrm{d}x\right)^{(p-1)/p}\left(\fint_{B_R}|\nabla_{\mathbb{H}}u-\nabla_{\mathbb{H}}v|^p\right)^{1/p}\\
&\leqslant\ cR^n\left(\fint_{B_R}|\nabla_{\mathbb{H}}u|^p+1\,\mathrm{d}x\right)^{(p-1)/p}\left(\fint_{B_R}|\nabla_{\mathbb{H}}u|^p+|\nabla_{\mathbb{H}}v|^p\right)^{1/p}\\
&\overset{(4.8),(4.16)}{\leqslant}c(n,p,\alpha,\beta)R^n\varepsilon^{(p-1)/p}\lambda.
\end{aligned}
$$

Using repeatedly Lemma 4.1, we deduce

$$
(\mu^2+|\nabla_{\mathbb{H}}u|^2)^{p/2}\leqslant\tilde{c}[(\mu^2+|\nabla_{\mathbb{H}}z|^2)^{p/2}+I+II+III]
$$

where I,II,III have been introduced in (4.17)–(4.19), for a suitable $\tilde{c}\equiv\tilde{c}(n,p,q,\alpha,\beta)$.

Let us consider the restricted maximal operator to the ball $10B$, denoted by M^{**} we have $M^{**}\equiv M^*_{10B}$. By the previous estimates we obtain immediately

$$
\begin{aligned}
&|\{x\in B_{R_0}\colon\ M^{**}(\mu^p+|\nabla_{\mathbb{H}}u|^p)(x)>A\lambda,\ M^*_{q/p}(\mu^2+|\nabla_{\mathbb{H}}\psi|^p)(x)\leqslant\varepsilon\lambda\}|\\
&\quad\leqslant\left|\left\{x\in B_{R_0}\colon\ M^{**}(\tilde{c}(\mu^2+|\nabla_{\mathbb{H}}z|^2)^{p/2})>\frac{A\lambda}{4\tilde{c}}\right\}\right|\\
&\qquad+\left|\left\{x\in B_{R_0}\colon\ M^{**}(\tilde{c}I)>\frac{A\lambda}{4\tilde{c}}\right\}\right|+\left|\left\{x\in B_{R_0}\colon\ M^{**}(\tilde{c}II)>\frac{A\lambda}{4\tilde{c}}\right\}\right|\\
&\qquad+\left|\left\{x\in B_{R_0}\colon\ M^{**}(\tilde{c}III)>\frac{A\lambda}{4\tilde{c}}\right\}\right|=:IV+V+VI+VII.
\end{aligned}
$$

Estimate for IV: by (4.14) we obtain $IV\leqslant c\lambda$ and therefore $IV=0$.

Estimate for V, VI, VII: we use estimate (4.3) for the maximal function and the estimates (4.17), (4.18) and (4.19) to conclude that there exists a constant $\bar{c} = \bar{c}(n,p,q,\alpha,\beta)$ such that

$$V \leqslant \frac{\tilde{c}}{c\lambda}\varepsilon^{(p-1)/p}R^n\lambda \leqslant \bar{c}\varepsilon^{(p-1)/p}|B_{R_0}|,$$
$$VI \leqslant \bar{c}\varepsilon^{(p-1)/p}|B_{R_0}|, \quad VII \leqslant \bar{c}\varepsilon^{(p-1)/p}|B_{R_0}|.$$

Taking ε and A small enough to have

$$|\{x \in B_{R_0}\colon\ M^{**}(\mu^p + |\nabla_{\mathbb{H}}u|^p)(x) > A\lambda\}| < \delta|B_{R_0} \cap B|,$$

following the argument of the proof of Lemma 10.3 of [32], by (2.4) we obtain

$$|\{x \in B_{R_0}\colon\ M^{*}(\mu^p + |\nabla_{\mathbb{H}}u|^p)(x) > A\lambda\}| < \delta|B_{R_0} \cap B|,$$

which contradicts (4.6). □

We are now able to give the proof of Theorem 1.2.

Proof. The proof of the theorem can be handled in a quite standard way, following [32]. We will sketch the main steps for the reader's convenience. We will start by choosing an exponent s such that $s > q$; this implies of course that from now on, all the constants depending on s will actually depend on q. We choose A with the aim of using Lemma 4.3. In this manner we determine the choice of the number ε, depending on the same quantities, once more in view of the application of Lemma 4.3. Now let us set

$$\begin{aligned} \mu_1(t) &:= |\{x \in B_{R_0}\colon\ M^*(\mu^p + |\nabla_{\mathbb{H}}u|^p)(x) > t\}|, \\ \mu_2(t) &:= |\{x \in B_{R_0}\colon\ M^*_{q/p}(|F|^p)(x) > t\}| \end{aligned} \tag{4.20}$$

and keep in mind that the maximal operators $M^*_{q/p}$ are restricted to the ball B_{2R_0}. The proof will proceed by iterating the function $\mu_1(\cdot)$ using information on $\mu_2(\cdot)$, that is getting information on the measure of the level sets of $|\nabla_{\mathbb{H}}u|$, in terms of those of $|\nabla_{\mathbb{H}}\psi|$. We choose the starting level λ_0 as

$$\lambda_0 := C \fint_{B_{2R_0}} (\mu^p + |\nabla_{\mathbb{H}}u|^p)\,\mathrm{d}x,$$

where C is a suitable constant depending on the doubling constant C_d and on c_w; the role of this constant in the sequel does not require any further detail. Therefore, using (4.4) and the fact that $A > 1$, we find for any $m \in \mathbb{N}$

$$\mu_1(A^m\lambda_0) \leqslant \mu_1(\lambda_0) \leqslant \frac{\delta}{2}|B_{R_0}|. \tag{4.21}$$

Now we want to use Lemma 4.2; more precisely, for every $m = 0, 1, 2, \dots$ we would like to apply it with the choices

$$\delta = \frac{1}{2A^{q/p}}$$

and

$$(4.22)\quad E := \{z \in B_{R_0} :\ M^*(\mu^p + |\nabla_{\mathbb{H}}|^p) > A^{m+1}\lambda_0,\ \text{and}\ M^*_{q/p}(|\nabla_{\mathbb{H}}\psi|^p) < \varepsilon A^m \lambda_0\},$$

$$(4.23)\quad G := \{z \in B_{R_0} :\ M^*(\mu^p + |\nabla_{\mathbb{H}}u|^p) > A^m \lambda_0\}.$$

Thus we first check if the assumptions for Lemma 4.2 hold. First of all, we can immediately see that $|E| \leqslant \mu_1(A^{m+1}\lambda_0)$, therefore, combining this information with (4.21), we readily have

$$|E| \leqslant \frac{\delta}{2}|B_{R_0}|,$$

which is the first assumption needed in the application of the lemma. The second assumption is exactly given by Lemma 4.3, which is applied with $\lambda \equiv A^m\lambda_0$; therefore, recalling that $|G| = \mu_1(A^m\lambda_0)$ and that $|E| \geqslant \mu_1(A^{m+1}\lambda_0) - \mu_2(A^m\varepsilon\lambda_0)$, the thesis of Lemma 4.2 gives

$$(4.24)\qquad \mu_1(A^{m+1}\lambda_0) \leqslant \frac{1}{2A^{q/p}}\mu_1(A^m\lambda_0) + \mu_2(A^m\varepsilon\lambda_0)$$

for any $m = 0, 1, 2, \dots$ Induction on the previous inequality easily gives

$$\mu_1(A^{m+1}\lambda_0) \leqslant \Big(\frac{1}{2A^{q/p}}\Big)^{m+1}\mu_1(\lambda_0) + \sum_{i=0}^{m}\Big(\frac{1}{2A}\Big)^{m-i}\mu_2(A^i\varepsilon\lambda_0).$$

Therefore, if we multiply the previous equation by $A^{q(m+1)/p}$ and sum over m from 0 to $M \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{m=0}^{M} A^{q(m+1)/p}\mu_1(A^{m+1}\lambda_0) &\leqslant \sum_{m=0}^{M}\frac{1}{2^{m+1}}\mu_1(\lambda_0) + \sum_{m=0}^{M}\sum_{i=0}^{m}A^{q(i+1)/p}\Big(\frac{1}{2}\Big)^{m-i}\mu_2(A^i\varepsilon\lambda_0)\\ &\leqslant \mu_1(\lambda_0) + \sum_{m=0}^{M}\sum_{i=0}^{m}A^{q(i+1)/p}\Big(\frac{1}{2}\Big)^{m-i}\mu_2(A^i\varepsilon\lambda_0).\end{aligned}$$

Interchanging the order of summation in the second term of the last inequality and exploiting the geometric series, we actually deduce after passing to the limit as $M \to \infty$

$$(4.25)\qquad \sum_{m=0}^{\infty} A^{q(m+1)/p}\mu_1(A^{m+1}\lambda_0) \leqslant \mu_1(\lambda_0) + 2A^{q/p}\sum_{m=0}^{\infty}2A^{qm/p}\mu_2(A^m\varepsilon\lambda_0).$$

Now we would like to turn the previous estimate into an estimate for the maximal function. This can be done in a standard way by applying the elementary inequality

$$\int_{B_{R_0}} g^q \,\mathrm{d}x = \int_0^\infty q\lambda^{q-1}(x \in B_{R_0} \colon\ g(x) > \lambda) \,\mathrm{d}\lambda,$$

which holds for $g \in L^q(B_{R_0})$, $g \geqslant 0$, $q \geqslant 1$, to the function $g \equiv M^*(\mu^p + |\nabla_{\mathbb{H}} u|^p)$; we just need to decompose the interval $[0, \infty)$ into intervals $[0, \lambda_0]$ and $[A^n\lambda_0, A^{n+1}\lambda_0]$ and exploit (4.25) together with the monotonicity of the functions μ_1, μ_2 and the L^p estimate for the maximal function. At the end, we come up with

$$\begin{aligned}
\int_{B_{R_0}} (\mu + |\nabla_{\mathbb{H}} u|)^q \,\mathrm{d}x &\leqslant c \int_{B_{R_0}} M^*(\mu^p + |\nabla_{\mathbb{H}} u|^p)^{q/p} \,\mathrm{d}x = c \int_0^\infty \lambda^{q/p-1} \mu_1(\lambda) \,\mathrm{d}\lambda \\
&= c \int_0^{\lambda_0} \lambda^{q/p-1} \mu_1(\lambda) \,\mathrm{d}\lambda + c \sum_{m=0}^{\infty} \int_{A^m\lambda_0}^{A^{m+1}\lambda_0} \lambda^{q/p-1} \mu_1(\lambda) \,\mathrm{d}\lambda \\
&\leqslant \lambda_0^{q/p} |B_{R_0}| + c\lambda_0^{q/p} \sum_{m=1}^{\infty} A^{qm/p} \mu_2(A^m \varepsilon \lambda_0) \\
&\leqslant c\biggl(\fint_{B_{2R_0}} (\mu^p + |\nabla_{\mathbb{H}} u|^p) \,\mathrm{d}x \biggr)^{q/p} |B_{R_0}| + \frac{A}{\varepsilon^{q/p}(A-1)} \int_0^\infty \lambda^{q/p-1} \mu_2(\lambda) \,\mathrm{d}\lambda \\
&\leqslant c\biggl(\fint_{B_{2R_0}} (\mu^p + |\nabla_{\mathbb{H}} u|^p) \,\mathrm{d}x \biggr)^{q/p} |B_{R_0}| + c \int_{B_{R_0}} M^*_{q/p}(1 + |\nabla_{\mathbb{H}} \psi|^p)^{q/p} \,\mathrm{d}x \\
&\leqslant c\biggl(\fint_{B_{2R_0}} (\mu^p + |\nabla_{\mathbb{H}} u|^p) \,\mathrm{d}x \biggr)^{q/p} |B_{R_0}| + c \int_{B_{2R_0}} |\nabla_{\mathbb{H}} \psi|^q \,\mathrm{d}x,
\end{aligned}$$

where the constants in the last line include the dependence on ε and A, and therefore, due to our choices, these constants finally depend on n, p, q, α, β. Therefore, after elementary manipulations, we finally come to the estimate

$$\begin{aligned}
&\biggl(\fint_{B_{R_0}} (\mu + |\nabla_{\mathbb{H}} u|)^q \,\mathrm{d}x \biggr)^{1/q} \\
&\qquad \leqslant c\biggl(\fint_{B_{2R_0}} (\mu^p + |\nabla_{\mathbb{H}} u|^p) \,\mathrm{d}x \biggr)^{q/p} + c\biggl(\fint_{B_{2R_0}} |\nabla_{\mathbb{H}} \psi|^q \,\mathrm{d}x \biggr)^{1/q},
\end{aligned}$$

which holds for any small radius R_0 fulfilling the condition $B_{2R_0} \Subset \Omega$. The conclusion comes due to a standard covering argument, in the spirit of [32]. □

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