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SOME MEAN CONVERGENCE AND COMPLETE CONVERGENCE THEOREMS FOR SEQUENCES OF m-LINEARLY NEGATIVE QUADRANT DEPENDENT RANDOM VARIABLES

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Abstract. The structure of linearly negative quadrant dependent random variables is extended by introducing the structure of m-linearly negative quadrant dependent random variables ($m = 1, 2, \ldots$). For a sequence of m-linearly negative quadrant dependent random variables $\{X_n, n \geqslant 1\}$ and $1 < p < 2$ (resp. $1 \leqslant p < 2$), conditions are provided under which $n^{-1/p} \sum\limits_{k=1}^{n} (X_k - EX_k) \to 0$ in L^1 (resp. in L^p). Moreover, for $1 \leqslant p < 2$, conditions are provided under which $n^{-1/p} \sum\limits_{k=1}^{n} (X_k - EX_k)$ converges completely to 0. The current work extends some results of Pyke and Root (1968) and it extends and improves some results of Wu, Wang, and Wu (2006). An open problem is posed.

Keywords: m-linearly negative quadrant dependence, mean convergence, complete convergence

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1. Introduction

The concept of negative quadrant dependent (NQD, for short) random variables was introduced by Lehmann in [7].

Definition 1.1. Two random variables X and Y are said to be NQD if for all $x, y \in \mathbb{R}$,

$$P(X \leqslant x, Y \leqslant y) \leqslant P(X \leqslant x)P(Y \leqslant y).$$

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A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be pairwise NQD if every pair of random variables in the sequence is NQD.

The concept of linearly negative quadrant dependent (LNQD, for short) random variables was introduced by Newman in [8].

Definition 1.2. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be LNQD if for all finite disjoint subsets $A, B \subset \mathbb{N}$ and $r_j, r_k > 0$, $\sum_{j\in A} r_j X_j$ and $\sum_{k\in B} r_k X_k$ are NQD.

Remark 1.1. It is easily seen that if $\{X_n, n \geqslant 1\}$ is a sequence of LNQD random variables, then $\{aX_n + b, n \geqslant 1\}$ is also a sequence of LNQD random variables for all choices of a and b.

The concept of LNQD random variables is much weaker (see [8]) than the concepts of independent random variables and negatively associated (NA, for short, cf. [4]) random variables. The convergence properties of LNQD sequences have been studied; we refer to [8] for the central limit theorem (CLT, for short), [13] for uniform rates of convergence in the CLT, [5] for the Hoeffding-type inequality, [6] for strong convergence, and [14] for exponential inequalities, complete convergence, and almost sure convergence.

Now we introduce the concept of m-linearly negative quadrant dependent (m-LNQD, for short) random variables.

Definition 1.3. Let $m \geqslant 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be m-LNQD if for all $n \geqslant 2$ and all choices of $i_1, \ldots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, \ldots, X_{i_n}$ are LNQD.

It is easily seen that this concept is a natural extension of the concept of LNQD random variables (wherein $m = 1$). Indeed, if $\{X_n, n \geqslant 1\}$ is m-LNQD for some $m \geqslant 1$, then $\{X_n, n \geqslant 1\}$ is m'-LNQD for all $m' > m$.

We now provide three examples of sequences of m-LNQD random variables. The first two examples are straightforward.

Example 1.1. Let $\{X_n, n \geqslant 1\}$ be a LNQD sequence of random variables and let $m \geqslant 2$. For $n \geqslant 1$, let $r \geqslant 1$ be such that $(r-1)m+1 \leqslant n \leqslant rm$ and let $Z_n = X_r$. Then $\{Z_n, n \geqslant 1\}$ is a sequence of m-LNQD random variables.

Example 1.2. Let $\{X_n, n \geqslant 1\}$ be a LNQD sequence of random variables and let $\{Y_{ij}, i \geqslant 1, 1 \leqslant j \leqslant m-1\}$ be an array of independent random variables which is independent of $\{X_n, n \geqslant 1\}$. Let $m \geqslant 2$. For $n \geqslant 1$, let $r \geqslant 1$ be such that

$(r-1)m+1 \leqslant n \leqslant rm$ and let

$$Z_n = \begin{cases} X_r & \text{if } n = (r-1)m+1, \\ Y_{r,n-(r-1)m-1} & \text{if } (r-1)m+2 \leqslant n \leqslant rm. \end{cases}$$

Then $\{Z_n, n \geqslant 1\}$ is a sequence of m-LNQD random variables.

The third example is more interesting as it is concerned with moving averages; these processes are important in time series analysis and for econometric applications. The moving average process smooths the data under consideration and provides a powerful tool for trend detection.

Example 1.3. Let $\{X_n, n \geqslant 1\}$ be a LNQD sequence of random variables and let $m \geqslant 2$. Then the sequence of moving averages $\{Z_n, n \geqslant 1\}$ defined by

$$Z_n = \frac{1}{m} \sum_{k=n}^{m+n-1} X_k, \quad n \geqslant 1,$$

is a sequence of m-LNQD random variables.

A sequence of random variables $\{U_n, n \geqslant 1\}$ is said to converge completely to a constant a if

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

This concept of complete convergence was introduced by Hsu and Robbins in [3].

Definition 1.4 ([1]). A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be uniformly integrable in the Cesàro sense if

$$\lim_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} E|X_k|^p I(|X_k|^p \geqslant x) = 0. \tag{1.1}$$

Chandra [1] showed that (1.1) implies

$$\sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} E|X_k|^p < \infty \tag{1.2}$$

and

$$\lim_{x\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} xP(|X_k|^p \geqslant x) = 0. \tag{1.3}$$

In [9], the notion of h-uniform integrability with respect to an array of constants $\{a_{nk}\}$ was introduced and it was shown that this notion is weaker than (1.1). In the current work, we will need a particular case of the notion of h-uniform integrability with respect to the array of constants $\{a_{nk} = 1/n, 1 \leqslant k \leqslant n, n \geqslant 1\}$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E|X_k|^p I(|X_k|^p \geqslant h(n)) = 0, \tag{1.4}$$

where $\{h(n), n \geqslant 1\}$ is a sequence of positive constants satisfying $h(n) \uparrow \infty$ and $h(n)/n \to 0$ as $n \uparrow \infty$; see also [11].

Remark 1.2.

(i) We refer to Remark 1 of [9], where it is shown that (1.1) implies (1.4).
(ii) Obviously, (1.4) implies

$$\lim_{n\to\infty} \sup_{y\geqslant h(n)} \frac{1}{n} \sum_{k=1}^{n} yP(|X_k|^p \geqslant y) = 0. \tag{1.5}$$

In this paper, we will establish an exponential inequality of Kolmogorov's type for m-LNQD random variables, and we will study mean convergence and complete convergence for sequences of m-LNQD random variables.

Throughout, C is a generic positive constant, whose value may vary from one place to another, and $I(A)$ is the indicator function of the event A.

2. Main results

In this section we give the statements of the main results; they will be proved in Section 4. Theorems 2.1 and 2.2 are mean convergence results and Theorem 2.3 is a complete convergence result.

Theorem 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-LNQD random variables. Then for $1 < p < 2$, conditions (1.2) and (1.5) imply*

$$n^{-1/p} \sum_{k=1}^{n} (X_k - EX_k) \to 0 \tag{2.1}$$

in L^1 and, hence, in probability as $n \to \infty$.

Theorem 2.2. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-LNQD random variables. Then for $1 \leqslant p < 2$, conditions (1.2) and (1.4) imply*

$$n^{-1/p} \sum_{k=1}^{n} (X_k - EX_k) \to 0 \tag{2.2}$$

in L^p and, hence, in probability as $n \to \infty$.

Remark 2.1. Wu, Wang, and Wu [15] obtained p-mean convergence for a sequence of NA random variables under the condition (1.1). Since (1.1) implies (1.2) and (1.4), and NA implies m-LNQD, Theorem 2.2 extends and improves the result of [15].

Corollary 2.1. *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed m-LNQD random variables with $E|X_1|^p < \infty$ for some $1 \leqslant p < 2$. Then (2.2) holds.*

Remark 2.2. Pyke and Root [10] obtained p-mean convergence for the partial sums from a sequence of independent and identically distributed random variables with $E|X_1|^p < \infty$ for some $1 \leqslant p < 2$. Therefore, Theorem 2.2 extends the result of [10].

Remark 2.3. Wan [12] obtained p-mean convergence and convergence in probability, respectively, for a sequence of pairwise NQD random variables under conditions (1.1) and (1.3). Therefore, we put forward the following open problem:

Open problem. Do Theorems 2.1 and 2.2 hold for a sequence of pairwise NQD random variables (instead of for a sequence of m-LNQD random variables)?

Theorem 2.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-LNQD random variables and let $1 \leqslant p < 2$. Suppose that $\sum_{n=1}^{\infty} (h(n)/n)^{\lambda(2-p)/p} < \infty$ for some $\lambda > p$. Then (1.2) and*

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > h(n)) < \infty \tag{2.3}$$

imply

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{k=1}^{n} (X_k - EX_k)\right| > \varepsilon n^{1/p}\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.4}$$

3. Lemmas

To prove the results in this paper, we need the following five lemmas.

Lemma 3.1 ([7]). *Let $\{X_n, n \geqslant 1\}$ be a sequence of pairwise* NQD *random variables. Let $\{f_n, n \geqslant 1\}$ be a sequence of increasing functions. Then $\{f_n(X_n), n \geqslant 1\}$ is a sequence of pairwise* NQD *random variables.*

Lemma 3.2 ([6]). *Suppose $\{X_n, n \geqslant 1\}$ is a sequence of* LNQD *random variables. Then for all $n \geqslant 1$*

$$E\exp\Big(\sum_{k=1}^{n} X_k\Big) \leqslant \prod_{k=1}^{n} E\exp(X_k).$$

Lemma 3.3. *Let $\{X_n, n \geqslant 1\}$ be a sequence of* LNQD *random variables with $EX_n = 0$ and $0 < B_n = \sum\limits_{k=1}^{n} EX_k^2 < \infty$, $n \geqslant 1$. Let $S_n = \sum\limits_{k=1}^{n} X_k$, $n \geqslant 1$. Then for all $n \geqslant 1$,*

$$P(|S_n| \geqslant x) \leqslant P\Big(\max_{1\leqslant k\leqslant n} |X_k| > y\Big) + 2\exp\Big(\frac{x}{y} - \frac{x}{y}\log\Big(1 + \frac{xy}{B_n}\Big)\Big) \quad \text{for all } x > 0, y > 0.$$

The proof of Lemma 3.3 can be obtained by applying Lemma 3.2 and following the approach of Fuk and Nagaev in [2]. We omit the details.

Lemma 3.4. *Let $\{X_n, n \geqslant 1\}$ be a sequence of m-*LNQD *random variables with zero means and finite second moments. Let $S_n = \sum\limits_{k=1}^{n} X_k$ and $B_n = \sum\limits_{k=1}^{n} EX_k^2$, $n \geqslant 1$. Then for all $n \geqslant m \geqslant 1$, $x > 0$, and $y > 0$,*

$$P(|S_n| \geqslant x) \leqslant mP\Big(\max_{1\leqslant k\leqslant n} |X_k| > y\Big) + 2m\exp\Big(\frac{x}{my} - \frac{x}{my}\log\Big(1 + \frac{xy}{mB_n}\Big)\Big).$$

Proof. For all $1 \leqslant k \leqslant n$, take $\tau = [n/m]$. Let

$$Y_k = \begin{cases} X_k & \text{if } 1 \leqslant k \leqslant n, \\ 0 & \text{if } k > n \end{cases} \quad \text{and} \quad T_{nj} = \sum_{i=0}^{\tau} Y_{mi+j} \quad (1 \leqslant j \leqslant m).$$

By Lemma 3.3, we have

$$
\begin{aligned}
P(S_n \geqslant x) &\leqslant P\Big(\bigcup_{j=1}^{m}[T_{nj} \geqslant x/m]\Big) \leqslant \sum_{j=1}^{m} P(T_{nj} \geqslant x/m) \\
&\leqslant \sum_{j=1}^{m} P\Big(\max_{0\leqslant i\leqslant \tau} Y_{mi+j} > y\Big) \\
&\quad + \sum_{j=1}^{m} \exp\Big(\frac{x}{my} - \frac{x}{my}\log\Big(1 + \frac{xy}{m\sum_{i=0}^{\tau} EY_{mi+j}^2}\Big)\Big) \\
&\leqslant mP\Big(\max_{1\leqslant k\leqslant n} X_k > y\Big) + m\exp\Big(\frac{x}{my} - \frac{x}{my}\log\Big(1 + \frac{xy}{m\sum_{k=1}^{n} EX_k^2}\Big)\Big) \\
&= mP\Big(\max_{1\leqslant k\leqslant n} X_k > y\Big) + m\exp\Big(\frac{x}{my} - \frac{x}{my}\log\Big(1 + \frac{xy}{mB_n}\Big)\Big).
\end{aligned}
$$

By a similar argument as above, we get

$$P(-S_n \geqslant x) \leqslant mP\Big(\max_{1\leqslant k\leqslant n} -X_k > y\Big) + m\exp\Big(\frac{x}{my} - \frac{x}{my}\log\Big(1 + \frac{xy}{mB_n}\Big)\Big).$$

Therefore, the proof is complete. □

The following lemma is similar to Lemma 2.2 of [11]. The proof follows essentially the same steps but for the sake of completeness we present it.

Lemma 3.5. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables satisfying* (1.2) *and* (1.5) *for some real number* $p > 0$. *Then the following statements hold:*

(i) *for all* $0 < \alpha < p$,

$$\lim_{n\to\infty} n^{-\alpha/p} \sum_{k=1}^{n} E|X_k|^{\alpha} I(|X_k|^p > n) = 0; \tag{3.1}$$

(ii) *for all* $\beta > p$,

$$\lim_{n\to\infty} n^{-\beta/p} \sum_{k=1}^{n} E|X_k|^{\beta} I(|X_k|^p \leqslant n) = 0. \tag{3.2}$$

P r o o f. First, we prove (3.1). Put $I = n^{-\alpha/p} \sum_{k=1}^{n} E|X_k|^\alpha I(|X_k|^p > n)$. Then

$$
\begin{aligned}
I &= n^{-\alpha/p} \sum_{k=1}^{n} E|X_k|^\alpha I(|X_k|^p > n) = n^{-\alpha/p} \sum_{k=1}^{n} \int_0^\infty P(|X_k|^\alpha I(|X_k|^p > n) > t)\,\mathrm{d}t \\
&= n^{-\alpha/p} \sum_{k=1}^{n} \int_0^{n^{\alpha/p}} P(|X_k|^\alpha I(|X_k|^p > n) > t)\,\mathrm{d}t \\
&\quad + n^{-\alpha/p} \sum_{k=1}^{n} \int_{n^{\alpha/p}}^\infty P(|X_k|^\alpha I(|X_k|^p > n) > t)\,\mathrm{d}t \\
&\leqslant \sum_{k=1}^{n} P(|X_k|^p > n) + n^{-\alpha/p} \sum_{k=1}^{n} \int_{n^{\alpha/p}}^\infty P(|X_k|^\alpha > t)\,\mathrm{d}t =: I' + I''.
\end{aligned}
$$

Since $h(n)/n \to 0$ as $n \to \infty$, there exists N_1 such that $h(n) \leqslant n$ if $n > N_1$. Hence, by taking $y = n$ in (1.5), we get $I' \to 0$ as $n \to \infty$. Letting $t = y^{\alpha/p}$, we have

$$
I'' = (\alpha/p) n^{-\alpha/p} \sum_{k=1}^{n} \int_n^\infty y^{\alpha/p-1} P(|X_k|^p > y)\,\mathrm{d}y.
$$

Let $\varepsilon > 0$. By (1.5), there exists N_2 such that $n^{-1} \sum_{k=1}^{n} yP(|X_k|^p > y) \leqslant \varepsilon$ if $n > N_2$. Then for $n > \max\{N_1, N_2\}$ we come to

$$
I'' \leqslant \varepsilon(\alpha/p) n^{1-\alpha/p} \int_n^\infty y^{\alpha/p-2}\,\mathrm{d}y = \varepsilon \frac{\alpha}{\alpha - p} n^{1-\alpha/p} y^{\alpha/p-1}|_n^\infty = \varepsilon \frac{\alpha}{p-\alpha}.
$$

Thus $\limsup_{n\to\infty} I'' \leqslant \varepsilon$ by $0 < \alpha < p$. Since $\varepsilon > 0$ is arbitrary, $I'' \to 0$ as $n \to \infty$. The proof of (3.1) is complete.

Next we prove (3.2). Again let $\varepsilon > 0$. Put $J = n^{-\beta/p} \sum_{k=1}^{n} E|X_k|^\beta I(|X_k|^p \leqslant n)$. Then

$$
\begin{aligned}
J &= n^{-\beta/p} \sum_{k=1}^{n} \int_0^\infty P(|X_k|^\beta I(|X_k|^p \leqslant n) \geqslant t)\,\mathrm{d}t \\
&= n^{-\beta/p} \sum_{k=1}^{n} \int_0^{n^{\beta/p}} P(|X_k|^\beta I(|X_k|^p \leqslant n) \geqslant t)\,\mathrm{d}t \\
&\leqslant n^{-\beta/p} \sum_{k=1}^{n} \int_0^{n^{\beta/p}} P(|X_k|^\beta \geqslant t)\,\mathrm{d}t.
\end{aligned}
$$

Letting $t = y^{\beta/r}$, we have for $n > N_1$

$$
\begin{aligned}
J &\leqslant (\beta/p) n^{-\beta/p} \sum_{k=1}^{n} \int_0^n y^{\beta/p-1} P(|X_k|^p \geqslant y)\,\mathrm{d}y \\
&= (\beta/p) n^{-\beta/p} \sum_{k=1}^{n} \int_0^{h(n)} y^{\beta/p-1} P(|X_k|^p \geqslant y)\,\mathrm{d}y \\
&\quad + (\beta/p) n^{-\beta/p} \sum_{k=1}^{n} \int_{h(n)}^{n} y^{\beta/p-1} P(|X_k|^p \geqslant y)\,\mathrm{d}y =: J' + J''.
\end{aligned}
$$

By $\beta/p > 1$, (1.2), and $h(n)/n \to 0$ as $n \to \infty$, we obtain

$$
\begin{aligned}
J' &\leqslant (\beta/p)(h(n))^{\beta/p-1} n^{-\beta/p} \sum_{k=1}^{n} \int_0^{h(n)} P(|X_k|^p \geqslant y)\,\mathrm{d}y \\
&\leqslant (\beta/p)(h(n)/n)^{\beta/p-1} \sup_{n\geqslant 1} n^{-1} \sum_{k=1}^{n} E|X_k|^p \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

For $n > \max\{N_1, N_2\}$ we have

$$
\begin{aligned}
J'' &\leqslant \varepsilon(\beta/p) n^{1-\beta/p} \int_{h(n)}^{n} y^{\beta/p-2}\,\mathrm{d}y = \varepsilon \frac{\beta}{\beta - p}\, n^{1-\beta/p}(n^{\beta/p-1} - (h(n))^{\beta/p-1}) \\
&= \varepsilon \frac{\beta}{\beta - p}(1 - (h(n)/n)^{\beta/p-1}).
\end{aligned}
$$

Thus $\limsup_{n\to\infty} J'' \leqslant \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $J'' \to 0$ as $n \to \infty$. The proof of (3.2) is complete. □

4. Proofs of Theorems 2.1, 2.2, and 2.3

With the lemmas of Section 3 accounted for, the main results will now be proved.

P r o o f of Theorem 2.1. Let

$$
Y_k = -tI(X_k \leqslant -t) + X_k I(|X_k| < t) + tI(X_k \geqslant t), \quad k \geqslant 1
$$

and

$$
Z_k = X_k - Y_k = (X_k + t)I(X_k \leqslant -t) + (X_k - t)I(X_k \geqslant t), \quad k \geqslant 1.
$$

By Lemma 3.1, it follows that both $\{Y_k, k \geqslant 1\}$ and $\{Z_k, k \geqslant 1\}$ are m-LNQD sequences. Let $\varepsilon > 0$ and without loss of generality we may assume that $0 < \varepsilon < 1$. We have

$$\begin{aligned} E\biggl(n^{-1/p}\Bigl|\sum_{k=1}^{n}(X_k - EX_k)\Bigr|\biggr) &= n^{-1/p}\int_0^{\infty} P\biggl(\Bigl|\sum_{k=1}^{n}(X_k - EX_k)\Bigr| \geqslant t\biggr)\,\mathrm{d}t \\ &\leqslant \varepsilon^{1/p} + n^{-1/p}\int_{(n\varepsilon)^{1/p}}^{\infty} P\biggl(\Bigl|\sum_{k=1}^{n}(X_k - EX_k)\Bigr| \geqslant t\biggr)\,\mathrm{d}t \\ &\leqslant \varepsilon^{1/p} + n^{-1/p}\int_{(n\varepsilon)^{1/p}}^{\infty} P\biggl(\Bigl|\sum_{k=1}^{n}(Z_k - EZ_k)\Bigr| \geqslant t/2\biggr)\,\mathrm{d}t \\ &\quad + n^{-1/p}\int_{(n\varepsilon)^{1/p}}^{\infty} P\biggl(\Bigl|\sum_{k=1}^{n}(Y_k - EY_k)\Bigr| \geqslant t/2\biggr)\,\mathrm{d}t =: \varepsilon^{1/p} + I_1 + I_2. \end{aligned}$$

Taking $\alpha = 1$ and $\beta = 2$ in Lemma 3.5, we get

$$\lim_{n\to\infty} n^{-1/p}\sum_{k=1}^{n} E|X_k|I(|X_k|^p > n) = 0 \tag{4.1}$$

and

$$\lim_{n\to\infty} n^{-2/p}\sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n) = 0. \tag{4.2}$$

Since $h(n)/n \to 0$ as $n \to \infty$, there exists N such that $h(n) \leqslant n\varepsilon$ if $n > N$. Since $|Z_k| \leqslant |X_k|I(|X_k| > t)$, we have

$$\begin{aligned} \max_{t\geqslant(n\varepsilon)^{1/p}} \frac{1}{t}\Bigl|\sum_{k=1}^{n} EZ_k\Bigr| &\leqslant \max_{t\geqslant(n\varepsilon)^{1/p}} \frac{1}{t}\sum_{k=1}^{n} E|Z_k| \\ &\leqslant \max_{t\geqslant(n\varepsilon)^{1/p}} \frac{1}{t}\sum_{k=1}^{n} E|X_k|I(|X_k| > t) \leqslant (n\varepsilon)^{-1/p}\sum_{k=1}^{n} E|X_k|I(|X_k|^p > n\varepsilon) \\ &= (n\varepsilon)^{-1/p}\sum_{k=1}^{n} E|X_k|I(|X_k|^p > n) + (n\varepsilon)^{-1/p}\sum_{k=1}^{n} E|X_k|I(n\varepsilon < |X_k|^p \leqslant n) \\ &\leqslant \varepsilon^{-1/p}n^{-1/p}\sum_{k=1}^{n} E|X_k|I(|X_k|^p > n) + \varepsilon^{-2/p}n^{-2/p}\sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n). \end{aligned}$$

Then by (4.1) and (4.2) we obtain $t^{-1}\Bigl|\sum\limits_{k=1}^{n} EZ_k\Bigr| \to 0$ whenever $n \to \infty$ uniformly for $t \geqslant (n\varepsilon)^{1/p}$. Hence, for sufficiently large n, we have that $\Bigl|\sum\limits_{k=1}^{n} EZ_k\Bigr| \leqslant t/4$ for all

$t \geqslant (n\varepsilon)^{1/p}$. By (4.1) and (4.2) we get

$$
\begin{aligned}
I_1 &\leqslant n^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} P\Big(\Big|\sum_{k=1}^{n} Z_k\Big| \geqslant t/4\Big)\,\mathrm{d}t \leqslant n^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} P\Big(\bigcup_{k=1}^{n} [|X_k| > t]\Big)\,\mathrm{d}t \\
&\leqslant n^{-1/p} \sum_{k=1}^{n} \int_{(n\varepsilon)^{1/p}}^{\infty} P(|X_k| > t)\,\mathrm{d}t \leqslant n^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k|^p > n\varepsilon) \\
&= n^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k|^p > n) + n^{-1/p} \sum_{k=1}^{n} E|X_k| I(n\varepsilon < |X_k|^p \leqslant n) \\
&\leqslant n^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k|^p > n) + \varepsilon^{-1/p} n^{-2/p} \sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n) \to 0 \\
&\hspace{28em} \text{as } n \to \infty.
\end{aligned}
$$

Next we prove that $I_2 \to 0$ as $n \to \infty$. Let $B_n = \sum\limits_{k=1}^{n} E(Y_k - EY_k)^2$, $x = t/2$, $y = t/(2m\eta)$, $\eta > 1$. By Lemma 3.4 we arrive at

$$
\begin{aligned}
I_2 &= n^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} P\Big(\Big|\sum_{k=1}^{n} (Y_k - EY_k)\Big| \geqslant t/2\Big)\,\mathrm{d}t \\
&\leqslant C n^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} P\Big(\max_{1 \leqslant k \leqslant n} |Y_k - EY_k| \geqslant t/(2m\eta)\Big)\,\mathrm{d}t \\
&\quad + C n^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} \Big(\frac{1}{t^2} B_n\Big)^{\eta}\,\mathrm{d}t =: I_3 + I_4.
\end{aligned}
$$

For all $t \geqslant (n\varepsilon)^{1/p}$, by Jensen's inequality, we have

$$
\begin{aligned}
\max_{1 \leqslant k \leqslant n} \frac{1}{t}|EY_k| &\leqslant \max_{1 \leqslant k \leqslant n} \Big\{\frac{1}{t} E|X_k| I(|X_k| \leqslant t) + P(|X_k| > t)\Big\} \\
&\leqslant \max_{1 \leqslant k \leqslant n} \Big\{\frac{1}{t} E|X_k| I(|X_k| \leqslant n^{1/p}) + \varepsilon^{-1/p} n^{-1/p} E|X_k| I(n^{1/p} < |X_k| \leqslant t) \\
&\hspace{22em} + P(|X_k| > t)\Big\} \\
&\leqslant \Big(\max_{1 \leqslant k \leqslant n} \frac{1}{t^2} EX_k^2 I(|X_k| \leqslant n^{1/p})\Big)^{1/2} + \varepsilon^{-1/p} n^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k| > n^{1/p}) \\
&\quad + \sum_{k=1}^{n} P(|X_k|^p > n\varepsilon) \leqslant \varepsilon^{-1/p} \Big(n^{-2/p} \sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n)\Big)^{1/2} \\
&\quad + \varepsilon^{-1/p} n^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k|^p > n) + \sum_{k=1}^{n} P(|X_k|^p > n\varepsilon) =: I_{31} + I_{32} + I_{33}.
\end{aligned}
$$

By (4.1) and (4.2) we find that $I_{31} \to 0$ and $I_{32} \to 0$ as $n \to \infty$. Recalling that $h(n) \leqslant n\varepsilon$ for $n > N$, by taking $y = n\varepsilon$ in (1.5) we get that $I_{33} \to 0$ as $n \to \infty$.

Therefore, by a similar argument as in the proof of $I_1 \to 0$, we have

$$\begin{aligned}
I_3 &\leqslant Cn^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} P\Big(\max_{1\leqslant k\leqslant n} |Y_k| \geqslant t/(4m\eta)\Big)\,\mathrm{d}t \quad (\text{since } |Y_k| \leqslant |X_k|)\\
&\leqslant Cn^{-1/p} \sum_{k=1}^{n} \int_{(n\varepsilon)^{1/p}}^{\infty} P(|X_k| \geqslant t/(4m\eta))\,\mathrm{d}t\\
&\leqslant Cn^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k|^p \geqslant n\varepsilon/(4m\eta)^p) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Finally, we prove that $I_4 \to 0$ as $n \to \infty$. By the C_r-inequality we get

$$\begin{aligned}
I_4 &\leqslant Cn^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} \left(\frac{1}{t^2} \sum_{k=1}^{n} EX_k^2 I(|X_k| \leqslant t) + \sum_{k=1}^{n} P(|X_k| > t)\right)^{\eta} \mathrm{d}t\\
&= Cn^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} \left(\frac{1}{t^2} \sum_{k=1}^{n} EX_k^2 I(|X_k| \leqslant (n\varepsilon)^{1/p})\right.\\
&\quad \left. + \frac{1}{t^2} \sum_{k=1}^{n} EX_k^2 I((n\varepsilon)^{1/p} < |X_k| \leqslant t) + \sum_{k=1}^{n} P(|X_k| > t)\right)^{\eta} \mathrm{d}t\\
&\leqslant Cn^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} \left(\frac{1}{t^2} \sum_{k=1}^{n} EX_k^2 I(|X_k| \leqslant (n\varepsilon)^{1/p})\right)^{\eta} \mathrm{d}t\\
&\quad + Cn^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} \left(\frac{1}{t} \sum_{k=1}^{n} E|X_k| I((n\varepsilon)^{1/p} < |X_k| \leqslant t)\right)^{\eta} \mathrm{d}t\\
&\quad + Cn^{-1/p} \int_{(n\varepsilon)^{1/p}}^{\infty} \left(\sum_{k=1}^{n} P(|X_k| > t)\right)^{\eta} \mathrm{d}t =: I_{41} + I_{42} + I_{43}.
\end{aligned}$$

By $\eta > 1$ and (4.2) we obtain

$$\begin{aligned}
I_{41} &\leqslant Cn^{-1/p} \Big(\sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n\varepsilon)\Big)^{\eta} \int_{(n\varepsilon)^{1/p}}^{\infty} \frac{1}{t^{2\eta}}\,\mathrm{d}t\\
&\leqslant C\Big(n^{-2/p} \sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n\varepsilon)\Big)^{\eta} \quad (\text{since } 0 < \varepsilon < 1)\\
&\leqslant C\Big(n^{-2/p} \sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n)\Big)^{\eta} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

By a similar argument as in the proof of $I_1 \to 0$ we find

$$\begin{aligned}
I_{42} &\leqslant Cn^{-1/p} \Big(\sum_{k=1}^{n} E|X_k| I(|X_k|^p > n\varepsilon)\Big)^{\eta} \int_{(n\varepsilon)^{1/p}}^{\infty} \frac{1}{t^{\eta}}\,\mathrm{d}t\\
&\leqslant C\Big(n^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k|^p > n\varepsilon)\Big)^{\eta} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Following the proof of $I_{33} \to 0$, we have

$$\max_{t \geqslant (n\varepsilon)^{1/p}} \sum_{k=1}^{n} P(|X_k| > t) \leqslant \sum_{k=1}^{n} P(|X_k|^p > n\varepsilon) \to 0 \quad \text{as } n \to \infty.$$

Hence, by a similar argument as in the proof of $I_1 \to 0$, we come to

$$I_{43} \leqslant C n^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k|^p > n\varepsilon) \to 0 \quad \text{as } n \to \infty.$$

The proof is complete. $\square$

Proof of Theorem 2.2. Let

$$Y_k = -t^{1/p} I(X_k < -t^{1/p}) + X_k I(|X_k| \leqslant t^{1/p}) + t^{1/p} I(X_k > t^{1/p}), \quad k \geqslant 1,$$

and

$$Z_k = X_k - Y_k = (X_k + t^{1/p}) I(X_k < -t^{1/p}) + (X_k - t^{1/p}) I(X_k > t^{1/p}), \quad k \geqslant 1.$$

Let $\varepsilon > 0$ and without loss of generality we may assume that $0 < \varepsilon < 1$. Since

$$\begin{aligned}
E\biggl(n^{-1/p}\biggl|\sum_{k=1}^{n}(X_k - EX_k)\biggr|\biggr)^p &= \frac{1}{n}\int_0^\infty P\biggl(\biggl|\sum_{k=1}^{n}(X_k - EX_k)\biggr| \geqslant t^{1/p}\biggr)\,\mathrm{d}t \\
&\leqslant \varepsilon + \frac{1}{n}\int_{n\varepsilon}^\infty P\biggl(\biggl|\sum_{k=1}^{n}(X_k - EX_k)\biggr| \geqslant t^{1/p}\biggr)\,\mathrm{d}t \\
&\leqslant \varepsilon + \frac{1}{n}\int_{n\varepsilon}^\infty P\biggl(\biggl|\sum_{k=1}^{n}(Z_k - EZ_k)\biggr| \geqslant t^{1/p}/2\biggr)\,\mathrm{d}t \\
&\quad + \frac{1}{n}\int_{n\varepsilon}^\infty P\biggl(\biggl|\sum_{k=1}^{n}(Y_k - EY_k)\biggr| \geqslant t^{1/p}/2\biggr)\,\mathrm{d}t =: \varepsilon + I_5 + I_6,
\end{aligned}$$

it suffices to show that $I_5 \to 0$ and $I_6 \to 0$ as $n \to \infty$.

Let N be such that $h(n) \leqslant n\varepsilon$ for $n > N$. Then by $|Z_k| \leqslant |X_k| I(|X_k| > t^{1/p})$ and (1.4) we obtain

$$\begin{aligned}
&\max_{t \geqslant n\varepsilon} t^{-1/p}\biggl|\sum_{k=1}^{n} EZ_k\biggr| \leqslant \max_{t \geqslant n\varepsilon} t^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k| > t^{1/p}) \\
&\leqslant (n\varepsilon)^{-1/p} \sum_{k=1}^{n} E|X_k| I(|X_k|^p > n\varepsilon) \leqslant \varepsilon^{-1}\frac{1}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n\varepsilon) \\
&\leqslant \varepsilon^{-1}\frac{1}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > h(n)) \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, for sufficiently large n, we have that $\left|\sum_{k=1}^{n} EZ_k\right| \leqslant t^{1/p}/4$ holds uniformly for $t \geqslant n\varepsilon$. Hence, we get

$$\begin{aligned} I_5 &\leqslant \frac{1}{n}\int_{n\varepsilon}^{\infty} P\Big(\Big|\sum_{k=1}^{n} Z_k\Big| \geqslant t^{1/p}/4\Big)\,\mathrm{d}t \leqslant \frac{1}{n}\int_{n\varepsilon}^{\infty} P\Big(\bigcup_{k=1}^{n}[|X_k| > t^{1/p}]\Big)\,\mathrm{d}t \\ &\leqslant \frac{1}{n}\sum_{k=1}^{n}\int_{n\varepsilon}^{\infty} P(|X_k| > t^{1/p})\,\mathrm{d}t \leqslant \frac{1}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n\varepsilon) \\ &\leqslant \frac{1}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > h(n)) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Next we prove that $I_6 \to 0$ as $n \to \infty$. Let $B_n = \sum_{k=1}^{n} E(Y_k - EY_k)^2$, $x = t^{1/p}/2$, $y = t^{1/p}/(2m\gamma)$, $\gamma > p$. By Lemma 3.4 we have

$$\begin{aligned} I_6 \leqslant \frac{C}{n}\int_{n\varepsilon}^{\infty} P\Big(\max_{1\leqslant k\leqslant n} |Y_k - EY_k| \geqslant t^{1/p}/(2m\gamma)\Big)\,\mathrm{d}t \\ + \frac{C}{n}\int_{n\varepsilon}^{\infty} (t^{-2/p}B_n)^{\gamma}\,\mathrm{d}t =: I_7 + I_8. \end{aligned}$$

For all $t \geqslant n\varepsilon$, by Jensen's inequality we get

$$\begin{aligned} \max_{1\leqslant k\leqslant n} t^{-1/p}|EY_k| &\leqslant \max_{1\leqslant k\leqslant n} \{t^{-1/p}E|X_k|I(|X_k| \leqslant t^{1/p}) + P(|X_k| > t^{1/p})\} \\ &\leqslant \max_{1\leqslant k\leqslant n} \{t^{-1/p}E|X_k|I(|X_k| \leqslant n^{1/p}) + \varepsilon^{-1/p}n^{-1/p}E|X_k|I(n^{1/p} < |X_k| \leqslant t^{1/p}) \\ &\qquad + P(|X_k| > t^{1/p})\} \\ &\leqslant \Big(\max_{1\leqslant k\leqslant n} t^{-2/p}EX_k^2 I(|X_k| \leqslant n^{1/p})\Big)^{1/2} + \varepsilon^{-1/p}n^{-1/p}\sum_{k=1}^{n} E|X_k|I(|X_k| > n^{1/p}) \\ &\quad + \sum_{k=1}^{n} P(|X_k|^p > n\varepsilon) \leqslant \varepsilon^{-1/p}\Big(n^{-2/p}\sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n)\Big)^{1/2} \\ &\quad + \varepsilon^{-1/p}n^{-1}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n) + \sum_{k=1}^{n} P(|X_k|^p > n\varepsilon) =: I_{71} + I_{72} + I_{73}. \end{aligned}$$

Recalling that (1.4) implies (1.5), by Lemma 3.5 we have that (1.2) and (1.4) imply (4.1) and (4.2). Hence, by a similar argument as in the proof of $I_{31} \to 0$ and $I_{33} \to 0$, we can prove that $I_{71} \to 0$ and $I_{73} \to 0$. Clearly, by (1.4) we have $I_{72} \to 0$. Therefore, by a similar argument as in the proof of $I_5 \to 0$, we arrive at

$$
\begin{aligned}
I_7 &\leqslant \frac{C}{n}\int_{n\varepsilon}^{\infty} P\Big(\max_{1\leqslant k\leqslant n}|Y_k| \geqslant t^{1/p}/(4m\gamma)\Big)\,\mathrm{d}t \\
&\leqslant \frac{C}{n}\sum_{k=1}^{n}\int_{n\varepsilon}^{\infty} P(|X_k| \geqslant t^{1/p}/(4m\gamma))\,\mathrm{d}t \\
&\leqslant \frac{C}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n\varepsilon/(4m\gamma)^p) \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Now by the C_r-inequality we come to

$$
\begin{aligned}
I_8 &\leqslant \frac{C}{n}\int_{n\varepsilon}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{n} EX_k^2 I(|X_k|\leqslant t^{1/p}) + \sum_{k=1}^{n} P(|X_k|>t^{1/p})\Big)^{\gamma}\,\mathrm{d}t \\
&= \frac{C}{n}\int_{n\varepsilon}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{n} EX_k^2 I(|X_k|\leqslant (n\varepsilon)^{1/p}) \\
&\quad + t^{-2/p}\sum_{k=1}^{n} EX_k^2 I((n\varepsilon)^{1/p} < |X_k|\leqslant t^{1/p}) + \sum_{k=1}^{n} P(|X_k|>t^{1/p})\Big)^{\gamma}\,\mathrm{d}t \\
&\leqslant \frac{C}{n}\int_{n\varepsilon}^{\infty}\Big(t^{-2/p}\sum_{k=1}^{n} EX_k^2 I(|X_k|\leqslant (n\varepsilon)^{1/p})\Big)^{\gamma}\,\mathrm{d}t \\
&\quad + \frac{C}{n}\int_{n\varepsilon}^{\infty}\Big(t^{-1}\sum_{k=1}^{n} E|X_k|^p I((n\varepsilon)^{1/p} < |X_k|\leqslant t^{1/p})\Big)^{\gamma}\,\mathrm{d}t \\
&\quad + \frac{C}{n}\int_{n\varepsilon}^{\infty}\Big(\sum_{k=1}^{n} P(|X_k|>t^{1/p})\Big)^{\gamma}\,\mathrm{d}t =: I_{81}+I_{82}+I_{83}.
\end{aligned}
$$

By a similar argument as in $I_{41}\to 0$ and $I_{43}\to 0$, we can prove that $I_{81}\to 0$ and $I_{83}\to 0$. By $\gamma > p \geqslant 1$ and (1.4) we get

$$
\begin{aligned}
I_{82} &\leqslant \frac{C}{n}\Big(\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n\varepsilon)\Big)^{\gamma}\int_{n\varepsilon}^{\infty} t^{-\gamma}\,\mathrm{d}t \\
&\leqslant C\Big(\frac{1}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n\varepsilon)\Big)^{\gamma} \to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

The proof is complete. □

P r o o f of Theorem 2.3. Let

$$
Y_k = -n^{1/p} I(X_k < -n^{1/p}) + X_k I(|X_k|\leqslant n^{1/p}) + n^{1/p} I(X_k > n^{1/p}), k\geqslant 1
$$

and

$$Z_k = X_k - Y_k = (X_k + n^{1/p})I(X_k < -n^{1/p}) + (X_k - n^{1/p})I(X_k > n^{1/p}), k \geqslant 1.$$

Then we have

$$\begin{aligned}\sum_{n=1}^{\infty} P\Big(\Big|\sum_{k=1}^{n}(X_k - EX_k)\Big| > \varepsilon n^{1/p}\Big) &\leqslant \sum_{n=1}^{\infty} P\Big(\Big|\sum_{k=1}^{n}(Z_k - EZ_k)\Big| > \varepsilon n^{1/p}/2\Big) \\ &+ \sum_{n=1}^{\infty} P\Big(\Big|\sum_{k=1}^{n}(Y_k - EY_k)\Big| > \varepsilon n^{1/p}/2\Big) =: I_9 + I_{10}.\end{aligned}$$

By $|Z_k| \leqslant |X_k| I(|X_k| > n^{1/p})$ and (2.3) we have

$$\begin{aligned}&n^{-1/p}\Big|\sum_{k=1}^{n} EZ_k\Big| \leqslant n^{-1/p}\sum_{k=1}^{n} E|Z_k| \leqslant n^{-1/p}\sum_{k=1}^{n} E|X_k| I(|X_k|^p > n) \\ &\leqslant \frac{1}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n) \leqslant \frac{1}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > h(n)) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Therefore, we find that

$$\begin{aligned}I_9 &\leqslant C + \sum_{n=1}^{\infty} P\Big(\Big|\sum_{k=1}^{n} Z_k\Big| > \varepsilon n^{1/p}/4\Big) \leqslant C + \sum_{n=1}^{\infty} P\Big(\bigcup_{k=1}^{n}[|X_k| > n^{1/p}]\Big) \\ &\leqslant C + \sum_{n=1}^{\infty}\sum_{k=1}^{n} P(|X_k| > n^{1/p}) \leqslant C + \sum_{n=1}^{\infty} n^{-1}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n) \\ &\leqslant C + \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > h(n)) < \infty.\end{aligned}$$

Let $B_n = \sum\limits_{k=1}^{n} E(Y_k - EY_k)^2$, $x = \varepsilon n^{1/p}/2$, $y = \varepsilon n^{1/p}/(2m\lambda)$, $\lambda > p$. By Lemma 3.4 we get

$$\begin{aligned}I_{10} &\leqslant C\sum_{n=1}^{\infty} P\Big(\max_{1\leqslant k\leqslant n}|Y_k - EY_k| > \varepsilon n^{1/p}/(2m\lambda)\Big) + C\sum_{n=1}^{\infty}(n^{-2/p}B_n)^{\lambda} \\ &=: I_{11} + I_{12}.\end{aligned}$$

By Jensen's inequality we have

$$
\begin{aligned}
n^{-1/p} \max_{1\leqslant k\leqslant n} |EY_k| &\leqslant n^{-1/p} \max_{1\leqslant k\leqslant n} E|X_k|I(|X_k|^p \leqslant n) + \max_{1\leqslant k\leqslant n} P(|X_k|^p > n) \\
&\leqslant \Big(n^{-2/p} \max_{1\leqslant k\leqslant n} EX_k^2 I(|X_k|^p \leqslant n)\Big)^{1/2} + \frac{1}{n} \max_{1\leqslant k\leqslant n} E|X_k|^p I(|X_k|^p > n) \\
&\leqslant \left(n^{-2/p} \sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n)\right)^{1/2} + \frac{1}{n} \sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > n) \\
&=: I_{11}' + I_{11}''.
\end{aligned}
$$

Clearly, by (2.3) we have that $I_{11}'' \to 0$ as $n \to \infty$. Note that (2.3) implies (1.4) and (1.4) implies (1.5). Hence, from (4.2) we have that $I_{11}' \to 0$ as $n \to \infty$. Therefore, by (2.3) we obtain

$$
\begin{aligned}
I_{11} &\leqslant C \sum_{n=1}^{\infty} P\Big(\max_{1\leqslant k\leqslant n} |Y_k| > \varepsilon n^{1/p}/(4m\lambda)\Big) \leqslant C \sum_{n=1}^{\infty} \sum_{k=1}^{n} P(|X_k| > \varepsilon n^{1/p}/(4m\lambda)) \\
&\leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} E|X_k|^p I(|X_k|^p > (\varepsilon/(4m\lambda))^p n) < \infty.
\end{aligned}
$$

Finally, we prove that $I_{12} < \infty$. By the C_r-inequality we see that

$$
\begin{aligned}
I_{12} \leqslant C \sum_{n=1}^{\infty} &\left(n^{-2/p} \sum_{k=1}^{n} EX_k^2 I(|X_k|^p \leqslant n)\right)^{\lambda} \\
&+ C \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} P(|X_k|^p > n)\right)^{\lambda} =: I_{12}' + I_{12}''.
\end{aligned}
$$

By $\lambda > p \geqslant 1$ and (2.3) we observe

$$
\begin{aligned}
I_{12}'' &\leqslant C \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} E|X_k|^p (|X_k|^p > n)\right)^{\lambda} \\
&\leqslant C \Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=1}^{n} E|X_k|^p (|X_k|^p > h(n))\Big)^{\lambda} < \infty.
\end{aligned}
$$

By $\sum_{n=1}^{\infty}(h(n)/n)^{\lambda(2-p)/p}<\infty$, (1.2) and (2.3) we have

$$\begin{aligned}I'_{12}&\leqslant C\sum_{n=1}^{\infty}\Big(n^{-2/p}\sum_{k=1}^{n}EX_k^2I(|X_k|^p\leqslant h(n))\Big)^{\lambda}\\&\quad+C\sum_{n=1}^{\infty}\Big(n^{-2/p}\sum_{k=1}^{n}EX_k^2I(h(n)<|X_k|^p\leqslant n)\Big)^{\lambda}\\&\leqslant C\sum_{n=1}^{\infty}(h(n)/n)^{\lambda(2-p)/p}\Big(\frac{1}{n}\sum_{k=1}^{n}E|X_k|^pI(|X_k|^p\leqslant h(n))\Big)^{\lambda}\\&\quad+C\Big(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}E|X_k|^pI(h(n)<|X_k|^p\leqslant n)\Big)^{\lambda}\\&\leqslant C\sum_{n=1}^{\infty}(h(n)/n)^{\lambda(2-p)/p}\Big(\sup_{n\geqslant 1}\frac{1}{n}\sum_{k=1}^{n}E|X_k|^p\Big)^{\lambda}\\&\quad+C\Big(\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}E|X_k|^pI(|X_k|^p>h(n))\Big)^{\lambda}<\infty.\end{aligned}$$

The proof is complete. □

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