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Some Common Fixed Point Theorems in Menger Spaces

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Abstract

In this paper, we prove some common fixed point theorems for occasionally weakly compatible mappings in Menger spaces. An example is also given to illustrate the main result. As applications to our results, we obtain the corresponding fixed point theorems in metric spaces. Our results improve and extend many known results existing in the literature.

Key words: Menger space, weakly compatible mappings, occasionally weakly compatible mappings, fixed point

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1 Introduction

In fixed point theory, contraction mapping theorems have always been an active area of research since 1922 with the celebrated Banach contraction fixed point theorem [7]. As a generalization of metric space, Karl Menger [30, 31] introduced the notion of probabilistic metric spaces (briefly, PM-spaces) in which the concept of distance is considered to be statistical or probabilistic rather than deterministic. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In fact the study of such spaces received an impetus with the pioneering works of Schweizer and Sklar [45, 46]. The first effort in this direction was made by Sehgal [47], who in his doctoral dissertation initiated the study of contraction mapping theorems in probabilistic metric spaces. In 1972, Sehgal and Bharucha-Reid [48] studied the Banach contraction principle of metric space into the complete Menger space (also see [5, 17]). In an interesting paper [22], Hicks observed that fixed point theorems for certain contraction mappings on a Menger space endowed with a triangular t-norm may be obtained from corresponding results in metric spaces. Further, Hicks and Sharma [23] proposed an axiom which is easy to verify and avoids the sufficient condition for the metrization of a PM-space postulates the existence of a certain kind of t-norms. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. The theory of PM-spaces is of fundamental importance in probabilistic functional analysis due to its extensive applications in random differential as well as random integral equations, one may recall Chang et al. [9].

In 1996, Jungck [26] introduced the notion of weakly compatible mappings which is more general than compatibility and proved fixed point theorems in absence of continuity of the involved mappings. In recent years, many mathematicians established a number of common fixed point theorems satisfying contractive type conditions and involving conditions on commutativity, completeness and suitable containment of ranges of the mappings. Al-Thagafi and Shahzad [3] introduced the notion of occasionally weakly compatible mappings in metric space, which is more general than weakly compatible mappings (also see [4]). Recently, Jungck and Rhoades [27] extensively studied the notion of occasionally weakly compatible mappings in semi-metric spaces. The notions of improving commutativity of self mappings have been extended to PM-spaces by many authors. For example, Singh and Jain [49] extended the notion of weak compatibility and Chauhan et al. [13] extended the notion of occasionally weak compatibility to PM-spaces. The fixed point theorems for occasionally weakly compatible mappings in different settings investigated by many researchers (e.g. [1, 6, 8, 10, 11, 12, 14, 15, 16, 19, 36, 37, 38, 39, 40, 41, 42, 43, 50]).

In 2009, Fang and Gao [21] proved some common fixed point theorems for a pair of weakly compatible mappings in Menger spaces satisfying strict contractive conditions with property (E.A). More recently, Ali et al. [2] improved and extended the results of Fang and Gao [21] without any requirement on containment of ranges amongst the involved mappings.

The object of this paper is to prove a common fixed point theorem for two pairs of occasionally weakly compatible mappings in Menger space. An example is furnished to illustrate the main result. We extend our main result to two families of occasionally weakly compatible mappings in Menger spaces. Our results improve and extend many known results in Menger as well as metric spaces.

2 Preliminaries

Definition 2.1 [46] A triangular norm $\triangle$ (briefly, t-norm) is a binary operation on the unit interval [0,1] satisfying the following conditions: for all $a, b, c, d \in [0, 1]$

1. $\triangle(a, 1) = a$,

2. $\triangle(a,b) = \triangle(b,a)$,
3. $\triangle(a,b) \leq \triangle(c,d)$, whenever $a \leq c$ and $b \leq d$,
4. $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$.

Definition 2.2 [46] A mapping $F\colon \mathbb{R} \to \mathbb{R}^+$ is said to be a distribution function if it is non-decreasing and left continuous with $\inf\{F(t)\colon t \in \mathbb{R}\} = 0$ and $\sup\{F(t)\colon t \in \mathbb{R}\} = 1$.

We denote by $\Im$ the set of all distribution functions while H denotes the specific distribution function defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\mathcal{F} : X \times X \to \Im$ is called a probabilistic distance on X and the value of $\mathcal{F}$ at $(x,y) \in X \times X$ is represented by $F_{x,y}$.

Definition 2.3 [46] The ordered pair $(X,\mathcal{F})$ is called a PM-space if X is a non-empty set and $\mathcal{F}$ is a probabilistic distance satisfying the following conditions: for all $x,y,z \in X$ and $t,s > 0$

1. $F_{x,y}(t) = H(t)$ if and only if $x = y$,
2. $F_{x,y}(t) = F_{y,x}(t)$,
3. if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$ then $F_{x,z}(t+s) = 1$.

The ordered triple $(X,\mathcal{F},\triangle)$ is called a Menger space if $(X,\mathcal{F})$ is a PM-space, $\triangle$ is a t-norm and the following inequality holds:

$$F_{x,z}(t+s) \geq \triangle\left(F_{x,y}(t), F_{y,z}(s)\right),$$

for all $x,y,z \in X$ and $t,s > 0$.

Every metric space (X,d) can always be realized as a PM-space. So PM-spaces offer a wider framework than that of metric spaces and are better suited to cover even wider statistical situations.

Lemma 2.1 [48] *Let (X,d) be a metric space. Define a mapping $\mathcal{F}\colon X \times X \to \Im$ by*

$$F_{x,y}(t) = H(t - d(x,y)),$$

for all $x,y \in X$ and $t > 0$. Then $(X,\mathcal{F},\min)$ is called the induced Menger space by (X,d) and it is complete if (X,d) is complete.

Definition 2.4 [20, 21] Let $F_1, F_2 \in \Im$. The algebric sum $F_1 \oplus F_2$ of F_1 and F_2 is defined by

$$(F_1 \oplus F_2)(t) = \sup_{t_1+t_2=t} \min\{F_1(t_1), F_2(t_2)\},$$

for all $t \in \mathbb{R}$.

Obviously,

$$(F_1 \oplus F_2)(2t) = \min\{F_1(t), F_2(t)\},$$

for all $t \geq 0$.

Definition 2.5 [49] Two self mappings A and S of a non-empty set X are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is, if $Ax = Sx$ for some $x \in X$, then $ASx = SAx$.

Definition 2.6 [27] Two self mappings A and S of a non-empty set X are said to be occasionally weakly compatible iff there is a point $x \in X$ which is a coincidence point of A and S at which A and S commute, that is, $ASx = SAx$.

Remark 2.1 The notion of occasionally weakly compatible mappings is more general than weak compatibility (see example, [3, 4]).

The following Lemma plays a key role in what follows.

Lemma 2.2 [27] *Two self mappings A and S of a non-empty set X are said to be occasionally weakly compatible if A and S have a unique point of coincidence, $w = Ax = Sx$, then w is the unique common fixed point of A and S.*

3 Results

Theorem 3.1 *Let A, B, S and T be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ with a continuous t-norm $\triangle$ on $[0,1] \times \{1\}$ satisfying*

$$F_{Ax,By}(t) > \min\left\{F_{Sx,Ty}(t), {}_{\frac{2}{k}}[F_{Sx,Ax} \oplus F_{Sx,By}](t), {}_{2}[F_{Ty,By} \oplus F_{Ty,Ax}](t)\right\}, \tag{3.1}$$

for any $x, y \in X$ with $x \neq y$, for all $t > 0$ with some k where $1 \leq k < 2$ and ${}_af(t)$ means $f(at)$. Then, if the pairs (A, S) and (B, T) are each occasionally weakly compatible, there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S and T in X.

Proof Since the pairs (A, S) and (B, T) are each occasionally weakly compatible, there exist points $u, v \in X$ such that $Au = Su$, $ASu = SAu$ and $Bv = Tv$, $BTv = TBv$. Now we assert that $Au = Bv$. Then on using inequality (3.1) with $x = u$ and $y = v$, we get, for some $t_0 > 0$,

$$F_{Au,Bv}(t_0) >$$
$$\min\left\{F_{Su,Tv}(t_0), [F_{Su,Au} \oplus F_{Su,Bv}]\left(\frac{2}{k}t_0\right), [F_{Tv,Bv} \oplus F_{Tv,Au}](2t_0)\right\}$$

or, equivalently,

$$F_{Au,Bv}(t_0) >$$
$$\min\left\{F_{Au,Bv}(t_0), [F_{Au,Au} \oplus F_{Au,Bv}]\left(\frac{2}{k}t_0\right), [F_{Bv,Bv} \oplus F_{Bv,Au}](2t_0)\right\}$$

and so,

$$F_{Au,Bv}(t_0) > \min\left\{F_{Au,Bv}(t_0), F_{Au,Bv}\left(\frac{2}{k}t_0\right), F_{Bv,Au}(2t_0)\right\}.$$

Since, $F_{Au,Bv}\left(\frac{2}{k}t_0\right) > F_{Au,Bv}(t_0)$ and $F_{Bv,Au}(2t_0) > F_{Bv,Au}(t_0)$ for some $t_0 > 0$ (see [21]). Then we obtain

$$F_{Au,Bv}(t_0) > \min\left\{F_{Au,Bv}(t_0), F_{Au,Bv}(t_0), F_{Bv,Au}(t_0)\right\},$$

which implies

$$F_{Au,Bv}(t_0) > F_{Au,Bv}(t_0),$$

a contradiction. Therefore $Au = Bv$, hence $Au = Su = Bv = Tv$. Moreover, if there is another point z such that $Az = Sz$. Then using inequality (3.1) it follows that $Az = Sz = Bv = Tv$, or $Au = Az$. Hence $w = Au = Su$ is the unique point of coincidence of A and S. By Lemma 2.2, w is the unique common fixed point of A and S. Similarly, there is a unique point $z \in X$ such that $z = Bz = Tz$. Suppose that $w \neq z$, by putting $x = w$ and $y = z$ in inequality (3.1), we get, for some $t_0 > 0$,

$$F_{Aw,Bz}(t_0) >$$
$$\min\left\{F_{Sw,Tz}(t_0), [F_{Sw,Aw} \oplus F_{Sw,Bz}]\left(\frac{2}{k}t_0\right), [F_{Tz,Bz} \oplus F_{Tz,Aw}](2t_0)\right\},$$

or, equivalently,

$$F_{w,z}(t_0) > \min\left\{F_{w,z}(t_0), [F_{w,w} \oplus F_{w,z}]\left(\frac{2}{k}t_0\right), [F_{z,z} \oplus F_{z,w}](2t_0)\right\},$$

and so,

$$F_{w,z}(t_0) > \min\left\{F_{w,z}(t_0), F_{w,z}\left(\frac{2}{k}t_0\right), F_{z,w}(2t_0)\right\}.$$

Since, $F_{w,z}\left(\frac{2}{k}t_0\right) > F_{w,z}(t_0)$ and $F_{z,w}(2t_0) > F_{z,w}(t_0)$ for some $t_0 > 0$. Then we obtain

$$F_{w,z}(t_0) > \min\left\{F_{w,z}(t_0), F_{w,z}(t_0), F_{z,w}(t_0)\right\}.$$

It implies $F_{w,z}(t_0) > F_{w,z}(t_0)$, which is a contradiction. Hence $w = z$. Therefore, w is the unique common fixed point of A, B, S and T. □

Example 3.1 Let $X = [0, 4]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$, define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases}$$

for all $x, y \in X$. Clearly $(X, \mathcal{F}, \triangle)$ be a Menger space with continuous t-norm $\triangle(a, b) = \min\{a, b\}$. Now we define the self mappings A, B, S and T by

$$A(x) = \begin{cases} 2, & \text{if } 0 \leq x \leq 2; \\ 0, & \text{if } 2 < x \leq 4. \end{cases} \qquad B(x) = \begin{cases} 2, & \text{if } 0 \leq x \leq 2; \\ 1, & \text{if } 2 < x \leq 4. \end{cases}$$

$$S(x) = \begin{cases} 2, \text{ if } 0 \le x \le 2; \\ 4, \text{ if } 2 < x \le 4. \end{cases} \qquad T(x) = \begin{cases} 2, \text{ if } 0 \le x \le 2; \\ 3, \text{ if } 2 < x \le 4. \end{cases}$$

Then A, B, S and T satisfy all the conditions of Theorem 3.1 for some $k \in [1, 2)$. That is, $A(2) = 2 = S(2), AS(2) = 2 = SA(2)$ and $B(2) = 2 = T(2)$, $BT(2) = 2 = TB(2)$ which shows that the pairs (A, S) and (B, T) are each occasionally weakly compatible. Hence, 2 is the unique common fixed point of A, B, S and T.

Moreover, $A(X) = \{0, 2\} \nsubseteq \{2, 3\} = T(X)$ and $B(X) = \{1, 2\} \nsubseteq \{2, 4\} = S(X)$. Also, it is noticed that all the involved mappings A, B, S and T are discontinuous at $x = 0$.

Now, we extend Theorem 3.1 to even number of self mappings in Menger space.

Theorem 3.2 *Let $P_1, P_2, \ldots, P_{2n}, A$ and B be self mappings of a Menger space $(X, \mathcal{F}, \triangle)$ with a continuous t-norm $\triangle$ on $[0, 1] \times \{1\}$ satisfying*

$$F_{Ax,By}(t) > \min \left\{ \begin{array}{c} F_{P_1P_3\ldots P_{2n-1}x, P_2P_4\ldots P_{2n}y}(t), \\ {}_{\frac{2}{k}}[F_{P_1P_3\ldots P_{2n-1}x, Ax} \oplus F_{P_1P_3\ldots P_{2n-1}x, By}](t), \\ {}_2[F_{P_2P_4\ldots P_{2n}y, By} \oplus F_{P_2P_4\ldots P_{2n}y, Ax}](t) \end{array} \right\}, \qquad (3.2)$$

for any $x, y \in X$ with $x \neq y$, for all $t > 0$ with some k where $1 \le k < 2$ and ${}_af(t)$ means $f(at)$. Assume that $(\star)$

$P_1(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})P_1$,
$P_1P_3(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})P_1P_3$,
$\vdots$
$P_1 \ldots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1 \ldots P_{2n-3}$,
$A(P_3 \ldots P_{2n-1}) = (P_3 \ldots P_{2n-1})A$,
$A(P_5 \ldots P_{2n-1}) = (P_5 \ldots P_{2n-1})A$,
$\vdots$
$AP_{2n-1} = P_{2n-1}A$,

similarly,

$P_2(P_4 \ldots P_{2n}) = (P_4 \ldots P_{2n})P_2$,
$P_2P_4(P_6 \ldots P_{2n}) = (P_6 \ldots P_{2n})P_2P_4$,
$\vdots$
$P_2 \ldots P_{2n-2}(P_{2n}) = (P_{2n})P_2 \ldots P_{2n-2}$,
$B(P_4 \ldots P_{2n}) = (P_4 \ldots P_{2n})B$,
$B(P_6 \ldots P_{2n}) = (P_6 \ldots P_{2n})B$,
$\vdots$
$BP_{2n} = P_{2n}B$.

Then, if the pairs $(A, P_1P_3 \ldots P_{2n-1})$ and $(B, P_2P_4 \ldots P_{2n})$ are each occasionally weakly compatible, it follows that $P_1, P_2, \ldots, P_{2n}$, A and B have a unique common fixed point in X.

Proof On taking $P_1P_3\dots P_{2n-1} = S$ and $P_2P_4\dots P_{2n} = T$, from Theorem 3.1 it follows that w is the unique common fixed point of A, B, $P_1P_3\dots P_{2n-1}$ and $P_2P_4\dots P_{2n}$. Now we assert that w is the fixed point of all the component mappings. By taking $x = P_3\dots P_{2n-1}w$, $y = w$, $P_1^{'} = P_1P_3\dots P_{2n-1}$ and $P_2^{'} = P_2P_4\dots P_{2n}$ in inequality (3.2), we get, for some $t_0 > 0$,

$$F_{AP_3\dots P_{2n-1}w,Bw}(t_0) > \min\left\{\begin{array}{c} F_{P_1'P_3\dots P_{2n-1}w,P_2'w}(t_0), \\ [F_{P_1'P_3\dots P_{2n-1}w,AP_3\dots P_{2n-1}w} \oplus F_{P_2'P_3\dots P_{2n-1}w,Bw}]\left(\frac{2}{k}t_0\right), \\ [F_{P_2'w,Bw} \oplus F_{P_2'w,AP_3\dots P_{2n-1}w}](2t_0) \end{array}\right\},$$

or, equivalently,

$$F_{P_3\dots P_{2n-1}w,w}(t_0) > \min\left\{\begin{array}{c} F_{P_3\dots P_{2n-1}w,w}(t_0), \\ [F_{P_3\dots P_{2n-1}w,P_3\dots P_{2n-1}w} \oplus F_{P_3\dots P_{2n-1}w,w}]\left(\frac{2}{k}t_0\right), \\ [F_{w,w} \oplus F_{w,P_3\dots P_{2n-1}w}](2t_0) \end{array}\right\},$$

and so,

$$F_{P_3\dots P_{2n-1}w,w}(t_0) > \min\left\{\begin{array}{c} F_{P_3\dots P_{2n-1}w,w}(t_0), F_{P_3\dots P_{2n-1}w,w}\left(\frac{2}{k}t_0\right), \\ F_{w,P_3\dots P_{2n-1}w}(2t_0) \end{array}\right\}.$$

Since

$$F_{P_3\dots P_{2n-1}w,w}\left(\frac{2}{k}t_0\right) > F_{P_3\dots P_{2n-1}w,w}(t_0)$$

and

$$F_{P_3\dots P_{2n-1}w,w}(2t_0) > F_{P_3\dots P_{2n-1}w,w}(t_0)$$

for some $t_0 > 0$. Then one obtains

$$F_{P_3\dots P_{2n-1}w,w}(t_0) > \\ > \min\left\{F_{P_3\dots P_{2n-1}w,w}(t_0), F_{P_3\dots P_{2n-1}w,w}(t_0), F_{w,P_3\dots P_{2n-1}w}(t_0)\right\}.$$

It implies,

$$F_{P_3\dots P_{2n-1}w,w}(t_0) > F_{P_3\dots P_{2n-1}w,w}(t_0),$$

which is a contradiction, therefore, $P_3\dots P_{2n-1}w = w$ and thus we conclude that $P_1w = w$. Continuing this procedure, we have

$$Aw = P_1w = P_3w = \dots = P_{2n-1}w = w.$$

In a similar manner, we can also prove

$$Bw = P_2w = P_4w = \dots = P_{2n}w = w.$$

That is, w is the unique common fixed point of $P_1, P_2, \dots, P_{2n}, A$ and B. □

The following result is a slight generalization of Theorem 3.2.

Corollary 3.1 *Let $\{T_\alpha\}_{\alpha\in J}$ and $\{P_i\}_{i=1}^{2n}$ be two families of self mappings of a Menger space $(X,\mathcal{F},\triangle)$ with a continuous t-norm $\triangle$ on $[0,1]\times\{1\}$ satisfying: for a fixed $\beta\in J$ such that*

$$F_{T_\alpha x,T_\beta y}(t) > \min\left\{\begin{array}{c} F_{P_1P_3\ldots P_{2n-1}x,P_2P_4\ldots P_{2n}y}(t), \\ {}_{\frac{2}{k}}[F_{P_1P_3\ldots P_{2n-1}x,T_\alpha x}\oplus F_{P_1P_3\ldots P_{2n-1}x,T_\beta y}](t), \\ {}_2[F_{P_2P_4\ldots P_{2n}y,T_\beta y}\oplus F_{P_2P_4\ldots P_{2n}y,T_\alpha x}](t)\end{array}\right\}, \quad (3.3)$$

for any $x,y\in X$ with $x\neq y$, for all $t>0$ with some k where $1\le k<2$ and ${}_af(t)$ means $f(at)$. Assume that $(\star\star)$

$P_1(P_3\ldots P_{2n-1}) = (P_3\ldots P_{2n-1})P_1,$
$P_1P_3(P_5\ldots P_{2n-1}) = (P_5\ldots P_{2n-1})P_1P_3,$
$\vdots$
$P_1\ldots P_{2n-3}(P_{2n-1}) = (P_{2n-1})P_1\ldots P_{2n-3},$
$T_\alpha(P_3\ldots P_{2n-1}) = (P_3\ldots P_{2n-1})T_\alpha,$
$T_\alpha(P_5\ldots P_{2n-1}) = (P_5\ldots P_{2n-1})T_\alpha,$
$\vdots$
$T_\alpha P_{2n-1} = P_{2n-1}T_\alpha,$

similarly,

$P_2(P_4\ldots P_{2n}) = (P_4\ldots P_{2n})P_2,$
$P_2P_4(P_6\ldots P_{2n}) = (P_6\ldots P_{2n})P_2P_4,$
$\vdots$
$P_2\ldots P_{2n-2}(P_{2n}) = (P_{2n})P_2\ldots P_{2n-2},$
$T_\beta(P_4\ldots P_{2n}) = (P_4\ldots P_{2n})T_\beta,$
$T_\beta(P_6\ldots P_{2n}) = (P_6\ldots P_{2n})T_\beta,$
$\vdots$
$T_\beta P_{2n} = P_{2n}T_\beta.$

Then, if the pairs $(T_\alpha, P_1P_3\ldots P_{2n-1})$ and $(T_\beta, P_2P_4\ldots P_{2n})$ are each occasionally weakly compatible, it follows that all $\{P_i\}$ and $\{T_\alpha\}$ have a unique common fixed point in X.

On taking $A=B$ and $S=T$ in Theorem 3.1, we get the following natural result for a pair of self mappings:

Corollary 3.2 *Let A and S be self mappings of a Menger space $(X,\mathcal{F},\triangle)$ with a continuous t-norm $\triangle$ on $[0,1]\times\{1\}$ satisfying*

$$F_{Ax,Ay}(t) > \min\left\{F_{Sx,Sy}(t), {}_{\frac{2}{k}}[F_{Sx,Ax}\oplus F_{Sx,Ay}](t), {}_2[F_{Sy,Ay}\oplus F_{Sy,Ax}](t)\right\}, \quad (3.4)$$

for any $x,y\in X$ with $x\neq y$, for all $t>0$ with some k where $1\le k<2$ and ${}_af(t)$ means $f(at)$. Then, if the pair (A,S) be occasionally weakly compatible, it follows that A and S have a unique common fixed point in X.

Remark 3.1 The conclusions of Theorem 3.1, Theorem 3.2, Corollary 3.1 and Corollary 3.2 remain true if we replace inequalities (3.1), (3.2), (3.3) and (3.4) by the following respectively: for all $x, y \in X$

$$F_{Ax,By}(t) > \min\left\{F_{Sx,Ty}(t), \tfrac{2}{k}[F_{Sx,Ax} \oplus F_{Sx,By}](t), \tfrac{2}{k}[F_{Ty,By} \oplus F_{Ty,Ax}](t)\right\} \tag{3.5}$$

$$F_{Ax,By}(t) > \min\left\{\begin{array}{c} F_{P_1P_3\ldots P_{2n-1}x, P_2P_4\ldots P_{2n}y}(t), \\ \tfrac{2}{k}[F_{P_1P_3\ldots P_{2n-1}x, Ax} \oplus F_{P_1P_3\ldots P_{2n-1}x, By}](t), \\ \tfrac{2}{k}[F_{P_2P_4\ldots P_{2n}y, By} \oplus F_{P_2P_4\ldots P_{2n}y, Ax}](t) \end{array}\right\} \tag{3.6}$$

$$F_{T_\alpha x,T_\beta y}(t) > \min\left\{\begin{array}{c} F_{P_1P_3\ldots P_{2n-1}x, P_2P_4\ldots P_{2n}y}(t), \\ \tfrac{2}{k}[F_{P_1P_3\ldots P_{2n-1}x, T_\alpha x} \oplus F_{P_1P_3\ldots P_{2n-1}x, T_\beta y}](t), \\ \tfrac{2}{k}[F_{P_2P_4\ldots P_{2n}y, T_\beta y} \oplus F_{P_2P_4\ldots P_{2n}y, T_\alpha x}](t) \end{array}\right\} \tag{3.7}$$

$$F_{Ax,Ay}(t) > \min\left\{F_{Sx,Sy}(t), \tfrac{2}{k}[F_{Sx,Ax} \oplus F_{Sx,Ay}](t), \tfrac{2}{k}[F_{Sy,Ay} \oplus F_{Sy,Ax}](t)\right\} \tag{3.8}$$

Remark 3.2 Theorem 3.1, Theorem 3.2, Corollary 3.1 and Corollary 3.2 (in view of Remark 3.1 improve and extend the results of Ali et al. [2] and Fang and Gao [21] whereas Theorem 3.2 and Corollary 3.1 generalize the results of Imdad et al. [24], Razani and Shirdaryazdi [44] and Singh and Jain [49] without any requirement on containment of ranges, continuity of the involved mappings and completeness of the whole space or any subspace.

4 Related results in metric spaces

As an application to our earlier proved results in Section 3, we can obtain the corresponding fixed point theorems in metric spaces. Now we utilize Lemma 2.1 due to Sehgal and Bharucha-Reid [48] for our next result.

Theorem 4.1 *Let A, B, S and T be self mappings of a metric space (X, d) satisfying: for any $x, y \in X$ with $x \neq y$*

$$d(Ax, By) < \max\left\{\begin{array}{c} d(Sx, Ty), \frac{k}{2}[d(Sx, Ax) + d(Sx, By)], \\ \frac{1}{2}[d(Ty, By) + d(Ty, Ax)] \end{array}\right\}, \tag{4.1}$$

where $1 \leq k < 2$ is a constant. Then, if the pairs (A, S) and (B, T) are each occasionally weakly compatible, there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$. Moreover, $z = w$, so that there is a unique common fixed point of A, B, S and T in X.

Proof Define $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$ and $\triangle(a, b) = \min\{a, b\}$, then $(X, \mathcal{F}, \triangle)$ is the Menger space induced by the (X, d). It is easy to verify that inequality (4.1) of Theorem 4.1 implies inequality (3.1) of Theorem 3.1. Hence, the conclusion of Theorem 4.1 easily follows from Theorem 3.1. □

Corollary 4.1 *Let $P_1, P_2, \ldots, P_{2n}, A$ and B of a metric space (X, d) satisfying the condition $(\star)$ of Theorem 3.2. Suppose that*

$$d(Ax, By) < \max \left\{ \begin{array}{c} d(P_1P_3 \ldots P_{2n-1}x, P_2P_4 \ldots P_{2n}y), \\ \frac{k}{2}[d(P_1P_3 \ldots P_{2n-1}x, Ax) + d(P_1P_3 \ldots P_{2n-1}x, By)], \\ \frac{1}{2}[d(P_2P_4 \ldots P_{2n}y, By) + d(P_2P_4 \ldots P_{2n}y, Ax)] \end{array} \right\}, \tag{4.2}$$

holds for any $x, y \in X$ with $x \neq y$ and for some k where $1 \leq k < 2$. Then, if the pairs $(A, P_1P_3 \ldots P_{2n-1})$ and $(B, P_2P_4 \ldots P_{2n})$ are each occasionally weakly compatible, it follows that $P_1, P_2, \ldots, P_{2n}, A$ and B have a unique common fixed point in X.

Corollary 4.2 *Let $\{T_\alpha\}_{\alpha \in J}$ and $\{P_i\}_{i=1}^{2n}$ be two families of self mappings of a metric space (X, d) satisfying the condition $(\star\star)$ of Corollary 3.1. Suppose that there exists a fixed $\beta \in J$ such that*

$$d(T_\alpha x, T_\beta y) < \max \left\{ \begin{array}{c} d(P_1P_3 \ldots P_{2n-1}x, P_2P_4 \ldots P_{2n}y), \\ \frac{k}{2}[d(P_1P_3 \ldots P_{2n-1}x, T_\alpha x) + d(P_1P_3 \ldots P_{2n-1}x, T_\beta y)], \\ \frac{1}{2}[d(P_2P_4 \ldots P_{2n}y, T_\beta y) + d(P_2P_4 \ldots P_{2n}y, T_\alpha x)] \end{array} \right\}, \tag{4.3}$$

holds for any $x, y \in X$ with $x \neq y$ and for some k where $1 \leq k < 2$. Then, if the pairs $(T_\alpha, P_1P_3 \ldots P_{2n-1})$ and $(T_\beta, P_2P_4 \ldots P_{2n})$ are each occasionally weakly compatible, it follows that all $\{P_i\}$ and $\{T_\alpha\}$ have a unique common fixed point in X.

Corollary 4.3 *Let A and S be self mappings of a metric space (X, d) satisfying*

$$d(Ax, Ay) < \max \left\{ \begin{array}{c} d(Sx, Sy), \frac{k}{2}[d(Sx, Ax) + d(Sx, Ay)], \\ \frac{1}{2}[d(Sy, Ay) + d(Sy, Ax)] \end{array} \right\}, \tag{4.4}$$

for any $x, y \in X$ with $x \neq y$ and for some k where $1 \leq k < 2$. Then, if the pair (A, S) be occasionally weakly compatible, it follows that A and S have a unique common fixed point in X.

Remark 4.1 The conclusions of Theorem 4.1, Corollaries 4.1–4.3 remain true if we replace inequalities (4.1), (4.2), (4.3) and (4.4) by the following respectively: for all $x, y \in X$

$$d(Ax, By) < \max \left\{ \begin{array}{c} d(Sx, Ty), \frac{k}{2}[d(Sx, Ax) + d(Sx, By)], \\ \frac{k}{2}[d(Ty, By) + d(Ty, Ax)] \end{array} \right\} \tag{4.5}$$

$$d(Ax, By) < \max \left\{ \begin{array}{c} d(P_1P_3 \ldots P_{2n-1}x, P_2P_4 \ldots P_{2n}y), \\ \frac{k}{2}[d(P_1P_3 \ldots P_{2n-1}x, Ax) + d(P_1P_3 \ldots P_{2n-1}x, By)], \\ \frac{k}{2}[d(P_2P_4 \ldots P_{2n}y, By) + d(P_2P_4 \ldots P_{2n}y, Ax)] \end{array} \right\} \tag{4.6}$$

$$d(T_\alpha x, T_\beta y) < \max\left\{\begin{array}{c} d(P_1P_3\dots P_{2n-1}x, P_2P_4\dots P_{2n}y), \\ \frac{k}{2}[d(P_1P_3\dots P_{2n-1}x, T_\alpha x) + d(P_1P_3\dots P_{2n-1}x, T_\beta y)], \\ \frac{k}{2}[d(P_2P_4\dots P_{2n}y, T_\beta y) + d(P_2P_4\dots P_{2n}y, T_\alpha x)] \end{array}\right\} \tag{4.7}$$

$$d(Ax, Ay) < \max\left\{\begin{array}{c} d(Sx, Sy), \frac{k}{2}[d(Sx, Ax) + d(Sx, Ay)], \\ \frac{k}{2}[d(Sy, Ay) + d(Sy, Ax)] \end{array}\right\} \tag{4.8}$$

Remark 4.2 Theorem 4.1, Corollaries 4.1-4.3 (in view of Remark 4.1) improve and extend the results of [18, 25, 26, 28, 29, 32, 33, 34, 35].

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