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Existence of Periodic Solutions for Nonlinear Neutral Dynamic Equations with Functional Delay on a Time Scale

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Abstract

Let $\mathbb{T}$ be a periodic time scale. The purpose of this paper is to use a modification of Krasnoselskii's fixed point theorem due to Burton to prove the existence of periodic solutions on time scale of the nonlinear dynamic equation with variable delay $x^{\triangle}(t) = -a(t)\,h(x^{\sigma}(t)) + c(t)x^{\widetilde{\triangle}}(t - r(t)) + G(t, x(t), x(t - r(t)))$, $t \in \mathbb{T}$, where $f^{\triangle}$ is the $\triangle$-derivative on $\mathbb{T}$ and $f^{\widetilde{\triangle}}$ is the $\triangle$-derivative on $(id - r)(\mathbb{T})$. We invert the given equation to obtain an equivalent integral equation from which we define a fixed point mapping written as a sum of a large contraction and a compact map. We show that such maps fit very nicely into the framework of Krasnoselskii–Burton's fixed point theorem so that the existence of periodic solutions is concluded. The results obtained here extend the work of Yankson [15].

Key words: fixed point, large contraction, periodic solutions, time scales, nonlinear neutral dynamic equations

2000 Mathematics Subject Classification: 34K13, 06E30, Secondary 34K30, 34L30

1 Introduction

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this paper, we are interested in the analysis of qualitative theory of periodic solutions of dynamic equations. Motivated by the papers [1]–[4], [7]–[13], [15] and the references therein, we consider the following totally nonlinear neutral dynamic equation with variable delay

$$x^{\triangle}(t) = -a(t)\,h(x^{\sigma}(t)) + c(t)x^{\widetilde{\triangle}}(t - r(t)) + G(t, x(t), x(t - r(t))),\ \ t \in \mathbb{T}. \tag{1.1}$$

Throughout this paper we assume that $r\colon \mathbb{T} \to \mathbb{R}$ and that $id - r\colon \mathbb{T} \to \mathbb{T}$ is strictly increasing so that the function $x\,(t - r(t))$ is well defined over $\mathbb{T}$. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [7] Theorem 3) to show the existence of periodic solutions on time scales for equation (1.1). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [7], the added term destroys a contraction already present in part of the equation but it replaces it with the so called a large contraction mapping which is suitable for fixed point theory. During the process we have to transform (1.1) into an integral equation written as a sum of two mapping; one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii fixed point theorem, to show the existence of a periodic solution for equation (1.1). In the special case $\mathbb{T} = \mathbb{R}$, Yankson in [15] shows that (1.1) has a periodic solutions by using Krasnoselskii–Burton's fixed point theorem.

In Section 2, we present some preliminary material that we will need through the remainder of the paper. We will state some facts about the exponential function on a time scale as well as the modification of Krasnoselskii's fixed point theorem established by Burton (see ([7] Theorem 3) and [8]). For details on Krasnoselskii's theorem we refer the reader to [14]. We present our main results on periodicity in Section 3. The results presented in this paper extend the main results in [15].

2 Preliminaries

The concept of time scales analysis is a fairly new idea. In 1988, it was introduced by the German mathematician Stefan Hilger in his Ph.D. thesis [11]. It combines the traditional areas of continuous and discrete analysis into one theory. After the publication of two textbooks in this area (by Bohner and Peterson, 2001, 2003, [5]–[6]), more and more researchers were getting involved in this fast-growing field of mathematics.

The study of dynamic equations brings together the traditional research areas of (ordinary and partial) differential and difference equations. It allows one to handle these two research areas at the same time, hence shedding light on the reasons for their seeming discrepancies. In fact, many new results for the continuous and discrete cases have been obtained by studying the more general time scales case (see [1]–[6], [12], [13] and the references therein).

The reader can find more details on the materials and basic properties used here in the first chapter of Bohner and Peterson book [6, p. 1–50] and can find good examples on dynamic equations in Chapter 2 of [7, p. 17–46].

We have studied dynamic nonlinear equations with functional delay on a time scale and have obtained some interesting results concerning the existence of periodic solutions (see [2]–[3]) and this work is a continuation. Here, we focus on the nonlinear dynamic equation with variable delay (1.1) which is challenging equation and, for our delight, have not yet been studied by mean of fixed point technic on time scales.

We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Atici et al. [4] and Kaufmann and Raffoul [12]. The following two definitions are borrowed from [4] and [12].

Definition 2.1 We say that a time scale $\mathbb{T}$ is periodic if there exist a $p > 0$ such that if $t \in \mathbb{T}$ then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale.

Below are examples of periodic time scales taken from [12].

Example 2.2 The following time scales are periodic.

(1) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]\,,\ h > 0$ has period $p = 2h$.

(2) $\mathbb{T} = h\mathbb{Z}$ has period $p = h$.

(3) $\mathbb{T} = \mathbb{R}$.

(4) $\mathbb{T} = \{t = k - q^m \colon k \in \mathbb{Z},\ m \in \mathbb{N}_0\}$, where $0 < q < 1$ has period $p = 1$.

Remark 2.3 [12] All periodic time scales are unbounded above and below.

Definition 2.4 Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scales with the period p. We say that the function $f \colon \mathbb{T} \to \mathbb{R}$ is periodic with period T if there exists a natural number n such that $T = np$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and T is the smallest number such that $f(t \pm T) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $T > 0$ if T is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

Remark 2.5 [12] If $\mathbb{T}$ is a periodic time scale with period p, then $\sigma(t \pm np) = \sigma(t) \pm np$. Consequently, the graininess function μ satisfies

$$\mu(t \pm np) = \sigma(t \pm np) - (t \pm np) = \sigma(t) - t = \mu(t)$$

and so, is a periodic function with period p.

Our first two theorems concern the composition of two functions. The first theorem is the chain rule on time scales ([5, Theorem 1.93]).

Theorem 2.6 (Chain Rule) *Assume* $\nu \colon \mathbb{T} \to \mathbb{R}$ *is strictly increasing and* $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ *is a time scale. Let* $\omega \colon \widetilde{\mathbb{T}} \to \mathbb{R}$. *If* $\nu^{\triangle}(t)$ *and* $\omega^{\widetilde{\triangle}}(\nu(t))$ *exist for* $t \in \mathbb{T}^k$, *then*

$$(\omega \circ \nu)^{\triangle} = \left(\omega^{\widetilde{\triangle}} \circ \nu\right) \nu^{\triangle}.$$

In the sequel we will need to differentiate and integrate functions of the form $f(t - r(t)) = f(\nu(t))$, where $\nu(t) := t - r(t)$. Our second theorem is the substitution rule ([5, Theorem 1.98]).

Theorem 2.7 (Substitution) *Assume $\nu\colon \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f\colon \mathbb{T} \to \mathbb{R}$ is rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\triangle}(t)\,\triangle t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\,\widetilde{\triangle} s.$$

A function $p\colon \mathbb{T} \to \mathbb{R}$ is said to be regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous function $f\colon \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R}$ while the set

$$\mathcal{R}^+ = \{f \in \mathcal{R}\colon 1 + \mu(t)f(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Let $p \in \mathcal{R}$ and $\mu(t) \neq 0$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t,s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log\left(1 + \mu(z)p(z)\right)\triangle z\right).$$

It is well known that if $p \in \mathcal{R}^+$, then $e_p(t,s) > 0$ for all $t \in \mathbb{T}$. Also, the exponential function $y(t) = e_p(t,s)$ is the solution to the initial value problem $y^{\triangle} = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma.

Lemma 2.8 [5] *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$;

(ii) $e_p(\sigma(t),s) = (1 + \mu(t)p(t))\,e_p(t,s)$;

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where*

$$\ominus p(t) = -\frac{p(t)}{1 + \mu(t)p(t)};$$

(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;

(vi) $e_p^{\triangle}(.,s) = pe_p(.,s)$ *and*

$$\left(\frac{1}{e_p(.,s)}\right)^{\triangle} = -\frac{p(t)}{e_p^{\sigma}(.,s)}.$$

Lemma 2.9 [1] *If $p \in \mathcal{R}^+$, then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\triangle u\right), \quad \forall t \in \mathbb{T}.$$

Corollary 2.10 [1] *If $p \in \mathcal{R}^+$ and $p(t) < 0$ for all $t \in \mathbb{T}$, then for all $s \in \mathbb{T}$ with $s \leq t$ we have*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(u)\,\triangle u\right) < 1.$$

Krasnoselskii (see [7] or [14]) combined the contraction mapping theorem and Shauder's theorem and formulated the following hybrid and attractive result.

Theorem 2.11 *Let M be a closed convex nonempty subset of a Banach space $(S, \|.\|)$. Suppose that A and B map M into S such that*

(i) $\forall x, y \in M \Rightarrow Ax + By \in M$,

(ii) A is continuous and AM is contained in a compact set,

(iii) B is a contraction with constant $\alpha < 1$.

Then there is a $z \in M$ with $z = Az + Bz$.

This is a captivating result and has a number of interesting applications. In recent year much attention has been paid to this theorem. Burton [7] observed that Krasnoselskii result can be more interesting in applications with certain changes and formulated in Theorem 2.14 below (see [7] for the proof).

Definition 2.12 Let (M, d) be a metric space and $B \colon M \to M$. B is said to be a large contraction if $\varphi, \psi \in M$, with $\varphi \neq \psi$ then $d(B\varphi, B\psi) < d(\varphi, \psi)$ and if for all $\varepsilon > 0$ there exists $\delta < 1$ such that

$$[\varphi, \psi \in M,\ d(\varphi, \psi) \geq \varepsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

Theorem 2.13 *Let (M, d) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and $L > 0$, such that $d(x, B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.*

Theorem 2.14 (Krasnoselskii–Burton) *Let M be a closed bounded convex non-empty subset of a Banach space $(S, \|.\|)$. Suppose that A, B map M into M and that*

(i) $\forall x, y \in M \Rightarrow Ax + By \in M$,

(ii) A is continuous and AM is contained in a compact subset of M,

(iii) B is a large contraction.

Then there is a $z \in M$ with $z = Az + Bz$.

It is obvious that if we want to apply the above theorem we need to construct two mappings, one is large contraction and the other is compact.

3 Existence of periodic solutions

We will state and prove our main result in this section. After we provide an example to illustrate our results. Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = np$ for some $n \in \mathbb{N}$. By the notation $[a, b]$ we mean $[a, b] = \{t \in \mathbb{T} \colon a \leq t \leq b\}$, unless otherwise specified. The intervals $[a, b)$, $(a, b]$ and (a, b) are defined similarly.

Define

$$C_T = \{\varphi \colon \mathbb{T} \to \mathbb{R} \mid \varphi \in C \text{ and } \varphi(t + T) = \varphi(t)\}$$

where C is the space of continuous real-valued functions on $\mathbb{T}$. Then $(C_T, \|.\|)$ is a Banach space with the supremum norm

$$\|\varphi\| = \sup_{t\in\mathbb{T}} |\varphi(t)| = \sup_{t\in[0,T]} |\varphi(t)| .$$

We will need the following lemma whose proof can be found in [12].

Lemma 3.1 *Let $x \in C_T$. Then $\|x^\sigma\| = \|x \circ \sigma\|$ exists and $\|x^\sigma\| = \|x\|$.*

In this paper we assume that h is continuous, $a \in \mathcal{R}^+$ is continuous, $a(t) > 0$ for all $t \in \mathbb{T}$ and

$$a(t+T) = a(t), \quad c(t+T) = c(t), \quad (id - r)(t+T) = (id - r)(t), \tag{3.1}$$

with c continuously delta-differentiable, r twice continuously delta-differentiable and id the identity function on $\mathbb{T}$. Since we are searching for periodic solutions, it is natural to ask that $G(t, x, y)$ is continuous and periodic in t and Lipschitz continuous in x and y. That is

$$G(t+T, x, y) = G(t, x, y), \tag{3.2}$$

and there are positive constants k_1, k_2 such that

$$|G(t, x, y) - G(t, z, w)| \leq k_1 \|x - z\| + k_2 \|y - w\|, \quad \text{for } x, y, z, w \in \mathbb{R}. \tag{3.3}$$

Also, we assume that for all $t \in [0, T]$,

$$r^\triangle(t) \neq 1. \tag{3.4}$$

Lemma 3.2 *Suppose (3.1), (3.2) and (3.4) hold. If $x \in C_T$, then x is a solution of equation (1.1) if and only if*

$$\begin{aligned} x(t) &= (1 - e_{\ominus a}(t, t-T))^{-1} \int_{t-T}^{t} a(s) H(x(s)) e_{\ominus a}(t, s) \triangle s \\ &+ \frac{c(t)}{1 - r^\triangle(t)} x(t - r(t)) \\ &- (1 - e_{\ominus a}(t, t-T))^{-1} \int_{t-T}^{t} R(s) x^\sigma (s - r(s)) e_{\ominus a}(t, s) \triangle s \\ &+ (1 - e_{\ominus a}(t, t-T))^{-1} \int_{t-T}^{t} G(s, x(s), x(s - r(s))) e_{\ominus a}(t, s) \triangle s, \end{aligned} \tag{3.5}$$

where

$$H(x(s)) = x^\sigma(s) - h(x^\sigma(s)), \tag{3.6}$$

$$R(s) = \frac{\left(c^\triangle(s) + c^\sigma(s) a(s)\right)\left(1 - r^\triangle(s)\right) + r^{\triangle\triangle}(s) c(s)}{(1 - r^\triangle(s))(1 - r^\triangle(\sigma(s)))}. \tag{3.7}$$

Proof Let $x \in C_T$ be a solution of (1.1). First we write this equation as

$$x^{\triangle}(t) + a(t)x^{\sigma}(t) = H(x(t)) + c(t)x^{\widetilde{\triangle}}(t - r(t)) + G(t, x(t), x(t - r(t))).$$

Multiply both sides of the above equation by $e_a(t, 0)$ and then integrate from $t - T$ to t to obtain

$$\begin{aligned}&\int_{t-T}^{t} (e_a(s,0)x(s))^{\triangle} \triangle s \\ &= \int_{t-T}^{t} a(s)H(x(s))\, e_a(s,0)\triangle s + \int_{t-T}^{t} c(s)x^{\widetilde{\triangle}}(s - r(s))\, e_a(s,0)\triangle s \\ &+ \int_{t-T}^{t} G(s, x(s), x(s - r(s)))\, e_a(s,0)\triangle s.\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&e_a(t,0)x(t) - e_a(t-T,0)x(t-T) \\ &= \int_{t-T}^{t} a(s)H(x(s))\, e_a(s,0)\triangle s + \int_{t-T}^{t} c(s)x^{\widetilde{\triangle}}(s - r(s))\, e_a(s,0)\triangle s \\ &+ \int_{t-T}^{t} G(s, x(s), x(s - r(s)))\, e_a(s,0)\triangle s.\end{aligned}$$

Divide both sides of the above equation by $e_a(t, 0)$. Since $x \in C_T$, we have

$$\begin{aligned}&x(t)(1 - e_{\ominus a}(t, t-T)) \\ &= \int_{t-T}^{t} a(s)H(x(s))\, e_{\ominus a}(s,0)\triangle s + \int_{t-T}^{t} c(s)x^{\widetilde{\triangle}}(s - r(s))\, e_{\ominus a}(t,s)\triangle s \\ &+ \int_{t-T}^{t} G(s, x(s), x(s - r(s)))\, e_{\ominus a}(t,s)\triangle s. \qquad (3.8)\end{aligned}$$

Here we have used Lemma 2.8 to simplify the exponentials. We want to pull the factor $x^{\widetilde{\triangle}}(s - r(s))$ from under the integral in (3.8). Clearly

$$\begin{aligned}&\int_{t-T}^{t} c(s)x^{\widetilde{\triangle}}(s - r(s))\, e_{\ominus a}(t,s)\triangle s \\ &= \int_{t-T}^{t} \left(1 - r^{\triangle}(s)\right) x^{\widetilde{\triangle}}(s - r(s)) \frac{c(s)}{(1 - r^{\triangle}(s))} e_{\ominus a}(t,s)\triangle s.\end{aligned}$$

Using the integration by parts formula

$$\int_{t-T}^{t} f^{\triangle}(s)g(s)\triangle s = (fg)(t) - (fg)(t-T) - \int_{t-T}^{t} f^{\sigma}(s)g^{\triangle}(s)\triangle s,$$

and by Theorems 2.6 and 2.7 we obtain

$$\begin{aligned}\int_{t-T}^{t} c(s)x^{\widetilde{\triangle}}(s - r(s))\, e_{\ominus a}(t,s)\triangle s &= \frac{c(t)}{1 - r^{\triangle}(t)} x(t - r(t))(1 - e_{\ominus a}(t, t-T)) \\ &- \int_{t-T}^{t} R(s)x^{\sigma}(s - r(s))\, e_{\ominus a}(t,s)\triangle s, \qquad (3.9)\end{aligned}$$

where R is given by (3.7). We obtain (3.5) by substituting (3.9) in (3.8). Since each step is reversible, the converse follows easily. This completes the proof. □

To apply Theorem 2.14, we need to define a Banach space $\mathbb{B}$, a closed bounded convex subset M_L of $\mathbb{B}$ and construct two mappings, one is a large contraction and the other is compact. So, we let $(\mathbb{B}, \|.\|) = (C_T, \|.\|)$ and $M_L = \{\varphi \in \mathbb{B} \colon \|\varphi\| \leq L,\ \varphi^{\triangle} \text{ is bounded}\}$, where L is positive constant. We express equation (3.5) as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) = (\mathbb{C}\varphi)(t),$$

where $A, B \colon M \to \mathbb{B}$ are defined by

$$\begin{aligned}(A\varphi)(t) &= \frac{c(t)}{1-r^{\triangle}(t)}\varphi(t-r(t)) \\ &\quad - \left(1-e_{\ominus a}(t,t-T)\right)^{-1}\int_{t-T}^{t} R(s)\varphi^{\sigma}(s-r(s))\, e_{\ominus a}(t,s)\triangle s \\ &\quad + \left(1-e_{\ominus a}(t,t-T)\right)^{-1}\int_{t-T}^{t} G(s,\varphi(s),\varphi(s-r(s)))\, e_{\ominus a}(t,s)\triangle s, \end{aligned} \tag{3.10}$$

and

$$(B\varphi)(t) = \left(1-e_{\ominus a}(t,t-T)\right)^{-1}\int_{t-T}^{t} a(s)H(\varphi(s))\, e_{\ominus a}(t,s)\triangle s. \tag{3.11}$$

We need the following assumptions

$$(k_1+k_2)L + |G(t,0,0)| \leq \beta L a(t), \tag{3.12}$$

$$|R(t)| \leq \delta L a(t), \tag{3.13}$$

$$\max_{t\in[0,T]}\left|\frac{c(t)}{1-r^{\triangle}(t)}\right| = \alpha, \tag{3.14}$$

$$J(\alpha+\beta+\delta) \leq 1, \tag{3.15}$$

$$\max(|H(-L)|, |H(L)|) \leq \frac{(J-1)L}{J}, \tag{3.16}$$

where α, β, δ and J are constants with $J \geq 3$.

We begin with the following proposition (see [1]) and for convenience we present, below, its proof. In the next proposition we prove that, for a well chosen function h, the mapping H in (3.6) is a large contraction on M_L. So, let us make the following assumptions on the function $h \colon \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous on $U_L = [-L, L]$ and differentiable on $(-L, L)$.

(H2) h is strictly increasing on U_L.

(H3) $\sup_{s\in(-L,L)} h'(s) \leq 1$.

Proposition 3.3 *Let $h \colon \mathbb{R} \to \mathbb{R}$ be a function satisfying (H1)–(H3). Then the mapping H in (3.6) is a large contraction on the set M_L.*

Proof Let $\phi, \varphi \in M_L$ with $\phi^\sigma \neq \varphi^\sigma$. Then $\phi^\sigma(t) \neq \varphi^\sigma(t)$ for some $t \in \mathbb{T}$. Define the set

$$D(\phi, \varphi) = \{t \in \mathbb{T}\colon \phi^\sigma(t) \neq \varphi^\sigma(t)\}.$$

Note that $\varphi^\sigma(t) \in U_L$ for all $t \in \mathbb{T}$ whenever $\varphi \in M_L$. Since h is strictly increasing

$$\frac{h(\varphi^\sigma(t)) - h(\phi^\sigma(t))}{\varphi^\sigma(t) - \phi^\sigma(t)} = \frac{h(\phi^\sigma(t)) - h(\varphi^\sigma(t))}{\phi^\sigma(t) - \varphi^\sigma(t)} > 0, \tag{3.17}$$

holds for all $t \in D(\phi, \varphi)$. On the other hand, for all $t \in D(\phi, \varphi)$, we have

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &= |\phi^\sigma(t) - h(\phi^\sigma(t)) - \varphi^\sigma(t) + h(\varphi^\sigma(t))| \\ &= |\phi^\sigma(t) - \varphi^\sigma(t)| \left|1 - \left(\frac{h(\phi^\sigma(t)) - h(\varphi^\sigma(t))}{\phi^\sigma(t) - \varphi^\sigma(t)}\right)\right|. \end{aligned} \tag{3.18}$$

For each fixed $t \in D(\phi, \varphi)$, define the set $U_t \subset U_L$ by

$$U_t = \begin{cases} (\varphi^\sigma(t), \phi^\sigma(t)), & \text{if } \phi^\sigma(t) > \varphi^\sigma(t), \\ (\phi^\sigma(t), \varphi^\sigma(t)), & \text{if } \varphi^\sigma(t) > \phi^\sigma(t), \end{cases} \quad \text{for } t \in D(\phi, \varphi).$$

The mean value theorem implies that for each fixed $t \in D(\phi, \varphi)$ there exists a real number $c_t \in U_t$ such that

$$\frac{h(\phi^\sigma(t)) - h(\varphi^\sigma(t))}{\phi^\sigma(t) - \varphi^\sigma(t)} = h'(c_t).$$

By (H2) and (H3), we have

$$1 \geq \sup_{t \in (-L, L)} h'(t) \geq \sup_{t \in U_t} h'(t) \geq h'(c_t) \geq \inf_{s \in U_t} h'(s) \geq \inf_{t \in (-L, L)} h'(t) \geq 0. \tag{3.19}$$

Consequently, by (3.17)–(3.19), we obtain

$$|(H\phi)(t) - (H\varphi)(t)| \leq \left|1 - \inf_{u \in (-L, L)} h'(u)\right| |\phi^\sigma(t) - \varphi^\sigma(t)|, \tag{3.20}$$

for all $t \in D(\phi, \varphi)$. Hence, the mapping H is a large contraction in the supremum norm. Indeed, fix $\epsilon \in (0,1)$ and assume that ϕ and φ are two functions in M_L satisfying

$$\|\phi - \varphi\| = \sup_{t \in D(\phi, \varphi)} |\phi(t) - \varphi(t)| \geq \epsilon.$$

If $|\phi^\sigma(t) - \varphi^\sigma(t)| \leq \epsilon/2$ for some $t \in D(\phi, \varphi)$, then from (3.19) and (3.20), we get

$$|(H\phi)(t) - (H\varphi)(t)| \leq |\phi^\sigma(t) - \varphi^\sigma(t)| \leq \frac{1}{2}\|\phi - \varphi\|. \tag{3.21}$$

Since h is continuous and strictly increasing, the function $h\left(u + \frac{\epsilon}{2}\right) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if

$$\frac{\epsilon}{2} < |\phi^\sigma(t) - \varphi^\sigma(t)| \quad \text{for some } t \in D(\phi, \varphi),$$

then from (H2) and (H3) we conclude that

$$1 \geq \frac{h(\phi^{\sigma}(t)) - h(\varphi^{\sigma}(t))}{\phi^{\sigma}(t) - \varphi^{\sigma}(t)} > \lambda,$$

where,

$$\lambda = \frac{1}{2L} \min \left\{ h\left(u + \frac{\epsilon}{2}\right) - h(u),\, u \in [-L, L] \right\} > 0.$$

Therefore, from (3.18), we have

$$|(H\phi)(t) - (H\varphi)(t)| \leq (1-\lambda) \|\phi - \varphi\|. \tag{3.22}$$

Consequently, it follows from (3.21) and (3.22) that

$$|(H\phi)(t) - (H\varphi)(t)| \leq \eta \|\phi - \varphi\|,$$

where

$$\eta = \max \left\{ \frac{1}{2}, 1 - \lambda \right\} < 1.$$

The proof is complete. □

We shall prove that the mapping $\mathbb{C}$ has a fixed point which solves (1.1), whenever its derivative exists.

Lemma 3.4 *For A defined in (3.10), suppose that (3.1)–(3.4) and (3.12)–(3.15) hold. Then $A\colon M_L \to M_L$ is continuous in the supremum norm and maps M_L into a compact subset of M_L.*

Proof We first show that $A\colon M_L \to M_L$. Clearly, if φ is continuous, then $A\varphi$ is. Evaluating (3.10) at $t + T$ gives

$$\begin{aligned} (A\varphi)(t+T) &= \frac{c(t+T)}{1 - r^{\triangle}(t+T)} \varphi(t+T-r(t+T)) \\ &- (1 - e_{\ominus a}(t+T,t))^{-1} \int_t^{t+T} R(s)\varphi^{\sigma}(s - r(s)) e_{\ominus a}(t+T,s) \triangle s \\ &+ (1 - e_{\ominus a}(t+T,t))^{-1} \int_t^{t+T} G(s, \varphi(s), \varphi(s-r(s))) e_{\ominus a}(t+T,s) \triangle s. \end{aligned} \tag{3.23}$$

Use Theorem 2.7 with $u = s - T$ and conditions (3.1) and (3.2) to get

$$(A\varphi)(t+T) = \frac{c(t)}{1 - r^{\triangle}(t)} \varphi(t - r(t))$$

$$- (1 - e_{\ominus a}(t+T,t))^{-1} \int_{t-T}^{t} h(u+T)\varphi^{\sigma}(u+T-r(u+T)) e_{\ominus a}(t+T,u+T) \triangle u$$

$$+ (1 - e_{\ominus a}(t+T,t))^{-1}$$

$$\times \int_{t-T}^{t} G(u+T, \varphi(u+T), \varphi(u+T-r(u+T))) e_{\ominus a}(t+T,u+T) \triangle u.$$

From Theorem 2.7, we have

$$\sigma(u+T-r(u+T)) = \sigma(u-r(u))+T,\ e_{\ominus a}(t+T,u+T) = e_{\ominus a}(t,u),$$
$$e_{\ominus a}(t+T,t) = e_{\ominus a}(t,t-T).$$

Thus (3.23) becomes

$$\begin{aligned}(A\varphi)(t+T) &= \frac{c(t)}{1-r^{\triangle}(t)}\varphi(t-r(t))\\ &\quad -(1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} R(u)\varphi^{\sigma}(u-r(u))e_{\ominus a}(t,u)\triangle u\\ &+(1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} G(u,\varphi(u),\varphi(u-r(u)))e_{\ominus a}(t,u)\triangle u = (A\varphi)(t).\end{aligned}$$

That is, $A\colon C_T \to C_T$. In view of (3.3) we arrive at

$$|G(t,x,y)| \le |G(t,x,y)-G(t,0,0)| + |G(t,0,0)| \le k_3\|x\| + k_4\|y\| + |G(t,0,0)|.$$

Note that from Corollary 2.10, we have

$$1-e_{\ominus a}(t,t-T) > 0.$$

So, for any $\varphi \in M_L$, we have

$$\begin{aligned}|(A\varphi)(t)| &\le \left|\frac{c(t)}{1-r^{\triangle}(t)}\right| |\varphi(t-r(t))|\\ &\quad + (1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} |R(s)|\,|\varphi^{\sigma}(u-r(s))|\, e_{\ominus a}(t,s)\triangle s\\ &\quad + (1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} |G(s,\varphi(s),\varphi(s-r(s)))|\, e_{\ominus a}(t,s)\triangle s\\ &\le \alpha L + (1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} \delta a(s) L e_{\ominus a}(t,s)\triangle s\\ &\quad + (1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} \left((k_1+k_2)\,L + |G(s,0,0)|\right) e_{\ominus a}(t,s)\triangle s\\ &\le \alpha L + \delta L\,(1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} a(s)e_{\ominus a}(t,s)\triangle s\\ &\quad + \beta L\,(1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} a(s)e_{\ominus a}(t,s)\triangle s\\ &= (\alpha+\beta+\delta)\,L \le \frac{L}{J} < L.\end{aligned}$$

Thus, $A\varphi \in M_L$. Consequently, we have $A\colon M_L \to M_L$.

We show that A is continuous in the supremum norm. Toward this, let $\varphi, \psi \in M_L$, and let

$$\begin{aligned}
\alpha' &= \max_{t\in[0,T]} \left(1 - e_{\ominus a}(t, t-T)\right)^{-1}, \\
\beta' &= \max_{t\in[t-T,t]} \left\{e_{\ominus a}(t,s)\right\}, \\
\rho &= \max_{t\in[0,T]} |G(t,0,0)|, \\
\gamma &= \max_{t\in[0,T]} \left\{a(t)\right\}, \\
\mu &= \max_{t\in[0,T]} \left|\frac{c^{\triangle}(t)}{1 - r^{\triangle}(\sigma(t))}\right|, \\
\mu' &= \max_{t\in[0,T]} \left|\frac{c^{\sigma}(t)}{1 - r^{\triangle}(\sigma(t))}\right|, \\
\upsilon &= \max_{t\in[0,T]} \left|\frac{r^{\triangle\triangle}(t)c(t)}{\left(1 - r^{\triangle}(t)\right)\left(1 - r^{\triangle}(\sigma(t))\right)}\right|.
\end{aligned} \tag{3.24}$$

Note that from $a(t) > 0$ we have $\max_{s\in[t-T,t]} \left\{e_{\ominus a}(t,s)\right\} \leq 1$. So,

$$\begin{aligned}
|(A\varphi)(t) - (A\psi)(t)| &\leq \left|\frac{c(t)}{1 - r^{\triangle}(t)}\right| |\varphi(t - r(t)) - \psi(t - r(t))| \\
&\quad + \left(1 - e_{\ominus a}(t, t-T)\right)^{-1} \\
&\quad \times \int_{t-T}^{t} |h(s)| \left|Q^{\sigma}(\varphi(s - r(s))) - Q^{\sigma}(\psi(s - r(s)))\right| e_{\ominus a}(t,s)\triangle s \\
&\quad + \left(1 - e_{\ominus a}(t, t-T)\right)^{-1} \\
&\quad \times \int_{t-T}^{t} \left|G(s, \varphi(s), \varphi(s - r(s))) - G(s, \psi(s), \psi(s - r(s)))\right| e_{\ominus a}(t,s)\triangle s \\
&\leq \alpha\|\varphi - \psi\| + \delta\|\varphi - \psi\| \left(1 - e_{\ominus a}(t, t-T)\right)^{-1} \int_{t-T}^{t} a(s)e_{\ominus a}(t,s)\triangle s \\
&\quad + (k_1 + k_2)\|\varphi - \psi\| \left(1 - e_{\ominus a}(t, t-T)\right)^{-1} \int_{t-T}^{t} e_{\ominus a}(t,s)\triangle s \\
&\leq \left(\alpha + \delta + (k_1 + k_2) T\alpha'\beta'\right) \|\varphi - \psi\|.
\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. Define $\theta = \epsilon/K$ with $K = \alpha + \delta + (k_1 + k_2)T\alpha'\beta'$, where k_1 and k_2 are given by (3.3). Then, for $\|\varphi - \psi\| < \theta$ we obtain

$$\|A\varphi - A\psi\| \leq K\|\varphi - \psi\| < \epsilon.$$

This proves that A is continuous.

It remains to show that A is compact. Let $\varphi_n \in M_L$, where n is a positive integer. Then, as above, we see that

$$\|A\varphi_n\| \leq L.$$

Moreover, a direct calculation shows that

$$(A\varphi_n)^{\triangle}(t) =$$
$$= \frac{c^{\triangle}(t)\varphi_n(t-r(t)) + c^{\sigma}(t)\varphi_n^{\triangle}(t-r(t))}{1-r^{\triangle}(\sigma(t))} + \frac{r^{\triangle\triangle}(t)c(t)\varphi_n(t-r(t))}{(1-r^{\triangle}(t))\,(1-r^{\triangle}(\sigma(t)))}$$
$$- R(t)\varphi_n^{\sigma}(t-r(t)) + G\,(t, \varphi_n(t), \varphi_n(t-r(t)))$$
$$+ a(t)\Bigg[(1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} R(s)\varphi_n^{\sigma}(s-r(s))e_{\ominus a}(t,s)\triangle s\Bigg]^{\sigma}$$
$$- a(t)\Bigg[(1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} G(s,\varphi_n(s),\varphi_n(s-r(s)))e_{\ominus a}(t,s)\triangle s\Bigg]^{\sigma}.$$

Let L' be the norm bound of $\varphi^{\triangle}$. By invoking (3.3), (3.24) and Lemma 3.1 we obtain

$$\begin{aligned}\left|(A\varphi_n)^{\triangle}(t)\right| &\le \mu L + \mu' L' + \upsilon L + \delta\gamma L + (k_1+k_2)L + \rho \\ &\quad + \gamma^2\alpha' T\delta\beta' L + \gamma\alpha' T\,[(k_1+k_2)L+\rho]\,\beta' \\ &\le (1+\gamma\alpha'\beta' T)\,[(k_1+k_2)L+\rho] \\ &\quad + [\mu+\upsilon+\delta\gamma\,(1+\gamma\alpha'\beta' T)]\,L + \mu' L' \le D,\end{aligned}$$

for some positive constant D. Hence the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli–Arzela theorem implies that a subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly to a continuous T-periodic function. Thus A is continuous and AM_L is contained in a compact subset of M_L. □

Lemma 3.5 *Let B be defined by (3.11) and that (H1)–(H3), (3.1) and (3.16) hold. Then $B\colon M_L \to M_L$ is a large contraction.*

Proof We first show that $B\colon M_L \to M_L$. Clearly, if φ is continuous, then $B\varphi$ is. Evaluate (3.11) at $t+T$ to have

$$(B\varphi)(t+T) = (1-e_{\ominus a}(t+T,t))^{-1}\int_{t}^{t+T} a(s)H(\varphi(s))e_{\ominus a}(t+T,s)\triangle s. \quad (3.25)$$

Use Theorem 2.7 with $u = s-T$ and condition (3.1) to get

$$(B\varphi)(t+T) =$$
$$= (1-e_{\ominus a}(t+T,t))^{-1}\int_{t-T}^{t} a(u)H(\varphi(u+T))e_{\ominus a}(t+T,u+T)\triangle u.$$

From Theorem 2.7, we have $\sigma(u+T) = \sigma(u)+T$, $e_{\ominus a}(t+T,u+T) = e_{\ominus a}(t,u)$ and $e_{\ominus a}(t+T,t) = e_{\ominus a}(t,t-T)$. Thus (3.25) becomes

$$(B\varphi)(t+T) =$$
$$= (1-e_{\ominus a}(t,t-T))^{-1}\int_{t-T}^{t} a(u)H(\varphi(u))e_{\ominus a}(t,u)\triangle u = (B\varphi)(t).$$

That is, $B\colon C_T \to C_T$. Note that from Corollary 2.10, we have

$$1 - e_{\ominus a}(t, t-T) > 0.$$

So, for any $\varphi \in M_L$, we get by (3.11) that

$$\begin{aligned}|(B\varphi)(t)| &\leq (1 - e_{\ominus a}(t,t-T))^{-1} \int_{t-T}^{t} a(s)\,|H(\varphi(s))|\,e_{\ominus a}(t,s)\triangle s \\ &\leq \max\left(|H(-L)|, |H(L)|\right)(1 - e_{\ominus a}(t,t-T))^{-1} \int_{t-T}^{t} a(s)e_{\ominus a}(t,s)\triangle s \\ &\leq \frac{(J-1)L}{J} < L.\end{aligned}$$

Thus $B\varphi \in M_L$. Consequently, we have $B\colon M_L \to M_L$.

It remains to show that B is large contraction with a unique fixed point in M_L. Form the proof of Proposition 3.3 we have for $\phi, \varphi \in M_L$, with $\phi \neq \varphi$

$$\begin{aligned}&|(B\phi)(t) - (B\varphi)(t)| \\ &\leq (1 - e_{\ominus a}(t,t-T))^{-1} \int_{t-T}^{t} a(s)\,|H(\phi(s)) - H(\varphi(s))|\,e_{\ominus a}(t,s)\triangle s \\ &\leq \|\phi - \varphi\|\,(1 - e_{\ominus a}(t,t-T))^{-1} \int_{t-T}^{t} a(s)e_{\ominus a}(t,s)\triangle s = \|\phi - \varphi\|.\end{aligned}$$

Then $\|B\phi - B\varphi\| \leq \|\phi - \varphi\|$. Now, let $\epsilon \in (0,1)$ be given and let $\phi, \varphi \in M_L$ with $\|\phi - \varphi\| \geq \epsilon$. From the proof of the proposition 3.3, we have found a $\eta < 1$, such that

$$\begin{aligned}&|(B\varphi)(t) - (B\psi)(t)| \\ &\leq (1 - e_{\ominus a}(t,t-T))^{-1} \int_{t-T}^{t} a(s)\,|H(\phi(s)) - H(\varphi(s))|\,e_{\ominus a}(t,s)\triangle s \\ &\leq \eta\|\phi - \varphi\|\,(1 - e_{\ominus a}(t,t-T))^{-1} \int_{t-T}^{t} a(s)e_{\ominus a}(t,s)\triangle s = \eta\|\varphi - \psi\|.\end{aligned}$$

Then $\|B\phi - B\varphi\| \leq \eta\,\|\varphi - \psi\|$. Consequently, B is a large contraction on M_L. □

Theorem 3.6 *Let $(C_T, \|.\|)$ be the Banach space of continuous T-periodic real valued functions on $\mathbb{T}$ and $M_L = \left\{\varphi \in C_T\colon \|\varphi\| \leq L, \varphi^{\triangle} \text{ is bounded}\right\}$, where L is positive constant. Suppose (H1)–(H3), (3.1)–(3.4) and (3.12)–(3.16) hold. Then equation (1.1) has a T-periodic solution φ in the subset M_L.*

Proof By Lemma 3.4, $A\colon M_L \to M_L$ is continuous and AM_L is contained in a compact set. Also, from Lemma 3.5, the mapping $B\colon M_L \to M_L$ is a large contraction. Next, note that if $\phi, \varphi \in M_L$, we have

$$\|A\phi + B\varphi\| \leq \|A\phi\| + \|B\varphi\| \leq \frac{L}{J} + \frac{(J-1)\,L}{J} = L.$$

Thus $A\phi + B\varphi \in M_L$. Clearly, all the hypotheses of the Krasnoselskii–Burton's theorem (Theorem 2.14) are satisfied. Thus there exists a fixed point $\varphi \in M_L$ such that $\varphi = A\varphi + B\varphi$. Hence the equation (1.1) has a T-periodic solution in M_L. □

References

[1] Adıvar, M., Raffoul, Y. N.: *Existence of periodic solutions in totally nonlinear delay dynamic equations.* Electronic Journal of Qualitative Theory of Differential Equations **2009**, 1 (2009), 1–20.

[2] Ardjouni, A., Djoudi, A.: *Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale.* Commun Nonlinear Sci Numer Simulat **17** (2012), 3061–3069.

[3] Ardjouni, A., Djoudi, A.: *Periodic solutions in totally nonlinear dynamic equations with functional delay on a time scale.* Rend. Sem. Mat. Univ. Politec. Torino **68**, 4 (2010), 349–359.

[4] Atici, F. M., Guseinov, G. Sh., Kaymakcalan, B.: *Stability criteria for dynamic equations on time scales with periodic coefficients.* In: Proceedings of the International Confernce on Dynamic Systems and Applications **3**, 3 (1999), 43–48.

[5] Bohner, M., Peterson, A.: Dynamic Equations on Time Scales, An Introduction with Applications. *Birkhäuser*, Boston, 2001.

[6] Bohner, M., Peterson, A.: Advances in Dynamic Equations on Time Scales. *Birkhäuser*, Boston, 2003.

[7] Burton, T. A.: *Liapunov functionals, fixed points and stability by Krasnoselskii's theorem.* Nonlinear Stud. **9**, 2 (2002), 181–190.

[8] Burton, T. A.: Stability by Fixed Point Theory for Functional Differential Equations. *Dover*, New York, 2006.

[9] Deham, H., Djoudi, A.: *Periodic solutions for nonlinear differential equation with functional delay.* Georgian Mathematical Journal **15**, 4 (2008), 635–642.

[10] Deham, H., Djoudi, A.: *Existence of periodic solutions for neutral nonlinear differential equations with variable delay.* Electronic Journal of Differential Equations **127**, (2010), 1–8.

[11] Hilger, S.: *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten.* Ph. D. thesis, *Universität Würzburg*, Würzburg, 1988.

[12] Kaufmann, E. R., Raffoul, Y. N.: *Periodic solutions for a neutral nonlinear dynamical equation on a time scale.* J. Math. Anal. Appl. **319 32**, 1 (2006), 315–325.

[13] Kaufmann, E. R., Raffoul, Y. N.: *Periodicity and stability in neutral nonlinear dynamic equation with functional delay on a time scale.* Electronic Journal of Differential Equations **27** (2007), 1–12.

[14] Smart, D. R.: *Fixed point theorems.* Cambridge Tracts in Mathematics **66**, *Cambridge University Press*, London–New York, 1974.

[15] Yankson, E.: *Existence of periodic solutions for totally nonlinear neutral differential equations with functional delay.* Opuscula Mathematica **32**, 3 (2012), 617–627.