Johannes Schleischitz

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Abstract. This paper introduces some methods to determine the simultaneous approximation constants of a class of well approximable numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$. The approach relies on results on the connection between the set of all s-adic expansions ($s \geq 2$) of $\zeta_1, \zeta_2, \ldots, \zeta_k$ and their associated approximation constants. As an application, explicit construction of real numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ with prescribed approximation properties are deduced and illustrated by Matlab plots.

1 Introduction

1.1 Basic facts and notations

This paper deals with the one parameter simultaneous approximation problem

$$\begin{aligned} |x| &\leq Q^{1+\theta} \\ |\zeta_1 x - y_1| &\leq Q^{-\frac{1}{k}+\theta} \\ &\vdots \\ |\zeta_k x - y_k| &\leq Q^{-\frac{1}{k}+\theta}, \end{aligned} \tag{1}$$

where $\zeta_1, \zeta_2, \ldots, \zeta_k$ are real numbers which we will assume to be linearly independent together with 1 and $x, y_1, y_2, \ldots, y_k$ are integers to be determined in dependence of the parameter $Q > 1$ in order to minimize θ. To be more precise, we define the function $\psi_j(Q)$ for $1 \leq j \leq k+1$ by setting $\psi_j(Q)$ the minimum over all $\theta \in \mathbb{R}$ such that there are j linearly independent vectors $(x, y_1, y_2, \ldots, y_k) \in \mathbb{Z}^{k+1}$ that satisfy the system (1). In the sequel we will restrict to approximation vectors with $x > 0$, which clearly is no loss of generality as $(x, y_1, \ldots, y_k) \mapsto (-x, -y_1, \ldots, -y_k)$ does not affect approximation constants. Another equivalent way to view the functions ψ_j is to consider the lattice $\Lambda = \{(x, \zeta_1 x - y_1, \ldots, \zeta_k x - y_k) : x, y_1, \ldots, y_k \in \mathbb{Z}\}$

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and the convex body (in fact the parallelepiped) $K(Q)$ defined as the set of points $(z_1, z_2, \dots, z_{k+1}) \in \mathbb{R}^{k+1}$ with

$$|z_1| \le Q \tag{2}$$

$$|z_i| \le Q^{-\frac{1}{k}}, \qquad 2 \le i \le k+1\,, \tag{3}$$

and to define $\lambda_j(Q)$ as the j-th *successive minimum* of Λ with respect to $K(Q)$. This j-th minimum is defined as the infimum over all $\lambda > 0$ for which the $\mathbb{R}$-span of $\lambda K(Q) \cap \Lambda$ has dimension at least j, or equivalently $\lambda K(Q)$ contains j linearly independent points of Λ.

With respect to these successive minima λ_j, the functions $\psi_j(Q)$ can also be determined by

$$Q^{\psi_j(Q)} = \lambda_j(Q)\,.$$

One has the inequalities

$$-1 \le \psi_1(Q) \le \psi_2(Q) \le \cdots \le \psi_{k+1}(Q) \le \frac{1}{k} \tag{4}$$

as we will show later, and Dirichlet's Theorem states

$$\psi_1(Q) < 0 \qquad \text{for all} \quad Q > 1\,. \tag{5}$$

Minkowski's second convex body theorem yields for any convex body K with volume $V(K)$ and any lattice Λ

$$\frac{2^{k+1}}{(k+1)!}\frac{\det(\Lambda)}{V(K)} \le \lambda_1\lambda_2\cdots\lambda_{k+1} \le 2^{k+1}\frac{\det(\Lambda)}{V(K)}\,,$$

see [1], so that in our special case, as $V(K(Q)) = 1$ for every Q, we have

$$c_1(\Lambda) \le \lambda_1(Q)\lambda_2(Q)\cdots\lambda_{k+1}(Q) \le c_2(\Lambda)$$

uniformly in the parameter Q. With $q := \log(Q)$ and taking logarithms, this yields

$$q\left|\sum_{i=1}^{k+1}\psi_i(Q)\right| \le C(\Lambda)\,, \tag{6}$$

with some constant $C(\Lambda)$ not depending on Q.

Another important property of the joint behaviour of the functions ψ_j is that for any given $1 \le s \le k$ there are arbitrarily large values $Q = Q(s)$ such that

$$\psi_s(Q) = \psi_{s+1}(Q) \tag{7}$$

provided that $1, \zeta_1, \zeta_2, \dots, \zeta_k$ are linearly independent over $\mathbb{Q}$, see Theorem 1.1 in [5]. To quantify the behaviour of $\psi_j(Q)$ Summerer and Schmidt introduced the quantities

$$\underline{\psi}_i := \liminf_{Q\to\infty}\psi_i(Q)\,, \qquad \overline{\psi}_i := \limsup_{Q\to\infty}\psi_i(Q)\,,$$

and gave the estimates

$$\underline{\psi}_j \geq \frac{j-k-1}{kj}\,, \qquad 1 \leq j \leq k+1 \tag{8}$$

$$\overline{\psi}_j \geq \frac{j-k}{k(j+1)}\,, \qquad 1 \leq j \leq k\,, \tag{9}$$

where (9) requires $1, \zeta_1, \zeta_2, \ldots, \zeta_k$ to be linearly independent over $\mathbb{Q}$ again. Each of these bounds will be shown to be best possible in Corollary 3. Moreover, (7) implies

$$\underline{\psi}_{i+1} \leq \overline{\psi}_i\,, \qquad 1 \leq i \leq k\,. \tag{10}$$

In order to study the dynamical behaviour of the functions $\psi_j(Q)$ it will be convenient to work with functions

$$L_j(q) = q\psi_j(Q)$$

as these functions are piecewise linear with slopes among $\{-1, \frac{1}{k}\}$. Therefore we have (4). Definition (6) is equivalent to

$$\left|\sum_{i=1}^{k+1} L_i(q)\right| \leq C(\Lambda)\,. \tag{11}$$

We also introduce the classical approximation constants $\omega_j, \widehat{\omega}_j$ defined by Jarník, Bugeaud in addition to $\underline{\psi}_j, \overline{\psi}_j$. For fixed $\zeta_1, \zeta_2, \ldots, \zeta_k$ and for every $X > 0$ define the functions $\omega_j(X)$ as the supremum over all real numbers ν (in fact the maximum) such that the system

$$|x| \leq X\,, \qquad |\zeta_i x - y_i| \leq X^{-\nu}, \qquad 1 \leq i \leq k\,, \tag{12}$$

has j linearly independent solutions $(x, y_1, \ldots, y_k) \in \mathbb{Z}^{k+1}$. The approximation constants $\omega_j, \widehat{\omega}_j$ are now defined as

$$\omega_j = \limsup_{X\to\infty} \omega_j(X)\,, \qquad \widehat{\omega}_j = \liminf_{X\to\infty} \omega_j(X)\,.$$

We will put $\omega := \omega_1, \widehat{\omega} := \widehat{\omega}_1$ and denote by $\Omega = (\omega, \omega_2, \ldots, \omega_{k+1}, \widehat{\omega}, \ldots, \widehat{\omega}_{k+1}) \in \mathbb{R}^{2k+2}$ the vector of classical approximation constants (relative to $\zeta_1, \zeta_2, \ldots, \zeta_k$). Very similarly to the proof of Theorem 1.4 in [5], which treats the special case $j = 1$, one obtains

$$(1+\omega_j)(1+\underline{\psi}_j) = (1+\widehat{\omega}_j)(1+\overline{\psi}_j) = \frac{k+1}{k}\,, \qquad 1 \leq j \leq k+1\,. \tag{13}$$

One just needs to replace "a solution" by "j linearly independent solutions" at any place it occurs in the proof. Combining (13) with (8), (9) for $1, \zeta_1, \zeta_2, \ldots, \zeta_k$ linearly independent over $\mathbb{Q}$ we obtain the bounds

$$\frac{1}{k} \leq \omega \leq \infty\,, \tag{14}$$

$$\frac{1}{k} \leq \omega_2 \leq 1\,, \tag{15}$$

$$0 \leq \omega_j \leq \frac{1}{j-1}\,, \qquad 3 \leq j \leq k+1 \tag{16}$$

for the constants ω_j as well as

$$\frac{1}{k} \leq \widehat{\omega} \leq 1\,, \tag{17}$$

$$0 \leq \widehat{\omega}_j \leq \frac{1}{j}\,, \qquad 2 \leq j \leq k\,, \tag{18}$$

$$0 \leq \widehat{\omega}_{k+1} \leq \frac{1}{k} \tag{19}$$

for the constants $\widehat{\omega}_j$. Each considered individually, these bounds again are best possible.

1.2 Outline of the results

In the present paper, we will put our focus on simultaneous approximation of numbers that allow good individual as well as simultaneous approximation. Liouville numbers, that is real numbers ζ for which the inequality

$$\left|\zeta - \frac{p}{q}\right| \leq \frac{1}{q^\eta}$$

has infinitely many rational solutions $\frac{p}{q}$ for arbitrarily large $\eta \in \mathbb{R}$, will be suitable examples since they all satisfy $\omega = \infty$, where $\omega = \omega_1$ is defined by (12) in the one-dimensional case.

In section 2, Propositions 1, 2, we establish a connection between the s-adic expansions ($s \geq 2$) of the components ζ_j of $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ and the approximation constants $\omega, \widehat{\omega}$. These results are then applied to the case where all ζ_j admit good approximations in one fixed base s independent of j. After these considerations for suitable arbitrary $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ we put our focus on Liouville numbers, using heavily the fact that $\omega = \infty$ in this case. Theorem 1 will allow to compute all classical approximation constants $\omega_j, \widehat{\omega}_j$ for a special type of Liouville numbers and the resulting Corollary 2 will lead us to the construction of vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ with prescribed approximation constants $\omega_j, \widehat{\omega}_j$ that are subject to certain restrictions. As consequences of these results we will be able to give an explicit example of a vector $\zeta_1, \zeta_2, \ldots, \zeta_k$ that shows a conjecture by Wolfgang Schmidt concerning successive minima of a lattice to be true. A non-constructive proof was given by Moshchevitin in a non-constructive way. Moreover we will construct cases where all functions ψ_j simultaneously take all possible values of their spectrum for arbitrarily large Q.

Inspired by methods used to deal with Liouville numbers, we then generalize Theorem 1 to a wider class of vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ for which $\omega < \infty$. This will be the subject of Theorems 2,3 and lead to many more explicit constructions of special cases of the Schmidt Conjecture.

In the last section we will first discuss the special case where $\underline{\psi}_{j+1} = \overline{\psi}_j$ for $1 \leq j \leq k$ and give a constructive existence proof for the degenerate case $\underline{\psi}_1 = -1$ in arbitrary dimension. Throughout the paper we will illustrate the derived results by Matlab plots of the functions L_j for the special cases we consider to visualize derived results. These plots shall also lead to some insight into the dynamical

behaviour of these functions in general. One should mention at this point that the plots often seem curved although the functions are piecewise linear, which is due to the non sufficient digital resolution, i.e. by zooming in one can see that they are indeed piecewise linear.

2 Results for Approximation constants

2.1 Estimates for $\omega, \widehat{\omega}$

In the sequel let $s \geq 2$ be an integer and $\zeta_i \in (0,1)$ for $1 \leq i \leq k$. For each $1 \leq i \leq k$ the non vanishing digits of the s-adic expansions of such ζ_i and $1-\zeta_i$ define two sequences $(a_n^{i,(s)})_{n\geq 1}$ and $(a_n'^{i,(s)})_{n\geq 1}$ by

$$\zeta_i = \sum_{n\geq 1} \alpha_{n,i}^{(s)} s^{-a_n^{i,(s)}}, \qquad a_1^{i,(s)} < a_2^{i,(s)} < \ldots, \quad 0 < \alpha_{n,i} \leq s-1 \tag{20}$$

$$1-\zeta_i = \sum_{n\geq 1} \beta_{n,i}'^{(s)} s^{-a_n'^{i,(s)}}, \qquad a_1'^{(s)} < a_2'^{(s)} < \ldots, \quad 0 < \beta_{n,i}' \leq s-1\,. \tag{21}$$

We call the sequence $a_{n,i}'^{(s)}$ the *dual expansion* of ζ_i in base s. Set $(b_n^{(s)})_{n\geq 1}$ the monotonically ordered sequence of all $(a_{n,i}^{i,(s)})_{n\geq 1}$ and similarly $(b_n'^{(s)})_{n\geq 1}$ the monotonically ordered sequence of all $(a_n'^{i,(s)})_{n\geq 1}$. The following Theorem expresses the simultaneous approximation constant ω of $\zeta_1, \zeta_2, \ldots, \zeta_k$ in terms of the s-adic presentations of ζ_i $(s = 2, 3, 4, \ldots)$ by using these two ordered sequences. The proof is introductory to the rest of the work and for this purpose quite detailed.

Proposition 1. *We have*

$$\omega \geq \max\left\{ \limsup_{\|(s,n)\|_\infty \to \infty} \frac{b_{n+1}^{(s)} - b_n^{(s)} - 1}{b_n^{(s)}}, \limsup_{\|(s,n)\|_\infty \to \infty} \frac{b_{n+1}'^{(s)} - b_n'^{(s)} - 1}{b_n'^{(s)}} \right\}, \tag{22}$$

and

$$\omega \leq \max\left\{ \limsup_{\|(s,n)\|_\infty \to \infty} \frac{b_{n+1}^{(s)} - b_n^{(s)}}{b_n^{(s)}}, \limsup_{\|(s,n)\|_\infty \to \infty} \frac{b_{n+1}'^{(s)} - b_n'^{(s)}}{b_n'^{(s)}} \right\}. \tag{23}$$

where $\|(A,B)\|_\infty := \max\{|A|, |B|\}$ *(or any other norm since they are all equivalent in* $\mathbb{R}^2$*).*

Proof. We first prove (22). By definition of $(b_n^{(s)})_{n\geq 1}$ as the mixed ordered sequence of the sequences $(a_n)_{n\geq 1}$, all numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ will have zeros at the positions $b_n^{(s)}+1, b_n^{(s)}+2, \ldots, b_{n+1}^{(s)}-1$ behind the comma in base s for any $s \geq 2$. Since multiplication of ζ_j by $s^{b_n^{(s)}}$ only shifts the comma $b_n^{(s)}$ positions to the right, this means, for any $1 \leq j \leq k$ all $s^{b_n^{(s)}}\zeta_j$ start with $b_{n+1} - b_n - 1$ zeroes in base s behind the comma. For this reason any pair (s,n) satisfies

$$\|s^{b_n(s)}\zeta_j\| = \left|s^{b_n^{(s)}}\zeta_j - \lfloor s^{b_n^{(s)}}\zeta_j \rfloor\right| \leq s^{-(b_{n+1}^{(s)} - b_n^{(s)} - 1)}$$

for any $1 \le j \le k$, where $\|\cdot\|$ denotes the smallest distance of a real number to an integer. Analogously, for all $1-\zeta_j$ and all pairs (s,n) we have

$$\|s^{b_n'^{(s)}}\zeta_j\| = \|s^{b_n'^{(s)}}(1-\zeta_j)\| \le |\lceil s^{b_n'^{(s)}}\zeta_j\rceil - s^{b_n'^{(s)}}\zeta_j| \le s^{-(b_{n+1}'^{(s)}-b_n'^{(s)}-1)}.$$

We conclude that for any pair (s,n)

$$\max_{1\le j\le k}\|x\zeta_j\| \le \max\left\{s^{-(b_{n+1}^{(s)}-b_n^{(s)}-1)}, s^{-(b_{n+1}'^{(s)}-b_n'^{(s)}-1)}\right\}, \tag{24}$$

with

$$x = s^{b_n^{(s)}} \quad \text{or} \quad x = s^{b_n'^{(s)}}.$$

Surely, $s^{b_n} \to \infty$ or $s^{b_n'} \to \infty$ is equivalent to $\|(s,n)\|_\infty \to \infty$, and we claim that (22) follows directly from the definition of the approximation constant ω. To see this we take a sequence of pairs (n,s) with $\|(s,n)\|_\infty \to \infty$, for which $\frac{b_{n+1}^{(s)}-b_n^{(s)}-1}{b_n^{(s)}}$ or $\frac{b_{n+1}'^{(s)}-b_n'^{(s)}-1}{b_n'^{(s)}}$ tend to the lim sup-values on the right hand side of (22). Putting $X_{\sigma(n,s)} := x_{\sigma(n,s)} := s^{b_n^{(s)}}$ or $X'_{\sigma(n,s)} := x_{\sigma(n,s)} := s^{b_n'^{(s)}}$, where σ is an arbitrary bijection $\mathbb{N}\times\mathbb{N}\to\mathbb{N}$, we obtain a sequence of X-values and x-values that leads via (24) to an approximation constant ω in (12) at least as large as both lim sup-values.

To prove (23), we first show the following assertion: It suffices to prove, that for any sufficiently large real parameter X there is a $s_0 = s_0(X)$, such that

$$\frac{b_2^{(s_0)}-b_1^{(s_0)}}{b_1^{(s_0)}} \ge \nu \quad \text{or} \quad \frac{b_2'^{(s_0)}-b_1'^{(s_0)}}{b_1'^{(s_0)}} \ge \nu\,, \tag{25}$$

where $\nu = \nu(s)$ is the largest exponent for which

$$\max_{1\le j\le k}\|\zeta_j s\| = s^{-\nu} \tag{26}$$

holds for all $s \le X$.

For any sequence $(X_i)_{i\ge1}$ let $(\nu_i)_{i\ge1}$ be the largest exponent, for which (26) holds with ν_i in place of ν for all $s \le X_i$. The existence of $s_0 = s_0(X)$ with (25) for any X implies the existence of a sequence $(\beta_i)_{i\ge1}$ with $\beta_i \ge \nu_i$ for all i with β_i of the shape $\frac{b_2^{(s_0)}-b_1^{(s_0)}}{b_1^{(s_0)}}$ hence of the shape of the expressions involved in (23) in the case $n=1$. By definition of ω we may choose the sequence $(X_i)_{i\ge1}$ such that $\lim_{i\to\infty}\nu_i = \limsup_{i\to\infty}\nu_i = \omega$. Furthermore we can assume without loss of generality that $(X_i)_{i\ge1}$ satisfies $s_i = X_i$ for any i, as the exponent ν in the definition of ω in (12) for a fixed x decreases with growing X. Combining all these observations we get $\limsup_{i\to\infty}\beta_i \ge \omega$ where β_i fits in the lim sup term of (23) if we set $(s_i,n_i) = (s_i,1)$, where s_i plays the role of s_0 above, for $X = X_i$.

It remains to prove that for such sequences we have

$$\lim_{i\to\infty}\|(s_i,n_i)\|_\infty = \lim_{i\to\infty}\|(s_i,1)\|_\infty = \limsup_{i\to\infty} s_i = \infty\,.$$

This, however, is easy to see. As $s_i = X_i$ the definition of s_i guarantees that the number $s_i = s_i(X_i)$ minimizes $\max_{1\le j\le k}\|\zeta_j s_i\|$ among all $s_i \le X_i$. On the other hand clearly $\liminf_{s\to\infty}\max_{1\le j\le k}\|\zeta_j\| = 0$ for any $\mathbb{Q}$-linearly independent $\zeta_1,\dots,\zeta_k$ and so by definition of $(s_i)_{i\ge1}$ we also have $\lim_{i\to\infty}\max_{1\le j\le k}\|\zeta_j s_i\| = 0$. Consequently the sequence $(s_i)_{i\ge1}$ cannot be bounded as only finitely many (strictly positive) values $\max_{1\le j\le k}\|\zeta_j s_i\|$ would appear, which proves

$$\limsup_{i\to\infty} s_i = \infty\,.$$

To complete the proof we have to find a value $s_0 = s_0(X)$ for which (25) holds. Note first, that for sufficiently large X and $s = s_0(X)$ we have $a_1^{j,(s)} = a_1'^{j,(s)} = 1$. Indeed for $s \ge \frac{1}{\min_i\|\zeta_i\|}$ and i_0 the index, for which the minimum is attained, we have $\{s\zeta_{i_0}\} \notin \{[0,\frac{1}{s}]\cup[\frac{s-1}{s},1)\}$, so the first digit after the comma in base s is neither 0 nor $(s-1)$. So we can assume X to be large enough to ensure $a_1^{j,(s)} = 1$ for all $1\le j\le k$ and hence $b_1^{(s)} = 1$ as well. It is now easy to see that putting $s_0 := s$ is an appropriate choice, since (26) says that all $s\zeta_j$ respectively $s(1-\zeta_j)$ start with $\lfloor\nu\rfloor$ digits zero in base s behind the comma. This yields

$$\frac{a_2^{j,(s)} - a_1^{j,(s)}}{a_1^{j,(s)}} = a_2^{j,(s)} - 1 \ge \lfloor\nu\rfloor + 1 \ge \nu$$

for all $1\le j\le k$, therefore

$$\frac{b_2^{(s)} - b_1^{(s)}}{b_1^{(s)}} = \frac{\min_j a_2^{j,(s)} - a_1^{j,(s)}}{a_1^{j,(s)}} \ge \nu\,,$$

respectively the same facts for $a_\cdot'^{,(s)}, b_\cdot'^{(s)}$. □

We easily deduce the following Corollary:

Corollary 1. *We have*

$$\omega \ge \max\left\{\sup_s\limsup_{n\ge1}\frac{b_{n+1}^{(s)} - b_n^{(s)} - 1}{b_n^{(s)}}, \sup_s\limsup_{n\ge1}\frac{b_{n+1}'^{(s)} - b_n'^{(s)} - 1}{b_n'^{(s)}}\right\}.$$

Similarly, we can give a lower bound for $\widehat{\omega}$ with respect to the s-adic representation of a real number.

Proposition 2. *For any $\zeta\in\mathbb{R}$ we have*

$$\widehat{\omega} \ge \max\left\{\sup_s\liminf_{n\ge1}\max_{1\le j\le n}\frac{b_{j+1}^{(s)} - b_j^{(s)} - 1}{b_{n+1}^{(s)}}, \sup_s\liminf_{n\ge1}\max_{1\le j\le n}\frac{b_{j+1}'^{(s)} - b_j'^{(s)} - 1}{b_{n+1}'^{(s)}}\right\}.$$

Proof. By definition of the supremum it is sufficient to prove

$$\widehat{\omega} \ge \mathscr{A}_s := \max\left\{\liminf_{n\ge1}\max_{1\le j\le n}\frac{b_{j+1}^{(s)} - b_j^{(s)} - 1}{b_{n+1}^{(s)}}, \liminf_{n\ge1}\max_{1\le j\le n}\frac{b_{j+1}'^{(s)} - b_j'^{(s)} - 1}{b_{n+1}'^{(s)}}\right\}$$

for any base s separately. So let s be fixed and put $b_n^{(s)} = b_n$. By definition of $\widehat{\omega}$ for arbitrary $\epsilon > 0$ and sufficiently large $X = X(\epsilon)$ we have to find an approximation vector $(x, y_1, \ldots, y_k) \in \mathbb{Z}^{k+1}$ with $x \leq X$ and

$$\max_{1\leq j\leq k} |\zeta_j x - y_j| \leq X^{-\mathscr{A}_s+\epsilon}. \tag{27}$$

For $\epsilon > 0$ and large X let n_0 be defined by $s^{b_{n_0}} \leq X < s^{b_{n_0+1}}$ respectively $s^{b'_{n_0}} \leq X < s^{b'_{n_0+1}}$. Put $x := s^{b_j}$ respectively $x := s^{b'_j}$ where j is the index, such that the inner maximum from the definition of $\mathscr{A}_s$ is attained for the given n_0. By definition of $(b_n)_{n\geq 1}$ as the mixed sequence the first $b_{j+1} - b_j - 1$ positions behind the comma of each $\zeta_t s^{b_j}, 1 \leq t \leq k$, respectively $(1-\zeta_t)s^{b_j}, 1 \leq t \leq k$, are zeros in base s. We infer that putting $y_t := \lfloor \zeta_t x \rfloor$ for all $1 \leq t \leq k$ respectively $y_t := \lceil \zeta_t x \rceil$ for all $1 \leq t \leq k$ we have

$$\max_{1\leq t\leq k} \|\zeta_t x\| = \max_{1\leq t\leq k} |\zeta_t x - y_t| \leq s^{b_j - b_{j+1}+1} \leq X^{\frac{b_j - b_{j+1}+1}{b_{n_0+1}}} \tag{28}$$

$$\text{resp.} \quad \max_{1\leq t\leq k} \|\zeta_t x\| = \max_{1\leq t\leq k} |(1-\zeta_t) x - y_t| \leq s^{b'_j - b'_{j+1}+1} \leq X^{\frac{b'_j - b'_{j+1}+1}{b'_{n_0+1}}}. \tag{29}$$

For the left hand side inequalities compare the proof of Proposition 3, the right hand side inequalities follow from $X < s^{b_{n_0+1}}$ and $X < s^{b'_{n_0+1}}$ respectively. As (28), (29) holds for every large X, we may let n tend to ∞ to conclude that (27) has a solution for all sufficiently large X. Hence the exponent of X in (28) and (29) respectively is larger than $\mathscr{A}_s - \epsilon$. □

Now we turn to simultaneous approximation of vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ with good approximation in one fixed simultaneous base $s \geq 2$, and we skip the dual expansion. We want to use Corollary 1 and Proposition 2 to give estimates for the simultaneous approximation constants $\omega, \widehat{\omega}$. With respect to the notation above, meaning that the s-adic digits of ζ_i are given by $(a_{n,i}^{(s)})_{n\geq 1}$ as in (20) and the ordered mixed sequence by $(b_n^{(s)})_{n\geq 1}$, we get

Lemma 1. *For any $s \geq 2$ we have*

$$\min_i \liminf_{n\geq 1} \left(\frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}} \right)^{1/k} \leq \limsup_{n\geq 1} \frac{b_{n+1}^{(s)}}{b_n^{(s)}} \leq \min_i \limsup_{n\geq 1} \frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}}.$$

Proof. The right hand inequality is trivial. For the left hand inequality keep s fixed and put $C := \min_i \liminf_{n\geq 1} \frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}}$ and choose n_0 large enough, such that for all i and all $n \geq n_0$ we have

$$\frac{a_{n+1}^i}{a_n^i} \geq C - \epsilon$$

(s has been dropped in the notation). For arbitrary $b_n, n \geq n_0$, there exist m, i_0 with $a_m^{i_0} = b_n$ by definition of $(b_n)_{n\geq 1}$. The interval $[a_m, (C-\epsilon)a_m]$ contains at

most k numbers b_i, since it contains at most one element of every sequence $(a_n^i)_{n\geq 1}$ for $1 \leq i \leq k$. By the pigeon hole principle there are two numbers b_j, b_{j+1} in the interval $[a_m^{i_0}, a_{m+1}^{i_0}]$ whose quotient $\frac{b_{j+1}}{b_j}$ is at least $(C-\epsilon)^{1/k}$. The lemma follows with $\epsilon \to 0$. □

In combination with Corollary 1 we observe

$$\omega \geq \min_i \liminf_{n\geq 1} \left(\frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}} \right)^{1/k} - 1, \qquad \forall s \geq 2\,.$$

Getting lower bounds for $\widehat{\omega}$ by just considering the sequences $a_{n+1}^{i,(s)}$ is more complicated and to some extent impossible as we will see in Corollary 3. In fact even if

$$\lim_{n\to\infty} \frac{a_{n+1}^{i,(s)}}{a_n^{i,(s)}} = \infty\,, \qquad 1 \leq i \leq k\,,$$

we can have $\widehat{\omega} = 1/k$, which is the weakest lower bound for $\widehat{\omega}$ by (17). We only mention that if we construct sequences $a_{n+1}^{i,(s)}$ for which $\lim_{n\to\infty} \frac{b_{n+1}^{(s)}}{b_n^{(s)}} = \infty$, Proposition 2 yields

$$\widehat{\omega} \geq \liminf_{n\geq 1} \frac{b_{n+1}^{(s)} - b_n^{(s)}}{b_{n+1}^{(s)}} = 1 - \frac{1}{\liminf_{n\geq 1} \frac{b_{n+1}^{(s)}}{b_n^{(s)}}} = 1\,,$$

and consequently $\widehat{\omega} = 1$ in view of (17).

2.2 The case $\omega = \infty$

In the following theorem, we compute the classical approximation constants $\omega_j, \widehat{\omega}_j$ for a special type of Liouville numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$, whose best approximation vectors $(x, y_1, y_2, \ldots, y_k)$ to $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ are easy to guess. The main arguments of the compilation will be carried out in the proofs of the following theorems.

Theorem 1. *Let k be a positive integer and for $1 \leq j \leq k$ let $\zeta_j = \sum_{n\geq 1} \frac{1}{q_{n,j}}$, where*

$$q_{1,1} < q_{1,2} < \cdots < q_{1,k} < q_{2,1} < q_{2,2} < \cdots < q_{2,k} < q_{3,1} < \cdots \tag{30}$$

are natural numbers, such that

$$q_{n,j} \mid q_{n,j+1} \quad \text{for} \quad 1 \leq j \leq k-1 \quad \text{and} \quad q_{n,k} \mid q_{n+1,1} \quad \text{for all} \quad n \geq 1 \tag{31}$$

and such that

$$\lim_{n\to\infty} \frac{\log(q_{n+1,1}) - \log(q_{n,k})}{\log(q_{n+1,k})} = \eta_1\,, \tag{32}$$

$$\lim_{n\to\infty} \frac{\log(q_{n+1,i}) - \log(q_{n+1,i-1})}{\log(q_{n+1,k})} = \eta_i\,, \quad 2 \leq i \leq k\,, \tag{33}$$

$$\lim_{n\to\infty} \frac{\log(q_{n+1,1})}{\log(q_{n,k})} = \eta_{k+1} = \infty\,, \tag{34}$$

where $\eta = (\eta_1, \eta_2, \cdots, \eta_{k+1}) \in \mathbb{R}^k \times \overline{\mathbb{R}}$ *satisfy*

$$\eta_1 + \eta_2 + \cdots + \eta_k = 1 \tag{35}$$

$$\eta_{k+1} > \eta_k \geq \eta_{k-1} \geq \ldots \geq \eta_1 > 0 \tag{36}$$

$$\eta_{k+1} = \infty\,. \tag{37}$$

Then the classical approximation constants relative to the vector $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_k)$ *are given by*

$$\begin{aligned}
\omega_1 &= \eta_{k+1} = \infty =: \wp_1(\eta)\\
\omega_2 &= \max\left\{\frac{\eta_k}{\eta_k + \eta_{k-1} + \cdots + \eta_1}, \frac{\eta_{k-1}}{\eta_{k-1} + \eta_{k-2} + \cdots + \eta_1}, \ldots, \frac{\eta_1}{\eta_1}\right\} =: \wp_2(\eta)\\
\omega_3 &= \max\left\{\frac{\eta_{k-1}}{\eta_k + \eta_{k-1} + \cdots + \eta_1}, \frac{\eta_{k-2}}{\eta_{k-1} + \eta_{k-2} + \cdots + \eta_1}, \ldots, \frac{\eta_1}{\eta_2 + \eta_1}\right\} =: \wp_3(\eta)\\
&\vdots\\
\omega_{k+1} &= \frac{\eta_1}{\eta_k + \eta_{k-1} + \cdots + \eta_1} =: \wp_{k+1}(\eta).
\end{aligned}$$

and

$$\begin{aligned}
\widehat{\omega}_1 &= \min\left\{\frac{\eta_k}{\eta_k + \eta_{k-1} + \cdots + \eta_1}, \frac{\eta_{k-1}}{\eta_{k-1} + \eta_{k-2} + \cdots + \eta_1}, \ldots, \frac{\eta_1}{\eta_1}\right\} =: \widehat{\wp}_1(\eta)\\
\widehat{\omega}_j &= 0, \qquad 2 \leq j \leq k+1.
\end{aligned}$$

Proof. We start with the constants ω_j and intend to prove the inequalities $\omega_j \geq \wp_j(\eta)$ and $\omega_j \leq \wp_j(\eta)$ separately for $1 \leq j \leq k+1$.
(1) $\omega_j \geq \wp_j(\eta)$:
Let $\wp_{j,l}$ be the l-th quotient of the maximum labelled $\wp_j(\eta)$. We give a detailed proof of $\omega_j \geq \wp_{j,1} = \eta_{k+2-j}$ and then mention how to generalize the proof to derive all the other inequalities $\omega_j \geq \wp_{j,l}$ for $l \neq 1$.

To prove $\omega_j \geq \wp_{j,1}$, we will construct j sequences of approximation vectors

$$\left(x^{(1,i)}, y_1^{(1,i)}, \ldots, y_k^{(1,i)}\right)_{i\geq 1}, \left(x^{(2,i)}, y_1^{(2,i)}, \ldots y_k^{(2,i)}\right)_{i\geq 1}, \ldots, \left(x^{(j,i)}, y_1^{(j,i)}, \ldots, y_k^{(j,i)}\right)_{i\geq 1}$$

which are linearly independent for each fixed $i \in \mathbb{N}$ and such that $\omega_j = \eta_{k+2-j}$ follows for $i \to \infty$. Indeed for p in a j-element subset of $\{1, 2, \ldots, k\}$ and any $\epsilon > 0$ we claim for i sufficiently large

$$\max_{1\leq t\leq k} -\frac{\log(|\zeta_t x^{(p,i)} - y_t^{(p,i)}|)}{\log(x^{(p,i)})} \geq \eta_{k+2-j} - \epsilon\,.$$

In analogy to the definition of $(b_n^s)_{n\geq 1}$ in subsection 2.1 let $(b_n)_{n\geq 1}$ be the combined sequence of the logarithms of the integers $q_{n,j}$ in increasing order, which means for any nonnegative integer M and $N \in \{1, 2, \ldots, k\}$ we have $b_{kM+N} = \log(q_{M,N})$. By (34) we have $\limsup \frac{b_{n+1}}{b_n} = \infty$ and thus by putting the first approximation

vector $(q_{n,i}, \lfloor \zeta_1 q_{n,i} \rfloor, \dots, \lfloor \zeta_k q_{n,i} \rfloor)$ with arbitrary i we may let n tend to infinity, to obtain $\omega = \infty$: indeed applying (31) we derive that all the remainder terms

$$\|\zeta_j q_{n,i}\| = \sum_{l:q_{n,l}>q_{n,i}} \frac{1}{q_{l,j}} q_{n,i} \le 2 \frac{q_{n,i}}{q_{n,i+1}}$$

are small due to (32)–(34). In order to estimate ω_j for $j \ge 2$ we construct a sequence of parameters X and approximation vectors $(x, y_1, \dots, y_k)$ with $x \le X$ explicitly. For the fixed choice $X^{(n)} := q_{n,k}$ we will get $\omega_j \ge \wp_{j,1}$. To see this let $x^{(1,n)} := X(n)$ and $y_t^{(1,n)} := \lfloor x^{(1,n)} \zeta_t \rfloor$ for $1 \le t \le k$. Define the second approximation vector by taking $x^{(2,n)} = q_{n,k-1}$ and again $y_t^{(2,n)} := \lfloor x^{(2,n)} \zeta_t \rfloor$. By means of (32), (33), (34) and the definition of $\wp_2(\eta)$ we claim that for each $C < \eta_{k+2-2} = \eta_k$

$$\left| \zeta_t x^{(2,n)} - y_t^{(2,n)} \right| \le (x^{(1,n)})^{-C} = (X^{(n)})^{-C} \tag{38}$$

holds for $n = n(C)$ large enough. This follows from

$$\left| \zeta_t x^{(2,n)} - y_t^{(2,n)} \right| = \left| q_{n,k-1}\zeta_t - \lfloor q_{n,k-1}\zeta_t \rfloor \right| = \sum_{i=n+1}^{\infty} \frac{q_{n,k-1}}{q_{i,t}}, \quad 1 \le t \le k-1,$$

$$\left| \zeta_k x^{(2,n)} - y_k^{(2,n)} \right| = \left| q_{n,k-1}\zeta_k - \lfloor q_{n,k-1}\zeta_k \rfloor \right| = \frac{q_{n,k-1}}{q_{n,k}} + \sum_{i=n+1}^{\infty} \frac{q_{n,k-1}}{q_{i,k}}$$

in view of the definition of ζ_t and our assumption (31). In every case all the values of $\zeta_t x^{(2,n)} - y_t^{(2,n)}$ for $1 \le t \le k$ are bounded by $\frac{q_{n,k-1}}{q_{n,k}}(1 + o(1))$. Using (32), (33), (34) this leads to (38).

Similarly, defining the j-th approximation vector for $2 \le j \le k$ by $x^{(j,n)} = q_{n,k+1-j}$ and for $j = k+1$ by $x^{(k+1,n)} = q_{n-1,k}$ and then putting $y_t^{(j,n)} := \lfloor x^{(j,n)} \zeta_t \rfloor$ yields the corresponding inequalities.

We now check that these vectors are linearly independent as required. To do this we prove that all the matrices $\boldsymbol{B}_n = \big(\boldsymbol{B}_n(i,j)\big)_{1 \le i,j \le k+1}$ obtained by writing the h-th approximation vector $(x^{(h,n)}, y_1^{(h,n)}, \dots, y_k^{(h,n)})$ in the h-th row, i.e.

$$\boldsymbol{B}_n = \begin{pmatrix} q_{n,k} & q_{n,k}\sum\limits_{i=1}^{n} q_{i,1}^{-1} & q_{n,k}\sum\limits_{i=1}^{n} q_{n,2}^{-1} & \cdots & q_{n,k}\sum\limits_{i=1}^{n} q_{i,k}^{-1} \\ q_{n,k-1} & q_{n,k-1}\sum\limits_{i=1}^{n} q_{i,1}^{-1} & q_{n,k-1}\sum\limits_{i=1}^{n} q_{n,2}^{-1} & \cdots & q_{n,k-1}\sum\limits_{i=1}^{n-1} q_{i,k}^{-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n,1} & q_{n,1}\sum\limits_{i=1}^{n} q_{i,1}^{-1} & q_{n,1}\sum\limits_{i=1}^{n-1} q_{n,k-1}^{-1} & \cdots & q_{n,1}\sum\limits_{i=1}^{n-1} q_{i,k}^{-1} \\ q_{n-1,k} & q_{n-1,k}\sum\limits_{j=1}^{n-1} q_{j-1,1}^{-1} & q_{n-1,k}\sum\limits_{j=1}^{n-1} q_{j-1,2}^{-1} & \cdots & q_{n-1,k}\sum\limits_{j=1}^{n-1} q_{j-1,k}^{-1} \end{pmatrix}$$

are nonsingular. Observe that if we subtract $\frac{\boldsymbol{B}_n(h,1)}{\boldsymbol{B}_n(h+1,1)}$ times the $(h+1)$-th row from the h-th row of the matrix $\boldsymbol{B}_n$, all entries in the new h-th line will be zero

apart from a one in position $(h, k+2-h)$. Starting with this process at $h=1$ and repeating it until $h=k$ we end up with the matrix

$$\boldsymbol{C}_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & 0 & \dots & 0 \\ q_{n-1,k} & q_{n-1,k}\sum\limits_{j=1}^{n-1} q_{j-1,1}^{-1} & q_{n-1,k}\sum\limits_{j=1}^{n-1} q_{j-1,2}^{-1} & \dots & q_{n-1,k}\sum\limits_{j=1}^{n-1} q_{j-1,k}^{-1} \end{pmatrix},$$

which is easily seen to have absolute value of the determinant equal to $q_{n-1,k} \neq 0$. Therefore also $\det(\boldsymbol{B}_n) = q_{n-1,k} \neq 0$, as required.

To obtain all the other inequalities $\omega_j \geq \wp_{j,i}$ for $2 \leq i$, where the upper bound of i depends on j, we proceed analogously. In the definition of $x^{(1,n)}$ we replace $q_{n,k}$ by $q_{n,k+1-i}$ and again for $1 \leq t \leq k$ we define $y_t^{(1,n)} = \lfloor x^{(1,n)}\zeta_t \rfloor$ for the first approximation vector. We define all the others by taking $x^{(2,t)} = q_{n,k+1-2}$, $x^{(3,n)} = q_{k+1-3,n}, \dots$ and again $y_t^{(i,n)} = \lfloor x^{(i,n)}\zeta_t \rfloor$ for $1 \leq t \leq k$ and $2 \leq i \leq k+1$. This construction yields the desired lower bounds (or 0 which is omitted in $\wp(\eta)$) as above again by (32), (33), (34).

$\omega_j \leq \wp_j(\eta)$:
We have to show that for $1 \leq j \leq k+1$ the approximation vectors

$$\left(x^{(j,n)}, y_1^{(j,n)}, \dots, y_k^{(j,n)}\right)$$

constructed in the first step of the proof are somehow best possible. We split the proof of this assertion in 3 steps. To simplify notation let $(c_n)_{n\geq 1} = (e^{b_n})_{n\geq 1}$ be the ordered mixed sequence $(q_{1,1}, q_{1,2}, \dots, q_{1,k}, q_{2,1}, \dots)$.

First step: For an arbitrary approximation vector $(x, y_1, \dots, y_k)$ let h be the index determined by $c_h \leq x < c_{h+1}$ and let g be the largest integer such that the index $g-1$ satisfies $c_{g-1} \mid x$. Since $x < c_{h+1}$ and consequently $c_{h+1} \nmid x$ we clearly have $g \leq h+1$. When $X \to \infty$ so does h and we claim that for $h \to \infty$

$$\max_{1\leq t\leq k} |\zeta_t x - y_t| \geq \frac{c_{g-1}}{c_g} - o\left(\frac{c_{g-1}}{c_g}\right), \qquad g > h+1-k \tag{39}$$

$$\max_{1\leq t\leq k} |\zeta_t x - y_t| \geq \frac{1}{c_{h+2-k}} - o\left(\frac{1}{c_{h+2-k}}\right), \qquad g \leq h+1-k\,. \tag{40}$$

Furthermore in the case $g \leq h+1-k$ (i.e. the assumption of (40)), the inequality $x < \frac{1}{2}c_{h+1}c_{h+1-k}^{-1}$ contradicts that

$$\max_{1\leq t\leq k} |\zeta_t x - y_t| < \frac{1}{2}\frac{1}{c_{h+1-k}} - o\left(\frac{1}{c_{h+1-k}}\right) \tag{41}$$

holds for $h \to \infty$.

Second step: Let X be a real parameter from the definition of the approximation constants ω_j and $m = m(X)$ be the index such that $c_m \leq X < \frac{c_{m+1}}{4}$. Then for

$1 \le j \le k+1$ a set of j vectors $(x^{(i)}, y_1^{(i)}, \dots, y_k^{(i)}), 1 \le i \le j$, satisfying the inequalities

$$x^{(i)} \le X\,, \qquad 1 \le i \le j\,, \tag{42}$$

$$\max_{1\le t\le k} |x^{(i)}\zeta_t - y_t^{(i)}| \le \frac{1}{2}\,, \qquad 1 \le i \le j\,, \tag{43}$$

can only be linearly independent if at least one $x^{(i)}$ is not divisible by c_{m+2-j}.

Third step: We intend to show by combining the first two steps and using (32)–(34), that for arbitrary X the choice of approximation vectors in the proof of $\omega_j \ge \wp_j(\eta)$ is somehow optimal, i.e. the approximation constants of this case cannot be improved.

Proof of first step: We first make the assumption $c_{h+1} \in (q_{n,k})_{n\ge1}$, and will explain at the end how to extend this easily to the case where c_{h+1} belongs to another sequence. This assumption is equivalent to $c_{h+1} = q_{n_1,k}$ for some $n_1 \in \mathbb{N}$ and it follows that $c_h = q_{n_1,k-1}$. By (31) we have $c_l \mid x$ for all $l \le g-1$ and $c_l \nmid x$ for all $l \ge g$, in particular $c_g \nmid x$. Recall $g \le h+1$. To prove the assertions we now consider the corresponding cases separately:

Case 1: $c_g > q_{n_1-1,k}$. Note that since $q_{n_1-1,k} = c_{h+1-k}$ this is equivalent to $g \ge h+2-k$ or $c_g \ge c_{h+2-k}$. We can write $x = x_1 + x_2$ with $0 < x_1 < c_g$ and $c_g \mid x_2$, since by our definition of g we have $x_1 \ne 0$. Denote by $\overline{g}$ the congruence class of g in the residue system $\{1, 2, \dots, k\}$ mod k. Note, that $c_{\overline{g}}$ is the smallest value c_g with g in the residue class $\overline{g}$, or equivalently $\zeta_{\overline{g}} = \sum_{l\ge0} c_{\overline{g}+kl}^{-1}$, which we will make use of. We claim that

$$\|x_1\zeta_{\overline{g}}\| \ge c_g^{-1}c_{g-1} - \sum_{l\ge1} c_{g+lk}^{-1}c_{g-1} \ge c_g^{-1}c_{g-1} - 2c_{h+1}c_{h+2}^{-1} \tag{44}$$

$$\{x_2\zeta_{\overline{g}}\} = \|x_2\zeta_{\overline{g}}\| = \Big\|x_2 \sum_{l\ge1} c_{g+kl}^{-1}\Big\| \le 2c_{h+1}c_{h+2}^{-1}, \tag{45}$$

where $\|\cdot\|$ denotes the closest distance to an integer an $\{\cdot\}$ the fractional part of a real number. Inequality (44) relies on the fact that $0 < \frac{x_1}{c_g} < 1$ and $c_{g-1} \mid x_1$, which is seen to be true because $c_{g-1} \mid x$, $c_{g-1} \mid c_g$ and $c_g \mid x_2$ by definition, so putting these together we get $c_{g-1} \mid x - x_2$, but $x - x_2 = x_1$. Combination of these two facts and recalling that exactly $g+k, g+2k, \dots$ are the indices greater than g belonging to the residue class $\overline{g}$ shows that $x_1\zeta_{\overline{g}}$ is of the form

$$x_1\zeta_{\overline{g}} = \frac{x_1}{c_g} + x_1\sum_{l\ge1} c_{g+lk}^{-1} = \frac{Kc_{g-1}}{c_g} + x_1\sum_{l\ge1} c_{g+lk}^{-1} \ge K\frac{c_{g-1}}{c_g} + x_1\sum_{l\ge1} c_{g+lk}^{-1}$$

with $K \in \{1, 2, \dots, \frac{c_g}{c_{g-1}} - 1\}$ (note $c_{g-1} \mid c_g$). The assertion now follows by a combination of $x_1 < c_g \le c_{h+1}$,

$$\sum_{l\ge1} c_{g+lk}^{-1} < c_{g+k}^{-1}\Big(1 + \frac{1}{2} + \frac{1}{4} + \cdots\Big) = 2c_{g+k}^{-1}, \tag{46}$$

and $c_{g+k}^{-1} \le c_{h+2}^{-1}$, which is true by the assumption of case 1.

Inequality (45) follows from the fact that for any $s \leq g$ by virtue of (31) we have $c_s \mid x_2$ which holds in particular for those c_s with s in the residue class $\overline{g}$. So all quantities $x_2 c_s^{-1}$ with $s \leq g$ are integers. Thus the sum of quantities of order smaller than $x_2 c_{g+k}^{-1}$ in $x_2 \zeta_{\overline{g}} = \sum_{l \geq 0} x_2 c_{\overline{g}+kl}^{-1}$, i.e. $x_2 \sum_{l \geq 1} c_{g+kl}^{-1}$, has the same fractional part as the entire sum. Now on the one hand we have $x_2 \leq x \leq c_{h+1}$, and on the other hand $c_{g+k} \geq c_{h+2}$ by the assumption of case 1. Together with (46) these assertions yield (45).

Summing (44) and (45) and noting that by (34) we have $c_{h+1} c_{h+2}^{-1} = o(c_g^{-1} c_{g-1})$, so that we can use the triangle inequality on the fractional parts, we further have

$$\|x\zeta_{\overline{g}}\| \geq c_g^{-1} c_{g-1} - 4 c_{h+1} c_{h+2}^{-1}.$$

By (36), (37) and (34) the expression $c_m^{-1} c_{m-1}$ is monotonically decreasing in m and the error term $4c_{h+1}c_{h+2}^{-1}$ is obviously $o(c_h c_{h+1}^{-1})$ by (34). Hence for $h \to \infty$ we obtain

$$\frac{c_{g-1}}{c_g} - o\left(\frac{c_h}{c_{h+1}}\right) \leq \|\zeta_{\overline{g}} x\| \leq \max_{1 \leq t \leq k} \|\zeta_t x\|.$$

This establishes (39) in this case as $\frac{c_h}{c_{h+1}} \leq \frac{c_{g-1}}{c_g}$ by (36) and (32)–(34).

If c_{h+1} belongs to another sequence $(q_{n,i})_{n \geq 1}$, which means $c_{h+1} = q_{n_1,i}$ with $i \neq 1$, we look at the case $c_g \geq q_{n_1-1,i}$ and again obtain

$$\|x_1 \zeta_{\overline{g}}\| \geq c_g^{-1} c_{g-1} - \sum_{l \geq 1} c_{g+lk}^{-1} c_{g-1} \geq c_g^{-1} c_{g-1} - 2 c_{h+1} c_{g+k}^{-1}$$
$$\{x_2 \zeta_{\overline{g}}\} = \|x_2 \zeta_{\overline{g}}\| = \Big\|x_2 \sum_{l \geq 1} c_{g+kl}^{-1}\Big\| \leq 2 c_{h+1} c_{g+k}^{-1},$$

as in the proof of the special case (without using $c_{h+2} \leq c_{g+k}$ as above from which we derived the weaker but sufficient conditions (44), (45)). However, by (32)–(34) we again have $c_{h+1} c_{g+k}^{-1} = o(c_g^{-1})$ as $h \to \infty$ (or equivalently $g \to \infty$ as $g \geq h+2-k$) and the rest of the argumentation is almost as above. Thus (39) holds in any case.

Case 2: $c_g \leq q_{n_1-1,k}$. In this case it is more convenient to work directly with the values $q_{\cdot,\cdot}$ instead of $c_\cdot$. As in case 1 let $x = x_1 + x_2$ with $0 < x_1 < q_{n_1,1}$ and $q_{n_1,1} \mid x_2$. Again $q_{n_1-1,k} \mid q_{n_1,1}$ and the definition of g ensures $x_1 \neq 0$. Analogously to the proof of (44), (45) in case 1 we deduce

$$\|x_1 \zeta_1\| \geq \frac{1}{q_{n_1,1}} - 2 q_{n_1,k} q_{n_1+1,1}^{-1}$$
$$0 \leq x_2 \zeta_1 \leq 2 q_{n_1,k} q_{n_1+1,1}^{-1}.$$

Using again the triangle inequality and (37), we again deduce

$$\|x\zeta_1\| \geq \frac{1}{q_{n_1,1}} - 4\frac{q_{n_1,k}}{q_{n_1+1,1}} = \frac{1}{q_{n_1,1}} - 4\frac{c_{h+1}}{c_{h+2}}.$$

But by (32)–(34) again $\frac{c_{h+1}}{c_{h+2}} = o(c_{h+2-k}^{-1}) = o(c_{n_1,1}^{-1})$ for $h \to \infty$ so that finally

$$\frac{1}{q_{n_1,k}} - o(c_{h+2-k}^{-1}) \leq \|\zeta_1 x\| \leq \max_{1 \leq t \leq k} \|\zeta_t x\|.$$

But $q_{n_1,1} = c_{h+2-k}$, so we have (40) in this case. If c_{h+1} belongs to another sequence $(q_{n,i})_{n\geq 1}$, $i \neq k$, we can apply very similar estimates with respect to $\zeta_{\overline{i+1}} = \zeta_{i+1}$ instead of ζ_1. So our assumption is no loss of generality in this case either. Thus (40) holds in any case.

We still have to prove that $x < \frac{1}{2}c_{h+1}c_{h+1-k}^{-1}$ contradicts (41). For simplicity we again discuss the case $c_{h+1} \in (q_{n,k})_{n\geq 1}$ first. Write $x = x_1 + x_2$ with $0 < x_1 < q_{n_1-1,k}$ and $q_{n_1-1,k} \mid x_2$. Note that again we have $x_1 \neq 0$ by the assumption $g \leq h+1-k$, so $c_g \leq c_{h+1-k} = q_{n_1-1,k}$, and the definition of g. Assume we have $x < \frac{1}{2}c_{h+1}c_{h+1-k}^{-1} = \frac{1}{2}q_{n_1,k}q_{n_1-1,k}^{-1}$. The fractional part of $x_2\zeta_k$ is $\sum_{l\leq 0} x_2 c_{h+1+lk}^{-1}$ as higher order summands are integers by definition of x_2. We split this expression in

$$\{x_2\zeta_k\} = \|x_2\zeta_k\| = \sum_{l\geq 0} x_2 c_{h+1+kl}^{-1} = x_2 c_{h+1}^{-1} + \sum_{l\geq 1} x_2 c_{h+1+kl}^{-1}$$

and using $x_2 \leq x$ we infer

$$\|x_2\zeta_k\| \leq \frac{1}{2}c_{h+1-k}^{-1} + \sum_{l\geq 1} c_{h+1}c_{h+1-k}^{-1}c_{h+1+kl}^{-1}\,, \tag{47}$$

which is obviously $\frac{1}{2}c_{h+1-k}^{-1} + o(c_{h+1-k}^{-1})$ as $h \to \infty$ by (34).

On the other hand, by definition of x_1 and $c_{h+1-k} = q_{n_1-1,k} \nmid x_1$ as $g \leq h+1-k$ by assumption and a very similar argument as in case 1 we have

$$\left\| x_1 \left(\frac{1}{q_{1,k}} + \frac{1}{q_{2,k}} + \cdots + \frac{1}{q_{n_1-1,k}} \right) \right\| \geq \frac{1}{q_{n_1-1,k}}\,.$$

On the other hand by $x_1 < q_{n_1-1,k}$ the sum of the remainder terms of $\zeta_k x_1$, i.e. $x_1 \sum_{l\geq 0} \frac{1}{q_{n_1+l,k}}$, is bounded above by $2\frac{q_{n_1-1,k}}{q_{n_1,k}}$ with very similar estimates as in (46). So

$$\|x_1\zeta_k\| \geq \frac{1}{q_{n_1-1,k}} - 2\frac{q_{n_1-1,k}}{q_{n_1,k}} \tag{48}$$

by a very similar argument as in case 1. As $n_1 \to \infty$, we have

$$\frac{q_{n_1-1,k}}{q_{n_1,k}} = o\left(\frac{1}{q_{n_1-1,k}}\right) = c_{h+1-k}^{-1} - o(c_{h+1-k}^{-1})$$

(note that $h \to \infty$ if $n_1 \to \infty$) and thus by (48)

$$\|x_1\zeta_k\| \geq c_{h+1-k}^{-1} - o(c_{h+1-k}^{-1}) \tag{49}$$

Using triangular inequality on (47), (49) thus gives

$$\max_{1\leq t\leq k} \|\zeta_t x\| \geq \|\zeta_k x\| \geq \frac{1}{2}c_{h+1-k}^{-1} - o(c_{h+1-k}^{-1}).$$

Our last assertion is proved in this case and the assumption $c_h \in (q_{n,k})_{n\geq 1}$ can obviously be dropped again.

Proof of second step: Without loss of generality assume that $c_m \in (q_{n,k})_{n\geq 1}$, the proof for the other cases is essentially the same. This means $c_m = q_{m_1,k}$ for

some $m_1 \in \mathbb{N}$ and consequently $c_{m+1} = q_{m_1+1,1}$. Suppose c_{m-j+2} divides $x^{(i)}$ for all $1 \le i \le j$. On the one hand, by our assumption the $(j-1)$ numbers $c_{m-j+2}, c_{m-j+3}, \dots, c_m$ belong to the sequences $q_{n,k}, q_{n,k-1}, \dots, q_{n,k-j+2}$. On the other hand, $c_u \mid c_{u+1}$ for all $u \ge 1$ combined with $c_{m-j+2} \mid x^{(i)}$ for all $1 \le i \le j$ implies that for all $s \le m-j+2$ the number c_s divides $x^{(i)}$. From these two facts we conclude that for $g \notin \{k, k-1, \dots, k+2-j\}$, i.e. $g \in G := \{1, 2, \dots, k+1-j\}$, the partial sum $x^{(i)} \sum_{r=1}^{m_1} \frac{1}{q_{r,g}}$ of $\zeta_g x^{(i)}$ is an integer, since every summand $\frac{x^{(i)}}{q_{r,g}}$ is. As terms of order lower than m_1 in $\zeta_g x^{(i)}$ for $g \in G \setminus \{1\}$ obviously add up to a quantity smaller than $\frac{1}{2}$, for $1 \le i \le j$ and $g \in G \setminus \{1\}$ we have

$$\|\zeta_g x^{(i)}\| = \zeta_g x^{(i)} - \lfloor \zeta_g x^{(i)} \rfloor = \sum_{r=m_1+1}^{\infty} \frac{1}{q_{r,g}} < \frac{1}{2}\,, \tag{50}$$

and combined with (43) eventually

$$y_g^{(i)} = \lfloor \zeta_g x^{(i)} \rfloor = x^{(i)} \sum_{r=1}^{m_1} \frac{1}{q_{r,g}}\,. \tag{51}$$

In view of our assumption $X < \frac{c_{m+1}}{4} = \frac{q_{m_1+1,1}}{4}$ the results (50), (51) are also valid for $g = 1$. To sum up, for all $g \in G$ we have (51), which obviously yields

$$\frac{x^{(a)}}{x^{(b)}} = \frac{y_g^{(a)}}{y_g^{(b)}}, \qquad g \in G, \quad 1 \le a, b \le j\,.$$

Thus in the matrix, whose i-th row is the i-th approximation vector

$$(x^{(i)}, y_1^{(i)}, \dots, y_k^{(i)}) \in \mathbb{Z}^{k+1}\,, \quad 1 \le i \le j\,,$$

the first $|G| = k - j + 3$ columns together have rank 1. The rank of the whole matrix therefore cannot exceed $1 + [(k+1) - (k-j+3)] = j - 1 < j$. This means the j rows are linearly dependent, a contradiction. So c_{m-j+2} cannot divide all the numbers $x^{(i)}$, as stated.

Proof of third step: We will prove for arbitrary j, that $\omega_j(X)$ is for $X \to \infty$ asymptotically bounded above by one of the fractions (depending on $\log(X)$) involved in the definition of $\wp_j(\eta)$, by which we mean that for any $\epsilon > 0$ and $X = X(\epsilon)$ large enough we have $\omega_j(X) < \wp_j(\eta) + \epsilon$. Since $\omega_j = \limsup_{X\to\infty} \omega_j(X)$, $\epsilon \to 0$ shows the required result.

So let X be arbitrary but fixed and let h be the index determined by $c_h \le X < c_{h+1}$. We first prove that without loss of generality we may restrict to the case where X lies an interval of the shape $[c_h, \frac{c_{h+1}}{4})$.

This is the case because the logarithm to the base $X = \frac{c_{h+1}}{4}$ of

$$D_{\boldsymbol{x}} := \max_{1 \le t \le k} |\zeta_t x - y_t|, \qquad \boldsymbol{x} := (x, y_1, \dots, y_k)$$

for vectors $\boldsymbol{x}$ with $|x| \le X = \frac{c_{h+1}}{4}$ is asymptotically the same as to the base c_{h+1}.

Indeed we have

$$\log_{c_{h+1}}(D_{\boldsymbol{x}}) = \frac{\log(D_{\boldsymbol{x}})}{\log(c_{h+1})}\,, \qquad \log_{\frac{c_{h+1}}{4}}(D_{\boldsymbol{x}}) = \frac{\log(D_{\boldsymbol{x}})}{\log(\frac{c_{h+1}}{4})}\,,$$
$$\lim_{h\to\infty} \frac{\log(c_{h+1})}{\log(\frac{c_{h+1}}{4})} = \lim_{c_h\to\infty} \frac{\log(c_{h+1})}{\log(\frac{c_{h+1}}{4})} = \lim_{c_h\to\infty} \frac{\log(c_{h+1})}{\log(c_{h+1}) - \log(4)} = 1\,,$$

and hence

$$\lim_{h\to\infty} \frac{\log_{c_{h+1}}(D_{\boldsymbol{x}})}{\log_{\frac{c_{h+1}}{4}}(D_{\boldsymbol{x}})} = 1\,. \tag{52}$$

On the other hand, since we can restrict to $\boldsymbol{x}$ belonging to some ω_j with $j \geq 2$, all expressions $\log_{c_{h+1}}(D_{\boldsymbol{x}})$ are bounded above by $2\omega_2 \leq 2$ (see (15)) for h sufficiently large. Together with (52) and since this holds for every vector $\boldsymbol{x}$ for which $x \leq X$, the definition of the quantities ω_j immediately implies that they remain unaffected by this change of base.

So let $h = h(X)$ be the index determined by $c_h \leq X < \frac{c_{h+1}}{4}$. By the second step of the proof (putting $m = h$) at least one of the j linearly independent approximation vectors $(x, y_1, y_2, \ldots, y_k) \in \mathbb{Z}^{k+1}$ has to satisfy the condition $c_{h-j+2} \nmid x$. Consider one of the j approximation vectors with this property. This means if we let $g-1$ be the largest index with $c_{g-1} \mid x$ as in step 1, we have $g - 1 \leq h - j + 1$, i.e. $g \leq h - j + 2$. Further let i be the index, for which $c_h = q_{N,i}$ belongs to the sequence $(q_{n,i})_{n\geq 1}$. At this point one should mention that we will repeatedly use step 1 in the following, neglecting the o-terms in the estimates (39), (40), (41) as they do not affect the asymptotic behaviour we aim to prove.

First we treat the case $c_{g-1} \geq q_{N-1,i}$ (case 1 step 1). Note, that $\frac{c_{m-1}}{c_m}$ is monotonically decreasing as m increases by (32)–(34) and (36), which we already used before. Thus by $g \leq h - j + 2$ and (39) we have

$$\max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{c_{h-j+1}}{c_{h-j+2}} - o\left(\frac{c_{h-j+1}}{c_{h-j+2}}\right) \qquad \text{for} \quad h \to \infty\,. \tag{53}$$

So $X \geq c_h$ implies

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\frac{\log(\frac{c_{h-j+1}}{c_{h-j+2}})}{\log(c_h)} = \frac{\log(c_{h-j+2}) - \log(c_{h-j+1})}{\log(c_h)}.$$

It is now easy to see by (32)–(34) that for h in a fixed residue class $\overline{h}$ of the residue system $\{1, 2, \ldots, k\}$ mod k, the right hand side tends to one of the fractions (depending on $\overline{h}$) in the definition of $\omega_j(\eta)$ or to zero as $h \to \infty$ or equivalently $X \to \infty$. Clearly, each expression in $\wp_j(\eta)$ is induced by some $\overline{h}$ in that way as well. This shows, that indeed we have $\omega_j(X) < \wp_j + \epsilon$ for any $\epsilon > 0$ and $X = X(\epsilon)$ large enough.

In the remaining case $c_{g-1} < q_{N-1,i}$ (case 2 step 1) by (40) and as c_{h+1} in (40) corresponds to $q_{N,i}$, $\max_{1\leq t\leq k} |x\zeta_t - y_t|$ is essentially bounded below by $\frac{1}{q_{N-1,i+1}}$ (omitting the lower order terms and i in the residue system $\{1, 2, \ldots, k\}$ mod k).

We distinguish three cases now.

If we have $i \notin \{k-1, k\}$, approximation relative to base X is bad, as in this case we have $\lim_{N\to\infty} \frac{\log(q_{N-1,i+1})}{\log(q_{N,i})} = 0$ as a consequence of (34), so again by $X \geq c_h = q_{N,i}$, the expression

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\log_{c_h} \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(q_{N-1,i+1})}{\log(c_h)} = \frac{\log(q_{N-1,i+1})}{\log(q_{N,i})}$$

tends to 0 as $X \to \infty$, and we are done again.

In the case $i = k$, or equivalently $q_{N,k} \leq X < q_{N+1,1}$, due to $c_{g-1} < q_{N-1,i} = q_{N-1,k}$ we know again by (40) that $\max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{1}{q_{N,1}}$ and so $X \geq q_{N,k}$ implies

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\log_{q_{N,k}} \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(q_{N,1})}{\log(q_{N,k})}.$$

Hence by (34), for $h \to \infty$ we have the asyptotic

$$\frac{\log(q_{N,1})}{\log(q_{N,k})} \sim \frac{\log(q_{N,1}) - \log(q_{N-1,k})}{\log(q_{N,k})}.$$

The right hand side, however, converges to $\eta_1 = \wp_{k+1}(\eta)$ for $N \to \infty$ by (32). This shows that $\omega_j(X) \leq \wp_{k+1}(\eta) + \epsilon$ for any $\epsilon > 0$ and $X = X(\epsilon)$ large enough and together with $\wp_{k+1} \leq \wp_j(\eta)$ for $1 \leq j \leq k+1$ we get $\omega_j \leq \wp_j(\eta)$ as desired.

We divide the remaining case $i = k-1$, which means $q_{N,k-1} \leq X < q_{N,k}$, again into two cases. If $x < \frac{1}{2}\frac{q_{N,k}}{q_{N-1,k}}$, it follows that $\max_{1\leq t\leq k} |x\zeta_t - y_t|$ is essentially bounded below by $\frac{1}{2}\frac{1}{q_{N-1,k}}$ in view of (41). This gives the estimate

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(\frac{1}{2}q_{N-1,k})}{\log(q_{N,k-1})} \leq \frac{\log(q_{N-1,k}) - \log(2)}{\log(q_{N,1})},$$

which tends to 0 for $X \to \infty$ by (34).

Otherwise $x \geq \frac{1}{2}\frac{q_{N,k}}{q_{N-1,k}}$, which clearly implies $\lim_{N\to\infty} \frac{\log(x)}{\log(q_{N,k})} = 1$ by (34). In particular for every $\epsilon > 0$ we have $\log(x) > (1+\epsilon)\log(q_{N,k})$ for $N = N(\epsilon)$ sufficiently large. Note $X \geq x$ and that $X \to \infty$ is equivalent to $N \to \infty$. Combination of these facts together with the fact that $\max_{1\leq t\leq k} |x\zeta_t - y_t|$ is essentially bounded below by $\frac{1}{q_{N,1}}$ by (40) yields the inequality

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(q_{N,1} + o(\log(q_{N,1})))}{\log(q_{N,k}(1+\epsilon))}$$

for any $\epsilon > 0$ and $N = N(\epsilon)$ sufficiently large. However, the right hand side is of the form $\frac{\log(q_{N,1})}{\log(q_{N,k})} + o\Big(\frac{\log(q_{N,1})}{\log(q_{N,k})}\Big)$ as $N \to \infty$ and $\epsilon \to 0$, which tends to $\eta_1 = \wp_{k+1}(\eta)$ for $N \to \infty$ as in the case $i = k$ above and so it is no improvement either. This shows step three.

Now it only remains to determine the approximation constants $\widehat{\omega}_j$. However, for $j \geq 2$ they are easily seen to be zero as a consequence of $\omega = \infty$. Indeed in this

case we have $\underline{\psi}_1 = -1$ by (13) and if for some $\epsilon > 0$ we had $\overline{\psi}_2 = \frac{1}{k} - \epsilon$, we would obtain

$$\sum_{j=1}^{k+1} \psi_j(q) = \psi_1(q) + \psi_2(q) + \sum_{j=3}^{k+1} \psi_j(q)$$
$$\leq (-1 + \frac{\epsilon}{3}) + (\frac{1}{k} - \epsilon + \frac{\epsilon}{3}) + (k-1)\frac{1}{k} < -\frac{\epsilon}{3}$$

for a sequence of arbitrary large values q, a contradiction to (6). So (13) again yields $\omega_j = 0$ for $2 \leq j \leq k+1$.

It remains to determine $\widehat{\omega}$. Let X be a real number of the form $X = \lfloor \frac{c_{h+1}}{4} - 1 \rfloor$, so that in particular we have $c_h \leq X < c_{h+1}$.

Putting $j = 1$ in (53) and noting $X = \frac{c_{h+1}}{4} - 1 \geq \frac{c_{h+1}}{5}$ in the case $g \geq h + 2 - k$ we obtain

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\log_{\frac{c_{h+1}}{5}} \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(c_{h+1}) - \log(c_h)}{\log(c_{h+1}) - \log(5)}. \quad (54)$$

If we now fix a residue class $\overline{h}$ for the values of h in the residue system $\{1, 2, \ldots, k\}$ mod k, the right hand side of (54) tends to one of the fractions in the definition of $\widehat{\varphi}(\eta)$ as $h \to \infty$. In fact, as $\overline{h}$ runs through the residue system $\{1, 2, \ldots, k\}$ mod k this induces a bijection between the residue classes $\overline{h}$ of the residue system $\{1, 2, \ldots, k\}$ mod k and the expressions of $\widehat{\omega}$.

In the case $g \leq h + 1 - k$ as $\max_{1\leq t\leq k} |x\zeta_t - y_t|$ is essentially bounded below by $\frac{1}{q_{N-1,i+1}}$ again, we have the upper estimate

$$-\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq -\log_{\frac{c_{h+1}}{5}} \max_{1\leq t\leq k} |x\zeta_t - y_t| \leq \frac{\log(q_{N,h})}{\log(q_{N+1,h+1}) - \log(5)}.$$

The right hand side, however, is smaller than the corresponding value in (54) for every h, so the case $g \leq h + 1 - k$ does never give any improvement. Thus, by its definition, the quantity $\widehat{\omega}$ can be estimated above by the minimum of the expressions of $\widehat{\varphi}(\eta)$, which simply is $\widehat{\varphi}(\eta)$.

On the other hand, fixing a residue classes $\overline{h}$ for h in the residue system $\{1, 2, \ldots, k\}$ mod k again and putting $x := c_h$ and $y_t := \lfloor \zeta_t c_h \rfloor$ for $1 \leq t \leq k$, we obtain a bijection between the resulting values

$$\lim_{h\in\overline{h}, h\to\infty} -\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t| = \lim_{h\in\overline{h}, X\to\infty} -\log_X \max_{1\leq t\leq k} |x\zeta_t - y_t|$$

as $\overline{h}$ runs through $\{1, 2, \ldots, k\}$ and the expressions involved in the definition of $\widehat{\varphi}_1(\eta)$. Hence $\widehat{\omega}$ is at least as large as the minimum of these expressions, which again is $\widehat{\omega}$. □

Note, that in the special case $k = 2$ we have $\eta_1 + \eta_2 = 1$, and the approximation constants of $\boldsymbol{\zeta}$ in Theorem 1 become

$$\omega = \infty\,, \qquad \omega_2 = 1\,, \qquad \omega_3 = \eta_1\,,$$
$$\widehat{\omega} = \eta_2\,, \qquad \widehat{\omega}_2 = 0\,, \qquad \widehat{\omega}_3 = 0\,.$$

Particularly, we see that $\widehat{\omega} + \omega_3 = 1$, which is easily seen by straightforward computation with repeated use of (13) to be equivalent to Jarník's identity in the form $\overline{\psi}_1 + 2\overline{\psi}_1\underline{\psi}_3 + \underline{\psi}_3 = 0$, see Theorem 1.5 in [5].

So far we have not asked for the numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ to be $\mathbb{Q}$-linearly independent together with 1, which is the usual assumption. For this purpose, we apply Theorem 1 in the special case $\zeta_j = \sum_{n\geq 1} 2^{-a_{n,j}}$ with suitable sequences $(a_{n,j})_{n\geq 1}$ for $1 \leq j \leq k$.

Corollary 2. *For $1 \leq j \leq k$ let $(a_{n,j})_{n\geq 1}$ be sequences with the properties*

$$a_{1,1} < a_{1,2} < \ldots < a_{1,k} < a_{2,1} < a_{2,2} < \ldots a_{2,k} < a_{3,1} < \ldots \tag{55}$$

and for $\eta \in \mathbb{R}^k \times \overline{\mathbb{R}}$ as in Theorem 1 put

$$\lim_{n\to\infty} \frac{a_{n+1,1} - a_{n,k}}{a_{n+1,k}} = \eta_1 \tag{56}$$

$$\lim_{n\to\infty} \frac{a_{n,i} - a_{n,i-1}}{a_{n,k}} = \eta_i, \quad 2 \leq i \leq k. \tag{57}$$

$$\lim_{n\to\infty} \frac{a_{n+1,1}}{a_{n,k}} = \eta_{k+1} = \infty \tag{58}$$

and $\zeta_j = \sum_{n\geq 1} 2^{-a_{n,j}}$ for $1 \leq j \leq k$. Then the corresponding approximation constants are given as in Theorem 1.

Proof. Clearly, if we put $q_{n,j} = 2^{a_{n,j}}$, all conditions of Theorem 1 are satisfied. □

Now one can easily prove that there are uncountably many vectors $\boldsymbol{\zeta} \in \mathbb{R}^k$, such that additionally $1, \zeta_1, \zeta_2, \ldots, \zeta_k$ are linearly independent over $\mathbb{Q}$. The arguments of the proof of the following Proposition 3 are suitable to prove the existence of vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ for which $1, \zeta_1, \zeta_2, \ldots, \zeta_k$ are linearly independent subject to certain approximation properties if this existence was established without the linear independence condition.

Proposition 3. *One can choose sequences $(a_{n,j})_{n\geq 1}$ in Corollary 2 such that*

$$\{1, \zeta_1, \ldots, \zeta_k\}$$

is linearly independent over $\mathbb{Q}$.

Proof. Note that in the case $k = 1$ (32)–(34) simply yield $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty$ and it follows from Liouville's Theorem that the corresponding number of the form $\zeta = \sum_{n\geq 1} 2^{-a_n}$ is transcendental, in particular $\{1, \zeta\}$ is linearly independent over $\mathbb{Q}$. In the case $k \geq 2$ consider the numbers $a_{1,j} = j$ for $1 \leq j \leq k$ and define

the sequences $(a_{n,j})_{n\geq 1}$ by the recurrence relations

$$a_{n+1,1} = \left\lfloor \frac{1}{\eta_1} n \cdot a_{n,k}(\eta_1) \right\rfloor \tag{59}$$

$$a_{n+1,2} = \left\lfloor \frac{1}{\eta_1} n \cdot a_{n,k}(\eta_1 + \eta_2) \right\rfloor \tag{60}$$

$$\vdots$$

$$a_{n+1,k} = \left\lfloor \frac{1}{\eta_1} n \cdot a_{n,k}(\eta_1 + \eta_2 + \cdots + \eta_k) \right\rfloor. \tag{61}$$

One checks that (59)–(61) imply (32)–(34). Now we prove that we can change (59)–(61) slightly such that $\{1, \zeta_1, \ldots, \zeta_k\}$ is linearly independent over $\mathbb{Q}$.

Let $(b_n)_{n\geq 1}$ be the ordered combined set of all $a_{n,j}$ defined as above. Note that we can obviously "disturb" the system (59)–(61) a little by adding one to the elements of the form b_a where $a \in A$ with A an arbitrary subset of $\mathbb{N}$, without violating (59)–(61). Noting that (59)–(61) imply $\lim_{n\to\infty} \frac{a_{n+1,1}}{a_{n,1}} = \infty$, by the considerations of the case $k = 1$ we know $\{1, \zeta_1\}$ is a $\mathbb{Q}$-linearly independent set, where ζ_1 is generated by the sequence $(a_{n,1})_{n\geq 1}$ defined above, ie $\zeta_1 = \sum_{n\geq 1} a_{n,1}^{-1}$. If we now consider the set of all sequences $\mathscr{A}_2 = \{(a'_{n,2})_{n\geq 1}\}$ which arise from $(a_{n,2})_{n\geq 1}$ as above by adding 1 to b_a with $a \in A_2$ for an arbitrary subset A_2 of $\mathbb{N}$, we see that $\mathscr{A}_2$ is uncountable. So there must be a number ζ'_2 generated by an $a'_{n,2} \in \mathscr{A}_2$ with the property that $\{1, \zeta_1, \zeta'_2\}$ are linearly independent over $\mathbb{Q}$, since the $\mathbb{Q}$-span of $\{1, \zeta_1\}$ is only countable. Now we proceed analoguously with sets $\mathscr{A}_j$ for $3 \leq j \leq k$ and finally get a $\mathbb{Q}$-linearly independent set $\{1, \zeta_1, \zeta'_2, \ldots, \zeta'_k\}$. As mentioned above, the set $\{\zeta_1, \zeta'_2, \ldots \zeta'_k\}$ fulfills all the requirements. □

Remark 1. Since algebraic numbers have countable cardinality one can readily generalize the proof above to show that we can even ask $\boldsymbol{\zeta}$ to be algebraically independent.

We now give some applications of the above theorem. Note that (4), (5) and (8) imply that for all $\epsilon > 0$ and sufficiently large $Q = Q(\epsilon) > 0$ we have the bounds

$$\begin{aligned} -1 \leq \psi_1(Q) &\leq 0 \\ \frac{j-k-1}{kj} - \epsilon \leq \psi_j(Q) &\leq \frac{1}{k}, \qquad 2 \leq j \leq k+1. \end{aligned}$$

In the first Corollary we construct $\zeta_1, \zeta_2, \ldots, \zeta_k$ for which each $\psi_j(Q)$ takes each of the values inside of the corresponding intervals $I_1 := (-1, 0), I_j := \left(\frac{j-k-1}{kj}, \frac{1}{k}\right)$ for arbitrarily large Q simultaneously for all $1 \leq j \leq k+1$. So roughly speaking in this case all ψ_j take their possible range of values for arbitrarily large (Q, ∞). In particular the bounds in (8) are best possible.

Corollary 3. *There exist $\zeta_1, \zeta_2, \ldots, \zeta_k$ for which the set $\{1, \zeta_1, \zeta_2, \ldots, \zeta_k\}$ is $\mathbb{Q}$ linearly independent and such that*

$$\begin{aligned} \omega_j &= \frac{1}{j-1}\,, && 1 \le j \le k+1\,,\\ \widehat{\omega} &= \frac{1}{k}\,, \\ \widehat{\omega}_j &= 0\,, && 2 \le j \le k+1\,. \end{aligned}$$

Proof. Note that by means of proposition 3 for every $\eta \in \mathbb{R}^{k+1}$ subject to the restrictions of Theorem 1 we can construct $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_k)$ together with 1 linearly independent over $\mathbb{Q}$ such that Theorem 1 holds. Putting $\eta_2 = \eta_3 = \cdots = \eta_{k+1} = \frac{1}{k}$ in Theorem 1 immediately gives all the stated equalities for this $\boldsymbol{\zeta}$. $\square$

Now we want to give our first explicit construction of special cases of Schmidt's conjecture, which was proved by Moshchevitin in a nonconstructive way in [3]. The conjecture states, that for each integer pair (k, i) with $k \ge 2, 1 \le i \le k-1$ there exists a vector $\boldsymbol{\zeta} \in \mathbb{R}^k$ with $\{1, \boldsymbol{\zeta}\}$ linearly independent over $\mathbb{Q}$ such that $\lim_{q\to\infty} \lambda_i(q) = 0$ and $\lim_{q\to\infty} \lambda_{i+2} = \infty$. Note that we cannot have $\lim_{q\to\infty} \lambda_i(q) = 0$, $\lim_{q\to\infty} \lambda_{i+1}(q) = \infty$ for any i because of the assumption of linear independence because of (7), see also the introduction in [3]. We now give a generalisation of this fact in the special case $i = 1$ for arbitrary $k \ge 2$.

Corollary 4. *Let $k \ge 2$ and $3 \le r \le k+1$ be integers. Then there exists $\boldsymbol{\zeta} \in \mathbb{R}^k$ with $\{1, \boldsymbol{\zeta}\}$ linearly independent over $\mathbb{Q}$ such that*

$$\begin{aligned} &\overline{\psi}_1 < 0\,,\\ &\underline{\psi}_j < 0 < \overline{\psi}_j\,, && 2 \le j \le r-1\,,\\ &\underline{\psi}_j > 0\,, && r \le j \le k+1\,. \end{aligned}$$

The case $r = 3$ clearly implies Schmidt's conjecture for $i = r - 2 = 1$.

Proof. We may assume $k \ge 3$, because for $k = 2$ the Corollary only states that $\overline{\psi}_1 < 0$ and $\underline{\psi}_3 > 0$ is possible, which only requires $\overline{\psi}_1 < 0$ by (6) and for any choice of $\eta = (\eta_1, \eta_2) \ne (1/2, 1/2)$ the construction of Theorem 1 gives an example. We apply Theorem 1 with η defined by

$$\eta_1 = \frac{\alpha^{k-1}}{1 + \alpha + \cdots + \alpha^{k-1}}, \tag{62}$$

$$\frac{\eta_j}{\eta_{j+1}} = \alpha, \qquad 1 \le j \le k-1. \tag{63}$$

The parameter $\alpha \in \{0, 1\}$ will be chosen later in dependence of (r, k). First note

that by (13) our inequalities translate to

$$\widehat{\omega} > \frac{1}{k}\,, \tag{64}$$

$$\widehat{\omega}_j < \frac{1}{k} < \omega_j\,, \qquad 2 \le j \le r-1\,, \tag{65}$$

$$\omega_j < \frac{1}{k}\,, \qquad r \le j \le k+1\,. \tag{66}$$

Now note, that the left hand side of (65) trivially holds by Theorem 1. With (62), (63) the quantity $\widehat{\wp}(\eta)$ of Theorem 1 becomes

$$\widehat{\omega} = \min\left\{1, \frac{1}{1+\alpha}, \dots, \frac{1}{1+\alpha+\cdots+\alpha^{k-1}}\right\} = \frac{1}{1+\alpha+\cdots+\alpha^{k-1}}.$$

Moreover the assumption $\alpha < 1$ implies that (64) holds. To obtain the remaining inequalities it is obviously sufficient to prove $\omega_r < \frac{1}{k} < \omega_{r-1}$ for some α, which by (62), (63) and Theorem 1 is equivalent to

$$\frac{\alpha^{r-2}}{1+\alpha+\cdots+\alpha^{r-2}} < \frac{1}{k} < \frac{\alpha^{r-3}}{1+\alpha+\cdots+\alpha^{r-3}}, \tag{67}$$

since the last term in $\wp_j(\eta)$ is easily seen to be the largest in $\wp_j$ in our special case of constant ratios. For $r = 3$ this reduces to $\frac{\alpha}{\alpha+1} < \frac{1}{k}$, which is obviously true if we choose any $\alpha \in (0, \frac{1}{k})$, so we can assume $r \ge 4$. Defining the functions

$$\Phi_u \colon \alpha \longmapsto \frac{\alpha^u}{1+\alpha+\cdots+\alpha^u}$$

shows that (67) in the cases left to consider is equivalent to $\phi_{u+1}(\alpha) < \frac{1}{k} < \phi_u(\alpha)$ for $1 \le u \le k-2$. It is easy to check that all these ϕ_u are continuous, $\phi_u(\alpha) > \phi_{u+1}(\alpha)$ and $\phi_u(0) = 0, \phi_u(1) = \frac{1}{u+1} > \frac{1}{k}$. Further more from

$$\begin{aligned}\Phi_u'(\alpha) &= \frac{u\alpha^{u-1}(1+\alpha+\cdots+\alpha^u) - \alpha^u(1+2\alpha+\cdots+u\alpha^{u-1})}{(1+\alpha+\cdots+\alpha^u)^2} \\ &= \frac{\alpha^{2u-2} + 2\alpha^{2u-3} + \cdots + (u-1)\alpha^u + u\alpha^{u-1}}{(1+\alpha+\cdots+\alpha^u)^2} > 0\end{aligned}$$

we deduce that they are monotonically increasing in α. Combination of these properties implies that for fixed u there exists some $t \in (0,1)$ such that $\phi_u(t) = \frac{1}{k}$ by intermediate value theorem. It further follows from these considerations on the one hand $\phi_u(\alpha) > \frac{1}{k}$ for $\alpha > t$, and on the other hand the existence of an interval $\alpha \in (t_0, t_1)$ with $t_0 < t < t_1$ such that $\phi_{u+1}(\alpha) < \frac{1}{k}$. Thus, for all $\alpha \in (t, t_1)$ we have $\phi_{u+1}(\alpha) < \frac{1}{k} < \phi_u(\alpha)$. □

2.3 The case $\omega < \infty$

We now aim to give similar results for vectors $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \dots, \zeta_k)$ whose components ζ_j have one-dimensional approximation constant $\omega < \infty$. Hence in particular

the simultaneous approximation constant ω is finite too, as by definition it cannot exceed the minimum of the one-dimensional constants. As in Theorem 1 each ζ_j will be the sum of the reciprocals of integers $q_{n,j}$ that satisfy (30), (31), but the conditions (32)–(37) will be altered in ways to obtain a symmetric situation in all ζ_j, which will be more convenient for the purposes of section 3. We start with proving the following

Theorem 2. *For $1 \le j \le k$ let $\zeta_j = \sum_{n\ge 1} \frac{1}{q_{n,j}}$ where $(q_{n,j})_{n\ge 1}$ are sequences of integers for which* (30), (31) *are satisfied and that for*

$$(b_n)_{n\ge 1} = \big(\log(q_{1,1}), \log(q_{2,1}), \dots, \log(q_{k,1}), \log(q_{1,2}), \dots\big)$$

the inequality

$$\liminf_{n\to\infty} \frac{b_{n+1}}{b_n} > 2$$

is satisfied. Then the first $(k-1)$ approximation constants relative to $\zeta_1, \zeta_2, \dots, \zeta_k$ are given by

$$\begin{aligned}
\omega &= \limsup_{n\to\infty} \frac{b_{n+1}-b_n}{b_n}, & \widehat{\omega} &= \liminf_{n\to\infty} \frac{b_n - b_{n-1}}{b_n}\,,\\
\omega_2 &= \limsup_{n\to\infty} \frac{b_n-b_{n-1}}{b_n}, & \widehat{\omega}_2 &= \liminf_{n\to\infty} \frac{b_{n-1} - b_{n-2}}{b_n}\,,\\
\omega_3 &= \limsup_{n\to\infty} \frac{b_{n-1}-b_{n-2}}{b_n}, & \widehat{\omega}_3 &= \liminf_{n\to\infty} \frac{b_{n-2} - b_{n-3}}{b_n}\,,\\
&\vdots & &\vdots\\
\omega_{k-1} &= \limsup_{n\to\infty} \frac{b_{n-k+3}-b_{n-k+2}}{b_n}, & \widehat{\omega}_{k-1} &= \liminf_{n\to\infty} \frac{b_{n-k+2} - b_{n-k+1}}{b_n}\,.
\end{aligned}$$

Further more we have the inequlities

$$\begin{aligned}
\omega_k &\ge \limsup_{n\to\infty} \frac{b_{n-k+2}-b_{n-k+1}}{b_n}, & \widehat{\omega}_k &\ge \liminf_{n\to\infty} \frac{b_{n-k+1} - b_{n-k}}{b_n}\,,\\
\omega_{k+1} &\ge \limsup_{n\to\infty} \frac{b_{n-k+1}-b_{n-k}}{b_n}, & \widehat{\omega}_{k+1} &\ge \liminf_{n\to\infty} \frac{b_{n-k} - b_{n-k-1}}{b_n}\,.
\end{aligned}$$

Proof. Denote the right hand side expressions by $\wp_j$ respectively $\widehat{\wp}_j$. We prove the inequalities $\omega_j \ge \wp_j$, $\widehat{\omega}_j \ge \widehat{\wp}_j$ for $1 \le j \le k+1$ and then $\omega_j \le \wp_j$, $\widehat{\omega}_j \le \widehat{\wp}_j$ for $1 \le j \le k-1$. This obviously yields the assertions of the Theorem.

Throughout the proof, for arbitrary X let the integer h be determined as the index for which $c_h \le X < c_{h+1}$. Obviously $X \to \infty$ is equivalent to $h \to \infty$, which will often be used implicitly. We first prove $\omega_j \ge \wp_j$ for $1 \le j \le k+1$.

Assume j arbitrary but fixed. Observe that for every h if we put $x^{(i)} = c_{h+1-i}$, $y_t^{(i)} = \lfloor \zeta_t x^{(i)} \rfloor$ for $1 \le t \le k$ and $1 \le i \le j$, we obtain

$$\max_{1\le i\le j} \max_{1\le t\le k} |\zeta_t x^{(j)} - y_t^{(j)}| = |\zeta_{t_0} x^{(j)} - y_{t_0}^{(j)}| = \frac{c_{h+1-j}}{c_{h+2-j}} + o\left(\frac{c_{h+1-j}}{c_{h+2-j}}\right) \tag{68}$$

as $h \to \infty$ with $t_0 := h + 2 - j$, as explained in the proof of Theorem 1. Choose an integer sequence $(h_r)_{r\geq 1}$ of values for h in (68) such that with $(X_r)_{r\geq 1} := (x_r)_{r\geq 1} := (c_{h_r})_{r\geq 1}$ we have

$$\begin{aligned}\lim_{r\to\infty} -\log_{X_r} \frac{c_{h_r+1-j}}{c_{h_r+2-j}} &= \limsup_{h\to\infty} -\log_{c_h} \frac{c_{h+1-j}}{c_{h+2-j}} \\ &= \limsup_{h\to\infty} \frac{\log(c_{h+2-j}) - \log(c_{h+1-j})}{\log(c_h)}\,,\end{aligned} \tag{69}$$

which is possible by definition of the lim sup. Leaving out the linearly independence condition, the so constructed vectors $\big(x^{(i)}, y_1^{(i)}, \ldots, y_k^{(i)}\big), 1 \leq i \leq j$, lead to the values $\wp_j$ as lower bounds for ω_j in view of (68). However, the missing linearly independence condition is now obtained exactly as in Theorem 1.

To prove $\widehat{\omega}_j \geq \widehat{\wp}_j$, consider any sequence $(h_r)_{r\geq 1}$ and the same approximation vectors as in the proof of $\omega_j \geq \wp_j$ but take logarithms to base $c_{h+1} > X$ instead of base c_h. This yields a lower estimate for $\omega_j(X)$ for $X \in [c_{h_r}, c_{h_{r+1}})$. As this is valid for any sequence $(h_r)_{r\geq 1}$ we obtain the lower bounds $\widehat{\wp}_j = \liminf_{X\to\infty} \omega_j(X)$ for $\widehat{\omega}_j$ by definition of $\liminf$, as we claimed. So far we have established the lower bounds $\wp_j$ (respectively $\widehat{\wp}_j$) for ω_j (respectively $\widehat{\omega}_j$) for $1 \leq j \leq k+1$.

For the upper bounds note first that with basically the same arguments as in the proof of step 3 in Theorem 1 we can restrict to the case $c_h \leq X \leq \frac{1}{4}c_{h+1}$. Further step 1 and step 2 of $\wp_j(\eta) \leq \omega_j$ in the proof of Theorem 1 remain valid in the present situation. Indeed, the estimates (39), (40) are already valid under the assumtion $\frac{b_{n+1}}{b_n} > 2$, which is weaker than (34) used in Theorem 1. The proof of step 2 is analoguous.

Now for every fixed X we divide all approximation vectors $(x, y_1, \ldots, y_k)$ with $x \leq X$ into two categories. Let g be the largest integer such that $c_{g-1} \mid x$ for an approximation vector $(x, y_1, \ldots, y_k)$ as in step 1 of Theorem 1. The distinction of vectors with $g > h+1-k$, which we will call vectors of category 1, and $g \leq h+1-k$, which we will call vectors of category 2, now leads to 2 cases.

Case 1: If for fixed X with $c_h \leq X < \frac{1}{4}c_{h+1}$ we have $g > h+1-k$ for an approximation vector (i.e. it belongs to category 1), (39) implies that

$$\omega_j(X) \leq -\log_{c_h} \frac{c_{h+1-j}}{c_{h+2-j}} = \frac{\log(c_{h+2-j}) - \log(c_{h+1-j})}{\log(c_h)}, \qquad 1 \leq j \leq k+1. \tag{70}$$

So we have that for every X the quantity $\wp_j$ is an upper bound for $\omega_j^1(X)$, by which we mean the supremum over all real numbers ν such that (12) has j linearly independent vector solutions all of which are of the first category. Hence if we define $\omega_j^1 = \limsup_{X\to\infty} \omega_j^1(X)$, we get

$$\omega_j^1 \leq \wp_j, \qquad 1 \leq j \leq k+1. \tag{71}$$

In order to give a connection between the approximation constants $\widehat{\omega}_j$ and approximation vectors of category 1, we define $\widehat{\omega}_j^1 := \liminf_{X\to\infty} \omega_j^1(X)$. We start with an arbitrary sequence $(X_r)_{r\geq 1}$ with corresponding subsequence $(c_{h_r})_{r\geq 1}$ of $(c_h)_{h\geq 1}$ and define a sequence $(X_r')_{r\geq 1}$ by putting $X_r' := \frac{1}{5}c_{h_r+1}$. In view of (39) and

observing that the fractions $\frac{c_m}{c_{m+1}}$ are monotonically decreasing by our assumption $\frac{b_{n+1}}{b_n} > 2$, we get the upper estimate for the approximation constants $\omega_j^1(X_r')$

$$\omega_j^1(X_r') \le -\log_{\frac{1}{5}X_{r+1}}\left(\frac{c_{h_r+2-j}}{c_{h_r+1-j}}\right) = \frac{\log(c_{h_r+2-j}) - \log(c_{h_r+1-j})}{\log(c_{h_r+1}) - \log(5)} \tag{72}$$

for $1 \le j \le k+1$. If we specify a sequence $(X_r)_{r\ge 1}$ for which the corresponding sequence $(h_r)_{r\ge 1}$ has the property

$$\begin{aligned}\lim_{r\to\infty} -\log_{h_r}\frac{c_{h_r+1-j}}{c_{h_r+2-j}} &= \liminf_{h\to\infty} -\log_{c_h}\frac{c_{h+1-j}}{c_{h+2-j}} \\ &= \liminf_{h\to\infty}\frac{\log(c_{h+2-j}) - \log(c_{h+1-j})}{\log(c_h)}\,,\end{aligned} \tag{73}$$

which is possible again by definition of lim inf, we put $n = h+1$ in the definition of $\widehat{\wp}_j$ so that the right hand side of (72) tends to $\widehat{\wp}_j$ as $r \to \infty$. Thus $\lim_{r\to\infty}\omega_j^1(X_{\underline{r}})$ exists and is bounded above by $\widehat{\wp}_j$. In particular

$$\widehat{\omega}_j^1 \le \widehat{\wp}_j, \qquad 1 \le j \le k+1. \tag{74}$$

Case 2: In the other case $g \le h+1-k$ (i.e. the vector belongs to category 2), consider first an arbitrary sequence $(X_r)_{r\ge 1}$ that tends monotonically to infinity and the corresponding subsequence $(c_{h_r})_{r\ge 1}$ of $(c_h)_{h\ge 1}$ determined by $c_{h_r} \le X_r < c_{h_{r+1}}$. Recall that without loss of generality we can assume $c_{h_r} \le X_r < \frac{1}{4}c_{h_r+1}$, as this does not affect the approximation constants. By (40) we have

$$\max_{1\le t\le k}|\zeta_t x - y_t| \ge \frac{1}{c_{h_r+2-k}} - o\left(\frac{1}{c_{h_r+2-k}}\right). \tag{75}$$

Furthermore define $\omega_j^2(X)$ as the supremum of all ν, such that (12) has j linearly independent vector solutions with at least one vector of category 2, for every fixed $X > 0$, and define $w_j^2 := \limsup_{X\to\infty} w_j^2(X)$. Taking logarithms to the bases $c_{h_r} < X_r$ for $r \to \infty$, (75) implies that for $1 \le j \le k+1$ the expression $\omega_j^2(X_r)$ is bounded above by

$$-\log_{X_r}\left(\frac{c_{h_r+2-k}}{c_{h_r+3-k}}\right) \le -\log_{c_{h_r}}\left(\frac{c_{h_r+2-k}}{c_{h_r+3-k}}\right) = \frac{\log(c_{h_r+3-k}) - \log(c_{h_r+2-k})}{\log(c_{h_r})}. \tag{76}$$

Note that the bound in (76) is valid for any sequence $(X_r)_{r\ge 1}$ and the corresponding sequence $(c_{h_r})_{r\ge 1}$. Hence on the one hand we have

$$\mathscr{A} := \limsup_{h\to\infty}\frac{\log(c_{h+3-k}) - \log(c_{h+2-k})}{\log(c_h)} \ge \limsup_{X\to\infty}\omega_j^2(X) = \omega_j^2 \tag{77}$$

simply by the definition of lim sup. Observe $\mathscr{A} = \wp_{k-1}$ (where n in the definition of $\wp_{k-1}$ corresponds to h in the definition of $\mathscr{A}$) and hence

$$\omega_j^2 \le \wp_{k-1}, \qquad 1 \le j \le k+1. \tag{78}$$

On the other hand, for any sequence $(X_r)_{r\geq 1}$ by (75) and our assumption $\frac{b_{n+1}}{b_n} = \frac{\log(c_{n+1})}{\log(c_n)} > 2$, for r sufficiently large and $1 \leq j \leq k-1$ we have

$$\begin{aligned} \omega_j^2(X_r) &\leq -\log_{X_r}\left(\frac{1}{c_{h_r+2-k}} - o\left(\frac{1}{c_{h_r+2-k}}\right)\right) \\ &< -\log_{X_r}\left(\frac{c_{h_r+2-k}}{c_{h_r+3-k}}\right) \leq \omega_{k-1}^1(X_r), \end{aligned} \tag{79}$$

where the right inequality is a consequence of the constructions $\omega_j \geq \wp_j$, $\widehat{\omega}_j \geq \widehat{\wp}_j$ in the case $j = k-1$ (see (68), (69)). Since this holds for any sequence $(X_r)_{r\geq 1}$, we have

$$\widehat{\omega}_j^2 := \liminf_{X\to\infty} \omega_j^2(X) \leq \liminf_{X\to\infty} \omega_{k-1}^1(X) = \widehat{\omega}_{k-1}^1 \leq \widehat{\omega}_j^1, \quad 1 \leq j \leq k-1. \tag{80}$$

We now combine the results above for the quantities $\omega_j^s, \widehat{\omega}_j^s$, $s \in \{1,2\}$ to derive the required upper bounds. As every approximation vector is either of category 1 or category 2 for fixed $X > 0$, the definitions of $\omega_j^s, s \in \{1,2\}$, imply $\omega_j(X) = \max\{\omega_j^1(X), \omega_j^2(X)\}$ for every $X > 0$. Observe that for any functions $f, g : \mathbb{R}^+ \mapsto \mathbb{R}$ we have

$$\max\left\{\limsup_{X\to\infty} f(X), \limsup_{X\to\infty} g(X)\right\} = \limsup_{X\to\infty} \max\left\{f(X), g(X)\right\}.$$

Applying this on $f(X) = \omega_j^1(X)$, $g(X) = \omega_j^2(X)$ implies that ω_j, which is by definition $\limsup_{X\to\infty} \omega_j(X)$, equals the maximum of $\omega_j^1 = \limsup_{X\to\infty} \omega_j^1(X)$ and $\omega_j^2 = \limsup_{X\to\infty} \omega_j^2(X)$. By (71), (78) for $1 \leq j \leq k-1$ this maximum is $\max\{\wp_j, \wp_{k-1}\} = \wp_j$, which proves the upper bounds for ω_j, $1 \leq j \leq k-1$.

It remains to check the upper estimates for the constants $\widehat{\omega}_j$. In view of $\omega_j(X) = \max\{\omega_j^1(X), \omega_j^2(X)\}$, (74) and (80), we obtain

$$\widehat{\omega}_j = \liminf_{X\to\infty} \max_{s=1,2}\{\omega_j^s(X)\} = \liminf_{X\to\infty} \omega_j^1(X) = \widehat{\omega}_j^1 \leq \widehat{\wp}_j$$

for $1 \leq j \leq k-1$. □

Remark 2. It is rather clear from the proof that Theorem 2 remains valid for $C = \infty$ too. We will need this later in Theorem 4.

Corollary 5. *Let the assumptions of Theorem 2 be satisfied and assume further the existence of the limit of the quotients $\frac{b_{n+1}}{b_n}$, i.e.*

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} =: C \geq 2.$$

Then the first $(k-1)$ approximation constants are given by

$$\begin{aligned} \omega &= C-1 & &\vdots \\ \omega_2 &= \frac{C-1}{C} = \widehat{\omega} & \omega_{k-1} &= \frac{C-1}{C^{k-2}} = \widehat{\omega}_{k-2} \\ \omega_3 &= \frac{C-1}{C^2} = \widehat{\omega}_2 & & \frac{C-1}{C^{k-1}} = \widehat{\omega}_{k-1}. \end{aligned}$$

For the remaining approximation constants we have the inequalities

$$\begin{aligned} \omega_k &\geq \frac{C-1}{C^{k-1}} \\ \omega_{k+1} \geq \widehat{\omega}_k &\geq \frac{C-1}{C^k} \\ \widehat{\omega}_{k+1} &\geq \frac{C-1}{C^{k+1}}. \end{aligned}$$

Proof. For every $h \geq 1$ we have

$$\lim_{n\to\infty} \frac{b_{n+h}}{b_n} = \lim_{n\to\infty} \frac{b_{n+h}}{b_{n+h-1}} \frac{b_{n+h-1}}{b_{n+h-2}} \cdots \frac{b_{n+1}}{b_n} = C^h$$

and hence Theorem 2 yields the claimed result. □

Remark 3. The bounds for $\omega_k, \widehat{\omega}_k, \omega_{k+1}, \widehat{\omega}_{k+1}$ could be improved further to

$$\begin{aligned} \frac{C-1}{C^{k-1}} \leq \omega_k &\leq \max\left\{\frac{C}{C^k-1}, \frac{C-1}{C^{k-1}}\right\} \\ \widehat{\omega}_k &= \frac{C-1}{C^k-1} \\ \omega_{k+1} &= \frac{1}{C^{k-1}} \\ \min\left\{\frac{1}{C^k-1}, \frac{C-1}{C^k}\right\} \leq \widehat{\omega}_{k+1} &\leq \frac{1}{C^k-1} \end{aligned}$$

by a rather long and technical proof that we will not present here. In particular in the case $C \geq \beta_k > 2$, where β_k is the largest real root of $P_k(x) = x^{k+1} - 2x^k - x + 1$, we have

$$\begin{aligned} \omega_k &= \frac{C-1}{C^{k-1}} \\ \widehat{\omega}_k &= \frac{C-1}{C^k-1} \\ \omega_{k+1} &= \frac{1}{C^{k-1}} \\ \widehat{\omega}_{k+1} &= \frac{1}{C^k-1}\,. \end{aligned}$$

Let us call the assumptions of Theorem 2 without the growth condition of $\frac{b_{n+1}}{b_n}$ the *basic assumptions* of Theorem 2 in the sequel. We can generalize the idea of the proof of Theorem 2 to get

Theorem 3. *Given the basic assumptions of Theorem 2, we consider some fixed $d \in \{1, 2, \ldots, k-1\}$ and define κ_d to be the largest real root of $P_d(x) := x^d - x^{d-1} - 1$. Then if*

1. $\frac{b_{n+1}}{b_n} > \kappa_d, \qquad$ *for all* $\quad n \geq 1,$

2. the sequence $(d_n)_{n\geq 1} := (b_{n+1} - b_n)_{n\geq 1}$ *is monotonically increasing*

are satisfied, the first $(k-d)$ *approximation constants are given by*

$$\begin{aligned}
\omega &= \limsup_{n\to\infty} \frac{b_{n+1}-b_n}{b_n}\,, & \widehat{\omega} &= \liminf_{n\to\infty} \frac{b_n - b_{n-1}}{b_n}\,,\\
\omega_2 &= \limsup_{n\to\infty} \frac{b_n-b_{n-1}}{b_n}\,, & \widehat{\omega}_2 &= \liminf_{n\to\infty} \frac{b_{n-1} - b_{n-2}}{b_n}\,,\\
\omega_3 &= \limsup_{n\to\infty} \frac{b_{n-1}-b_{n-2}}{b_n}\,, & \widehat{\omega}_3 &= \liminf_{n\to\infty} \frac{b_{n-2} - b_{n-3}}{b_n}\,,\\
&\vdots & &\vdots\\
\omega_{k-d} &= \limsup_{n\to\infty} \frac{b_{n-k+d+2}-b_{n-k+d+1}}{b_n}\,, & \widehat{\omega}_{k-d} &= \liminf_{n\to\infty} \frac{b_{n-k+d+1} - b_{n-k+d}}{b_n}\,.
\end{aligned}$$

Furthermore we have the inequalities

$$\begin{aligned}
\omega_{k-d+1} &\geq \limsup_{n\to\infty} \frac{b_{n-k+d+1}-b_{n-k+d}}{b_n}\,, & \widehat{\omega}_{k-d+1} &\geq \liminf_{n\to\infty} \frac{b_{n-k+d+1} - b_{n-k+d}}{b_n}\,,\\
&\vdots & &\vdots\\
\omega_{k+1} &\geq \limsup_{n\to\infty} \frac{b_{n-k+1}-b_{n-k}}{b_n}\,, & \widehat{\omega}_{k+1} &\geq \liminf_{n\to\infty} \frac{b_{n-k} - b_{n-k-1}}{b_n}\,.
\end{aligned}$$

Proof. We proceed as in the proof of Theorem 2 using the fact that d_n is increasing in place of the equivalent fact that $\frac{c_m}{c_{m+1}}$ is monotonically decreasing, which we deduced from the stronger assumptions in Theorem 2 (where it was infered from the stronger assumptions in Theorem 2) up to equation (76). Instead of (76), by our weaker assumption $\frac{b_{n+1}}{b_n} > \kappa_d$ instead of $\frac{b_{n+1}}{b_n} > 2 > \kappa_d$, we obtain the weaker upper bound

$$\omega_j^2(X_r) \leq -\log_{X_r}\left(\frac{1}{c_{h_r+d+1-k}}\right) < -\log_{X_r}\left(\frac{c_{h_r+d+1-k}}{c_{h_r+d+2-k}}\right) \leq \omega_{k-d}^1(X_r),$$

which yields $\wp_{k-d}$ as an upper bound for ω_j instead of $\wp_{k-1}$. We proceed analogously again till (79), instead of which we obtain

$$\omega_j^2(X_r) \leq -\log_{X_r}\left(\frac{1}{c_{h_r+d+1-k}}\right) < -\log_{X_r}\left(\frac{c_{h_r+d+1-k}}{c_{h_r+d+2-k}}\right) \leq \omega_{k-d}^1(X_r).$$

This gives $\widehat{\wp}_{k-d}$ as an upper bound for $\widehat{\omega}_j$ instead of $\widehat{\wp}_{k-1}$. The remainder of the proof is essentially the same as in Theorem 2. □

Remark 4. We clearly have $\lim_{d\to\infty} \kappa_d = 1$. On the other hand $1 + \frac{1}{d} < \kappa_d < 2$ which can esily be derived using the well known monotonic convergence of $(1+\frac{1}{n})^n$ to the Euler number $e \approx 2.71$.

Again, we easily deduce the following

Corollary 6. *Let the basic assumptions of Theorem 2 and condition 1 from Theorem 3 be satisfied. Let us further assume*

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = C \geq \kappa_d.$$

Then the first $(k-d)$ approximation constants are given by

$$\begin{aligned} \omega &= C-1 & &\vdots \\ \omega_2 &= \frac{C-1}{C} = \widehat{\omega} & \omega_{k-d} &= \frac{C-1}{C^{k-d-1}} = \widehat{\omega}_{k-d-1} \\ \omega_3 &= \frac{C-1}{C^2} = \widehat{\omega}_2 & &\frac{C-1}{C^{k-d}} = \widehat{\omega}_{k-d}\,. \end{aligned}$$

Further more we have the inequalities

$$\begin{aligned} \omega_{k-d+1} &\geq \frac{C-1}{C^{k-d}} \\ \omega_{k-d+2} \geq \widehat{\omega}_{k-d+1} &\geq \frac{C-1}{C^{k-d+1}} \\ &\vdots \\ \omega_{k+1} \geq \widehat{\omega}_k &\geq \frac{C-1}{C^k} \\ \widehat{\omega}_{k+1} &\geq \frac{C-1}{C^{k+1}}. \end{aligned}$$

Let us illustrate the results of Corollary 5 and Corollary 6 in the case $k=3$, $C=2$. In the first plot we put

$$\zeta_1' = 2^{-1} + 2^{-15}, \quad \zeta_2' = 2^{-3} + 2^{-31}, \quad \zeta_3' = 2^{-7}$$

which (for numerical purposes) are the initial terms of

$$\zeta_1 = \sum_{n\geq 0} 2^{-2^{3n+1}+1}, \quad \zeta_2 = \sum_{n\geq 0} 2^{-2^{3n+2}+1}, \quad \zeta_3 = \sum_{n\geq 1} 2^{-2^{3n}+1},$$

which clearly satisfy the conditions of Corollary 5 with $C=2$. Notice the special behaviour of $L_k = L_3$, $L_{k+1} = L_4$ in comparison to the first $(k-1)=2$ functions which behave as predicted in Corollary 5.

The assumptions of Corollary 6 are weaker in the sense that either $C<2$ or the quotients $\frac{b_{n+1}}{b_n}$ converge to $C=2$ without being strictly larger than 2 for every sufficiently large n. To illustrate this latter case we may put

$$\zeta_1' = 2^{-2} + 2^{-9}, \quad \zeta_2' = 2^{-3} + 2^{-17}, \quad \zeta_3' = 2^{-5} + 2^{-33}.$$

which are the initial terms of

$$\zeta_1 = \sum_{n\geq 0} 2^{-a_{3n+1}}, \quad \zeta_2 = \sum_{n\geq 0} 2^{-a_{3n+2}}, \quad \zeta_3 = \sum_{n\geq 1} 2^{-a_{3n}}$$

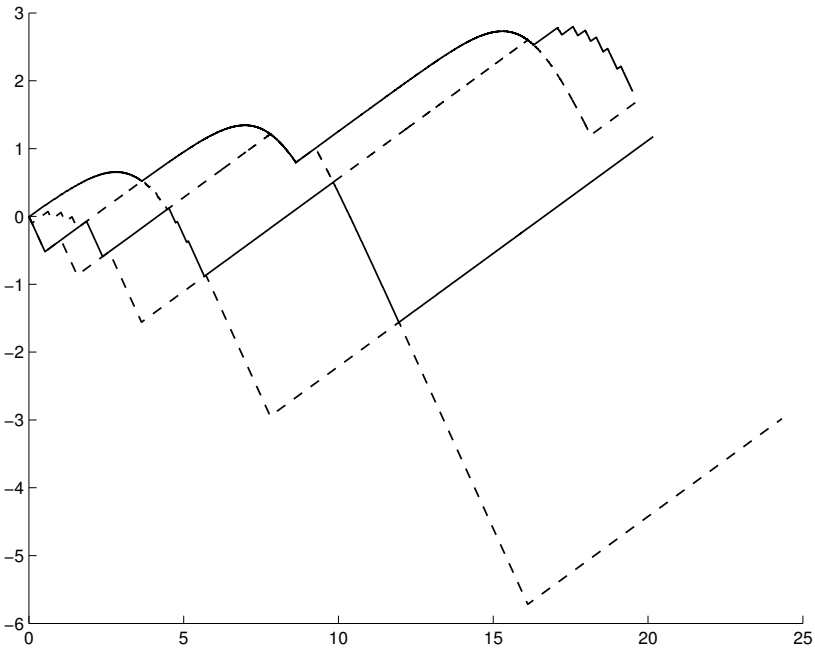

Figure 1: $k = 3$, $C = 2$; illustrates Corollary 5

with $a_1 = 2$ and $a_{n+1} = 2a_n - 1$ for all $n \geq 1$, which fulfills the conditions of Corollary 6 with $d = 2$. Indeed, we will see a different behaviour of L_2 compared to the previous picture. Only $L_{k-d} = L_1$ has the predicted shape. We can apply Corollary 6 to construct many more cases of Schmidt's conjecture explicitely. For simplicity of the proof we first deduce another easy Corollary from Corollary 6.

Corollary 7. *For $k \geq 2$ let $1 \leq d \leq k-1$ be an arbitrary integer. For any $C > \kappa_d$ there exists a sequence of positive integers $(b_n)_{n\geq 1}$ such that $\frac{b_{n+1}}{b_n} > C$ for all n and $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = C$. If $(a_{n,j})_{n\geq 1}$ for $1 \leq j \leq k$ are k sequences satisfying (55) such that $(b_n)_{n\geq 1}$ is their ordered mixed sequence, then for $\zeta_j = \sum_{n\geq 1} 2^{-a_{n,j}}$ the result of Corollary 6 is valid.*

Furthermore we can choose the sequence $(b_n)_{n\geq 1}$ such that $1, \zeta_1, \ldots, \zeta_k$ are $\mathbb{Q}$-linearly independent.

Proof. The sequence $(b_n)_{n\geq 1}$ defined by $b_1 = S$ and $b_{n+1} = \lceil Cb_n \rceil$ with S sufficiently large that $b_2 > b_1$ clearly satisfies the stated properties. Putting $q_{n,j} = 2^{a_{n,j}}$ we see that the assumptions of Corollary 6 are satisfied, since it clearly makes no difference if we take the logarithm to base 2 instead of e as the quotients $\frac{b_{n+1}}{b_n}$ don't change. By a variation of $(b_n)_{n\geq 1}$ as in the proof of Corollary 3 we can guarantee the linear independence. □

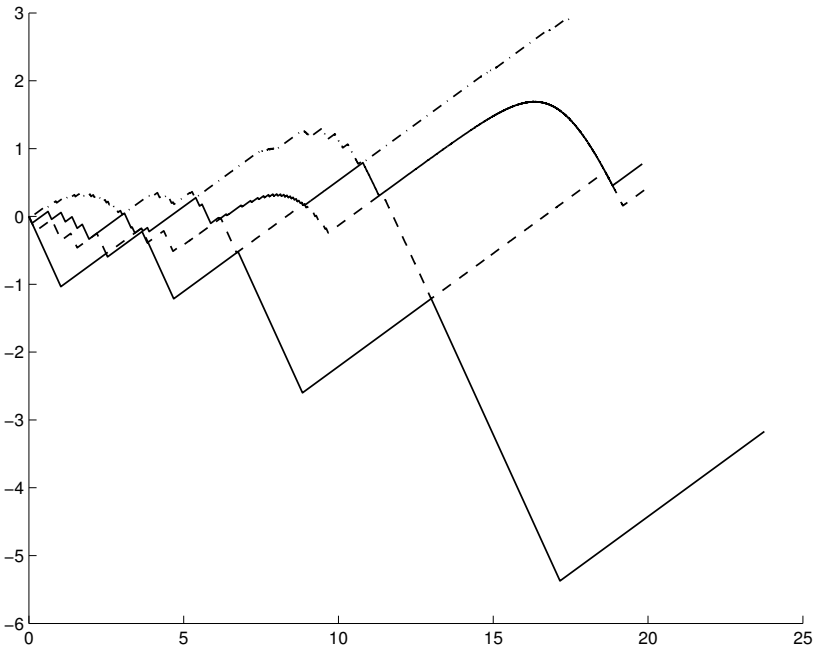

Figure 2: $k = 3$, $C = 2$; illustrates Corollary 6

Corollary 8. *With the notation of Corollary 7 there exists a constant $R(k)$ such that the following holds:*

- *$R(k) > \frac{k}{\log(k)}$ for k sufficiently large.*
- *For fixed $3 \leq T \leq R(k)$ there is some $C_0 = C_0(T)$, such that there exists a sequence $(b_n)_{n\geq 1}$ of positive integers satisfying $\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = C_0$ such that the corresponding vector $(\zeta_1, \ldots, \zeta_k)$ constructed via Corollary 7 with $C = C_0$ has approximation constants that satisfy*

$$\overline{\psi}_{T-2} < 0 \quad \textit{and} \quad \underline{\psi}_T > 0. \tag{81}$$

A possible choice of $R(k)$ is $R(k) := k-1+(k-2)\frac{\log\left(k^{\frac{1}{k-2}}-1\right)}{\log(k)}$. This provides explicit examples *for Schmidt's Conjecture.*

Proof. Let k be an arbitrary but fixed integer. In view of (13) for a given $3 \leq T \leq R(k)$ we need to find $C_0 = C_0(T)$ such that a vector $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ that arises from Corollary 7 with $C = C_0$ satisfies $\widehat{\omega}_{T-2} > \frac{1}{k} > \omega_T$ to obtain (81). We will implicitly identify such a vector $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ with the resulting value C from the limit of the quotients $\frac{b_{n+1}}{b_n}$ in Corollary 7. This is well defined as the approximation constants we consider don't depend on the choice of the exact vector but depend only on C by Corollary 7. As was shown there the set of such vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$

is nonempty for every $C > \kappa_d > 1$ and we can assume $(\zeta_1, \dots, \zeta_k)$ together with 1 to be $\mathbb{Q}$-linearly independent.

For any positive integer u define the function $\Psi_u(x) = \frac{x-1}{x^u}$. Each function $\Psi_.$ is easily seen to be continuous and Ψ_u increases on $[1, \frac{u}{u-1}]$ and decreases on $[\frac{u}{u-1}, \infty)$ with limit 0 as $x \to \infty$.

We use the notation of Theorem 3, in particular κ_d is the largest real root of $P_d(x) = x^d - x^{d-1} - 1$. We first prove that we can choose $C_0 = C_0(T)$ such that (81) holds for a given T that has the property

$$\Psi_{T-1}(\kappa_{k-T}) > \frac{1}{k} \tag{82}$$

with the constructions of Corollary 7 and the particular choice $C = C_0$.

Put $u = T-1$. If (82) is valid, the facts about the functions Ψ_u show that there is $x > \kappa_{k-T}$ such that $\Psi_{T-1}(x) = \frac{1}{k}$ and Ψ_{T-1} already decreases at x. Furthermore there is an interval $C_0 \in (x, x+\delta)$ such that $\Psi_{T-1}(C_0) < \frac{1}{k} < \Psi_{T-2}(C_0)$. Since $C_0 > x \geq \kappa_{k-T}$ we can apply Corollary 7 with $d := k - T, C := C_0$ and obtain

$$\omega_T = \Psi_{T-1}(C_0) < \frac{1}{k} < C_0 \frac{1}{k} = C_0 \Psi_{T-1}(C_0) = \Psi_{T-2}(C_0) < \widehat{\omega}_{T-2}\,,$$

as intended.

Now assume k is fixed, $1 \leq T \leq k+1$ and that for $C_0 := \kappa_{k-T}$

$$\frac{C_0 - 1}{C_0^{k-1} - C_0^{k-2}} > \frac{1}{k} \tag{83}$$

holds. By definition of $C_0 = \kappa_{k-T}$ we have $C_0^{T-1} = C_0^{k-1} - C_0^{k-2}$ and (83) further implies

$$\psi_{T-1}(C_0) = \frac{C_0 - 1}{C_0^{T-1}} = \frac{C_0 - 1}{C_0^{k-1} - C_0^{k-2}} > \frac{1}{k}$$

for such T, i.e. (82). Combining what we have shown so far, (81) can be obtained for T and $C_0(T) = \kappa_{k-T}$, provided (83) holds. We now show that (83) is true for $3 \leq T \leq R(k)$.

In view of $C_0 > 1$, inequality (83) is equivalent to $\kappa_{k-T} = C_0 \leq k^{\frac{1}{k-2}}$. Since P_{k-T}, whose largest root is $\kappa_{k-T} > 1$, increases on the interval $[1, \infty)$, this is equivalent to $P_{k-T}(k^{\frac{1}{k-2}}) \geq 0$, i.e.

$$k^{\frac{k-T}{k-2}} - k^{\frac{k-T-1}{k-2}} - 1 > 0\,.$$

Basic rearrangements show this is equivalent to

$$T \leq k - 1 + (k-2)\frac{\log\left(k^{\frac{1}{k-2}} - 1\right)}{\log(k)} =: R(k)\,.$$

To finish up the proof we are left to show that $R(k) > \frac{k}{\log(k)}$ holds for k sufficiently large. To see this we claim that for sufficiently large k we have

$$R(k) > \frac{k}{2}\left[1 + \frac{\log(k^{\frac{1}{k-2}} - 1)}{\log(k)}\right] > \frac{k}{2}\left[1 + \frac{\log(k^{\frac{1}{k}} - 1)}{\log(k)}\right] > \frac{k}{\log(k)}\,.$$

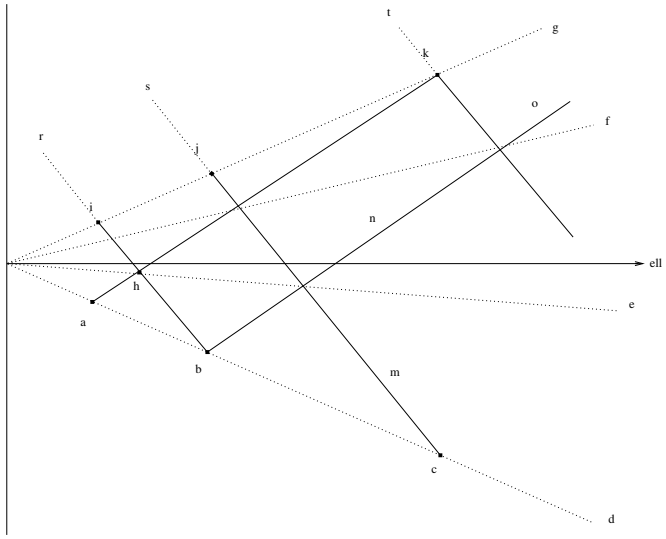

Figure 3: $\underline{\psi}_{j+1} = \overline{\psi}_j$, special case

For $k \geq 4$ only the right most inequality is non trivial. It is equivalent to $k(k^{\frac{1}{k}} - 1) > e^2$ or $k^{\frac{1}{k}} > \frac{e^2}{k} + 1$. However, this is easily seen to be true by $k^{\frac{1}{k}} > 1 + \frac{M}{k}$ for $k \geq k_0(M)$ for arbitrary M, which follows from the monotonic convergence of $\lim_{k\to\infty}(1 + \frac{M}{k})^k = e^M$ for every M. Putting $M > e^2$ arbitrary proves the claim. □

Remark 5. We have $\lim_{k\to\infty} \frac{R(k)}{k} = 0$, but the convergence is very slow, in particular we have seen it is slower than $\frac{1}{\log(k)}$.

3 The case $\underline{\psi}_{j+1} = \overline{\psi}_j$

In this section we want to treat the case of $\zeta_1, \zeta_2, \ldots, \zeta_k$ that define functions ψ_j with the property

$$\underline{\psi}_{j+1} = \overline{\psi}_j, \qquad 1 \leq j \leq k. \tag{84}$$

We will exclude the generic case where all approximation constants $\underline{\psi}_j, \overline{\psi}_j$ are zero, and call the cases where $\underline{\psi}_1 = -1$ the *degenerate cases*. Note, that the numbers Corollary 5 deals with lead to functions ψ_j that satisfy the equalities of (84) for $1 \leq j \leq k-2$, but not for $j \in \{k-1, k\}$ in the case $C < \infty$ (see the bounds given in the remark following Corollary 5). A *special case* of (84), for which an idealized picture is shown below, is of particular interest. If $(x^{(i)})_{i\geq 1}$ denotes the sequence of the first coordinates of approximation vectors $(x, y_1, \ldots, y_k)$, then we consider the special case where there is a sequence of $(x^{(i)})_{i\geq 1}$ with (ideally) constant value of $\frac{\log(x^{(i+1)})}{\log(x^{(i)})}$ such that

$$\max_{1\leq j\leq k} |\zeta_j x^{(i)} - y_j| \sim (x^{(i)})^{-\omega}.$$

This sequence causes the second minimum to attain the value ω_2, and so on. Thus in particular we have

$$\lim_{i\to\infty} \frac{\log(x^{(i+1)})}{\log(x^{(i)})} = \frac{\omega_j}{\omega_{j+1}}, \qquad 1 \le j \le k \tag{85}$$

$$\lim_{i\to\infty} -\frac{\log\big(\max_{1\le t\le k}|x^{(i)}\zeta_t - y_t^{(i)}|\big)}{\log(x^{(i)})} = \omega. \tag{86}$$

Let the equistence of a sequence $(x^{(i)})_{i\ge 1}$ such that (85), (86) holds and additionally for $i \ge i_0$ every $k+1$ consecutive approximation vectors $(x^{(j)}, y_1^{(j)}, \ldots, y_k^{(j)})$ belonging to $x^{(i)}, x^{(i+1)}, \ldots, x^{(i+k)}$ (i.e. $j \in \{i, i+1, \ldots, i+k\}$) are linearly independent be our definition of the *special case* mentioned above.

Roy shows, that numbers he defines as *extremal numbers* ζ in the introduction of [4] satisfy the property of the special case of (84) for $k = 2$ and $\zeta_1 = \zeta, \zeta_2 = \zeta^2$ and yield $\omega_j = \gamma^{j-1}$ for $1 \le j \le 3$ and $\widehat{\omega}_3 = \gamma^3$ with $\gamma := \frac{\sqrt{5}-1}{2}$. We are interested in other particular cases of the special case of (84).

It follows in general by (85), (86) that all the values $\underline{\psi}_\cdot, \overline{\psi}_\cdot$ are determined by the value $\underline{\psi}_1$ (or equivalently ω). This holds for the degenerate case in particular. However, we will show in Theorem 4 that this phenomenon holds for *all* $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ in the degenerate case of (84). By virtue of Corollary 5 we can easily provide concrete examples for the degenerate case. Before we do so, for the sake of completeness we give a general result about the degenerate case of (84).

Proposition 4. *Assume the approximation functions arising from $\zeta_1, \ldots, \zeta_k$ satisfy $\underline{\psi}_1 = -1$ and* (84)*. Then they already satisfy*

$$\underline{\psi}_1 = -1 \tag{87}$$

$$\overline{\psi}_1 = \frac{1-k}{2k} = \underline{\psi}_2 \tag{88}$$

$$\overline{\psi}_j = \frac{1}{k} = \underline{\psi}_{j+1}, \qquad 2 \le j \le k, \tag{89}$$

$$\overline{\psi}_{k+1} = \frac{1}{k}, \tag{90}$$

and hence in particular fall under the special case.

Proof. First note that in general if $\underline{\psi}_1 = -1$ we have $\overline{\psi}_j = \frac{1}{k}$ for $2 \le j \le k+1$ by means of (6), see the proof of Theorem 1. Consequently by (84) we have (89) and (90).

For (88) note first that $\frac{1-k}{2k}$ is always a lower bound as established in (8), (9). So by (10) it suffices to prove $\overline{\psi}_1 \le \frac{1-k}{2k}$.

Suppose we had $\overline{\psi}_1 > \frac{1-k}{2k}$. This means for some sequence $(q_n)_{n\ge 1}$ tending to infinity we have $\psi_1(q_n) > V$ for some $V > \frac{1-k}{2k}$. Putting $V_0 := 2(V - \frac{1-k}{2k}) > 0$ and using $\psi_2(q_n) \ge \psi_1(q_n)$ we have

$$\lim_{n\to\infty} \sum_{j=1}^{k+1} \psi_j(q_n) > 2V + (k-1)\left[\frac{1}{k} - \frac{V_0}{2(k-1)}\right] = \frac{V_0}{2} > 0,$$

a contradiction to (6) since $\lim_{n\to\infty} q_n = \infty$. □

Theorem 4. *For any $k \geq 2$ there exist real numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ as in Corollary 5 with $C = \infty$ together with 1 linearly independent over $\mathbb{Q}$ that satisfy the degenerate case of* (84) *and hence* (87)–(90) *by Proposition 4.*

Proof. Using (13) we obtain the equivalent system

$$\omega = \infty \tag{91}$$
$$\widehat{\omega} = 1 = \omega_2 \tag{92}$$
$$\widehat{\omega}_j = 0 = \omega_{j+1}, \qquad 2 \leq j \leq k \tag{93}$$
$$\widehat{\omega}_{k+1} = 0\,. \tag{94}$$

In the case $k \geq 3$, we can just apply Corollary 5 with $C = \infty$ because then we obviously have

$$\lim_{C\to\infty} \frac{C-1}{C} = 1, \qquad \lim_{C\to\infty} \frac{C-1}{C^j} = 0, \quad j \geq 2,$$

which gives (91)–(94). However, in the case $k = 2$ and $C = \infty$ we can also apply Corollary 5 with a slightly more sophisticated argumentation. Of course we directly infer (91) and the left equation in (92) follows as for $k \geq 3$. From (91) we can immediately deduce $\widehat{\omega}_j = 0$ for $j = 2, 3$ as in the proof of Theorem 1, which is a rephrasing of the left hand side of (93) and (94). By the left equation in (92) and Jarník's identity $\omega_3 + \widehat{\omega} = 1$ (see comments between the end of proof of Theorem 1 and Corollary 2) we get $\omega_3 = 0$, i.e. the right hand side of (93). For the missing right hand equation of (92) note that $\omega_2 \geq \widehat{\omega}$ is always true by (10) and on the other hand $\omega_2 \leq 1$ by (15), so by the left hand equality in (92) we infer the right hand equality of (92). □

This allows to show that the bounds in (14)–(19) are best possible if considered independently by using only three types of vectors $(\zeta_1, \zeta_2, \ldots, \zeta_k)$ depending on the dimension k. These types are:

- a set of together with 1 $\mathbb{Q}$-linearly independent *algebraic* numbers $\zeta_1, \zeta_2, \ldots, \zeta_k$ (leading to the generic case)
- $\zeta_1, \zeta_2, \ldots, \zeta_k$ as in Corollary 5 with $C = \infty$, for example $\zeta_j = \sum_{n\geq 1} \frac{1}{(nk+j)!}$ for $1 \leq j \leq k$
- $\zeta_1, \zeta_2, \ldots, \zeta_k$ as in Corollary 3

Corollary 9. *The bounds* (14)–(19) *are all (each for itself) optimal among $(\zeta_1, \ldots, \zeta_k)$ that are $\mathbb{Q}$-linearly independent together with* 1*.*

Proof. In Corollary 3 we have seen, that the upper bounds in (14), (15), (16) as well as the lower bounds in (17), (18), (19) cannot be improved.

In Theorem 4 we've just seen, that the left hand side of (16) and the right hand side of (17) are optimal.

However, all the other bounds are $1/k$ and it is well known that all constants $\omega_j, \widehat{\omega}_j$ are equal to $1/k$ in the generic case. To give concrete examples, an implication of Schmidt's subspace theorem says, that for all $\mathbb{Q}$-linearly independent algebraic numbers all approximation constants take the value $1/k$ (which follows already from $\omega = 1/k$ by (13) and (6)). So the lower bounds of (14), (15) such as the upper bounds of (18), (19) cannot be improved either, and the list is complete. □

Let $\zeta_1, \zeta_2, \ldots, \zeta_k$ be real numbers that lead to a special case of (84), i.e. (85), (86) hold. It follows directly from (85), (86) that all the constants $\omega_j, \widehat{\omega}_j$ only depend on ω. It is easy to check that more precisely we have

$$\frac{(1+\omega)^{k+1}}{\omega} = \frac{(1+\widehat{\omega}_{k+1})^{k+1}}{\widehat{\omega}_{k+1}} \tag{95}$$

$$\omega_j = \omega^{1-\frac{j-1}{k+1}} \widehat{\omega}_{k+1}^{\frac{j-1}{k+1}}, \qquad 1 \le j \le k+1. \tag{96}$$

Using this we now prove a lower bound for $\widehat{\omega}$ in dependence of ω.

Proposition 5. *In the special case of (84) for $k \ge 2$ we have*

$$\frac{\omega}{\omega+1} < \widehat{\omega} \le 1.$$

Proof. The right hand side inequality is just (17).

Suppose for some $k \ge 2$ we had $\widehat{\omega} \le \frac{\omega}{\omega+1}$. Putting $j = 2$ in (96) (note $\omega_2 = \widehat{\omega}$ by definition) we have

$$\widehat{\omega}_{k+1} \le \left[\left(\frac{\omega}{\omega+1}\right)\omega^{-\frac{k}{k+1}}\right]^{k+1} = \frac{\omega}{(\omega+1)^{k+1}}\,. \tag{97}$$

Denote

$$f_k(x) := \frac{(x+1)^{k+1}}{x}, \qquad k \ge 1\,.$$

Differentiating shows that f_k decreases on $x \in (0, \frac{1}{k})$ and increases on $x \in (\frac{1}{k}, \infty)$, so its global minimum on $(0, \infty)$ is at $x = \frac{1}{k}$. Combining this with $\widehat{\omega}_{k+1} < \frac{1}{k}$, (95) and (97) we obtain

$$f_k(\omega) = f_k(\widehat{\omega}_{k+1}) \ge f_k\left(\frac{1}{f_k(\omega)}\right), \qquad k \ge 1.$$

Putting $z := \frac{1}{f_k(\omega)}$ this gives $\frac{1}{z} \ge f_k(z)$, which is false, as $\frac{1}{z}$ is an expression in the binomial expansion of $f_k(z)$. □

Remark 6. One can prove that for $k \ge 2$ we have $\lim_{\omega\to\infty} \omega + 1 - \frac{\omega}{\widehat{\omega}} = 0$.

Observe, that in Corollary 6 with arbitrary C we always have $\widehat{\omega} = \frac{\omega}{\omega+1}$. Proposition 5 shows that given ω the resulting special case of (84) leads to a larger value of $\widehat{\omega}$. It may be conjectured that among all $\zeta_1, \zeta_2, \ldots, \zeta_k$ linearly independent together with 1 with prescribed $\omega = \omega_0$, the quantity $\widehat{\omega}$ is maximised for the special case of (84) with the value $\omega = \omega_0$.

Remark 7. Observe that the inequality $\frac{\widehat{\omega}^2}{1-\widehat{\omega}} \leq \omega$ always holds as established by Jarník, see Theorem 1 page 331 in [2]. So together with Proposition 5 in the special case of (84) we have

$$\frac{\widehat{\omega}^2}{1-\widehat{\omega}} \leq \omega \leq \frac{\widehat{\omega}}{1-\widehat{\omega}}\,.$$

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Author's address:
Johannes Schleischitz: Anzengrubergasse 23/2, 1050 Vienna
E-mail: johannes.schleischitz@univie.ac.at