Applications of Mathematics

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Applications of Mathematics, Vol. 58 (2013), No. 4, 405–422

Persistent URL: http://dml.cz/dmlcz/143338

58 (2013) APPLICATIONS OF MATHEMATICS No. 4, 405–422

OPTIMAL DESIGN OF THE COOLING PLUNGER CAVITY

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(Received May 5, 2011)

Abstract. An axisymmetric system of mould, glass piece, plunger and plunger cavity is considered. The state problem is given as a stationary head conduction process. The system includes the glass piece representing the heat source and is cooled inside the plunger cavity by flowing water and outside by the environment of the mould. The design variable is taken to be the shape of the inner surface of the plunger cavity.

The cost functional is the second power of the norm in the weighted space L^2_r of difference of trace of temperature from given constant, which is evaluated on the outward boundary of the plunger.

Existence and uniqueness of the state problem solution and existence of a solution of the optimization problem are proved.

Keywords: shape optimization, heat-conducting fluid, energy transfer

MSC 2010: 49Q10, 76D55, 93C20

1. Introduction

This work concerns the optimal design of the shape of the plunger cavity which controls the cooling process of the glass piece during the manufacturing process. The goal of optimization is to find such a shape of the inner plunger cavity which allows us to control down the plunger temperature in such a way to achieve a constant distribution of temperature across the surface of the moulding device at the moment of separation of the plunger from the moulded piece.

The mathematical model is a strong idealization of the non-stationary periodical problem of heat conduction. We study the problem of stationary conduction of heat for mean values of this periodical process with cooling by stationary flowing water.

This work was supported by the Research plan No. MSM 4674788501 funded by Ministry of Education, Youth and Sports of the Czech Republic.

In view of the fact that the system of mould, glass piece, plunger and plunger cavity is considered to be axisymmetric we assume planar stationary flow of water in planes involving the z axis. Now it is suitable to formulate the problem in cylindrical coordinates r, φ, z. We assume that the heat conduction and the flow pattern do not depend on the angle φ so we get a two-dimensional problem in the weighted Sobolev space.

The cost functional is defined as the second power of the norm in the weighted L^2_r space of the difference of the trace of temperature and the given constant evaluated on the outward boundary of the plunger.

In Section 1 we define a weak formulation of the state problem in cylindrical coordinates with reduced angle coordinate and prove the existence of its unique solution. Further we formulate the problem of the optimal design for the plunger cavity shape and prove the existence of solution.

2. Formulation of the problem

To formulate the state problem we start from the abstract formulation introduced by authors Haslinger and Neittaanmäki [1].

We rotate the system to the horizontal position to be able to describe the optimized plunger cavity surface by a function of one variable.

We define

$$F_2^e(x) = \begin{cases} 0 & \text{for } x \in [0, x_2^e], \\ f_2^e(x) & \text{for } x \in [x_2^e, 1], \end{cases} \tag{1.1}$$

where $x_2^e \in [s_{\min}, 1]$ ($s_{\min} > 0$ is a fixed constant given by the minimal thickness of the plunger wall), $f_2^e \in C^{(0),1}([x_2^e, 1])$, $f_2^e(x_2^e) = 0$ and $0 \leqslant f_2^e(x) \leqslant f_1(x) - s_{\min}$, $|f_2^{e\prime}(x)| < C_D$ for $x \in\,]x_2^e, 1]$, where f_1 is a fixed given increasing function which represents the outward shape of the plunger. Further we assume that $a \leqslant f_2^e(x) - s_2$ for $x \in [x_3^e, 1]$, where $a > 0$ represents the radius of the supply tube and $s_2 > 0$ is the minimal admissible split width between the inner wall of the plunger cavity and the supply tube, $x_3^e \in\,]x_2, 1]$ is the depth of insertion of the tube.

Remark. The condition $|f_2^{e\prime}(x)| < C_D$ yields a non-smooth shape of the real 3D plunger. It can be omitted and replaced by a small rotation of the system in negative sense in the proof of existence of a solution of the optimal design problem.

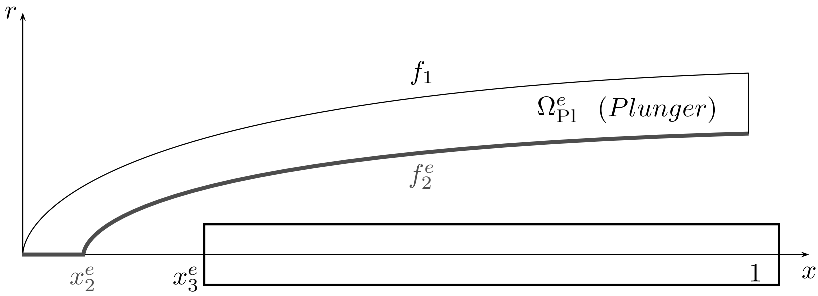

Figure 1. Scheme of plunger with optimized part of boundary.

Further we define the set of admissible functions as

$$U_{\text{ad}}^e = \Big\{ F_2^e(x) \in C^{(0),1}([0,1]);\ F_2^e(x) = \begin{cases} 0 & \text{for } x \in [0, x_2^e], \\ f_2^e(x) & \text{for } x \in [x_2^e, 1], \end{cases}$$
$$x_2^e \in [s_{\min}, 1],\ s_{\min} > 0,\ f_2^e \in C^{(0),1}([x_2^e, 1]),\ f_2^e(x_2^e) = 0,$$
$$0 \leqslant f_2^e(x) \leqslant f_1(x) - s_{\min},\ |f_2^{e\prime}(x)| < C_D \text{ for } x \in \,]x_2^e, 1],$$
$$f_1 \text{ given},\ a \leqslant f_2^e(x) - s_2 \text{ for } x \in [x_3^e, 1],\ a > 0,\ s_2 > 0,\ x_3^e \in \,]x_2, 1] \Big\},$$

where the function F_2^e describes the technological constraint for the inner cavity surface.

We consider the region Ω_{Pl}^e which depends on the design function $F_2^e(x)$, and which is defined by the formula

$$\Omega_{\text{Pl}}^e = \{(x, r) \in R^2;\ F_2^e(x) < r < f_1(x) \text{ for } x \in [0, 1]\}.$$

Denote by Θ the set of all admissible regions $\Omega_{\text{Pl}}^e \subset R^2$ with Lipschitz boundaries.

We define the convergence on the set Θ.

We say that a sequence $\Omega_{Pl}^n \in \Theta$ converges to a region $\Omega_{\text{Pl}} \in \Theta$ if, and only if, the sequence of functions ${}^nF_2^e(x)$ converges uniformly to the function $F_2^e(x)$ in $[0, 1]$.

Let us consider the union of four planar regions $\Omega = \Omega_{\text{Pl}}^e \cup \Omega_{\text{Gl}} \cup \Omega_{\text{Ca}}^e \cup \Omega_{\text{Mo}}$ which represents the planar cross section of the system mould, glass piece, plunger and the cooling canal of the plunger. Region Ω_{Pl}^e represents the plunger, region Ω_{Gl} the cooled glass piece, region Ω_{Ca}^e the cooling canal inside the plunger, where cooling water flows, and region Ω_{Mo} represents the mould.

Furthermore, we denote by Γ_1 the boundary between the plunger Ω_{Pl}^e and the moulded piece Ω_{Gl} and by Γ_2^e the boundary between the plunger Ω_{Pl}^e and the plunger cavity Ω_{Ca}^e. We denote by Γ_3 part of the boundary connecting the system mould, the moulded piece and the plunger with presser, by Γ_4 part of the axis of symmetry (see Figure 2), by Γ_5 part of the boundary formed by the tube. Γ_6 is the notation for part of the boundary between the moulded piece Ω_{Gl} and the mould Ω_{Mo} and Γ_7 is the outward boundary of the mould, which is surrounded by the external environment. Γ_{in} denotes part of the boundary, where cooling water comes into the cooling canal of the plunger, and Γ_{out} part of the boundary where water exits.

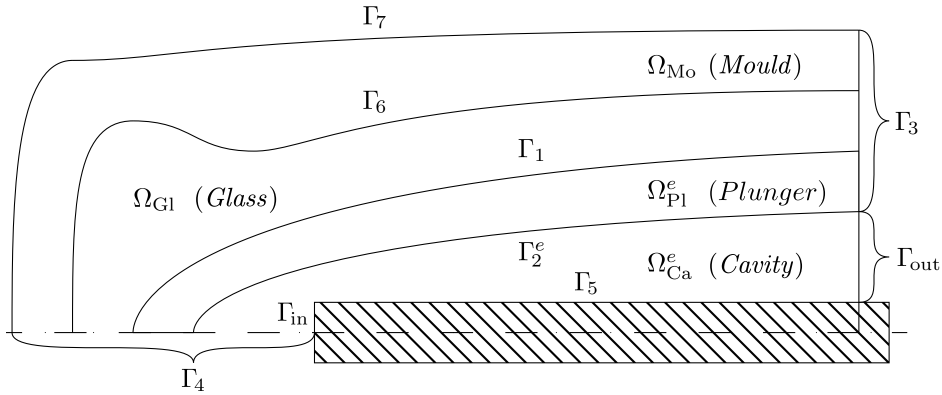

Figure 2. Scheme of the system mould, glass piece, plunger, cavity of plunger and supply tube.

In the three dimensional region G^e_{Ca} which is created by rotation of Ω^e_{Ca} around the x axis, we assume an axisymmetric incompressible potential flow of water, which is axisymmetric with the x axis. We split the boundary ∂G^e_{Ca} into the union of four parts as

$$\partial G^e_{\mathrm{Ca}} = \Gamma_2^{\mathrm{3D}} \cup \Gamma_5^{\mathrm{3D}} \cup \Gamma_{\mathrm{in}}^{\mathrm{3D}} \cup \Gamma_{\mathrm{out}}^{\mathrm{3D}}, \tag{1.2}$$

where Γ_2^{3D}, Γ_5^{3D}, $\Gamma_{\mathrm{in}}^{\mathrm{3D}}$, and $\Gamma_{\mathrm{out}}^{\mathrm{3D}}$ denote respectively parts of boundary of $\partial G^e_{\mathrm{Ca}}$ created by rotation of Γ_2^e, Γ_5, Γ_{in}, and Γ_{out}, around the x axis.

The potential Φ is given as a solution of the Neumann problem

$$\Delta\Phi = 0 \quad \text{in } G^e_{\mathrm{Ca}}, \tag{1.3}$$

$$\frac{\partial\Phi}{\partial n} = g \quad \text{on } \partial G^e_{\mathrm{Ca}}, \tag{1.4}$$

where $g \in L^2(\partial G^e_{\mathrm{Ca}})$, representing the normal component of the velocity at the entrance to and the exit of the plunger cavity, is in the form

$$g = \begin{cases} 0 & \text{on } \Gamma_2^{\mathrm{3D}} \cup \Gamma_5^{\mathrm{3D}}, \\ h_{\mathrm{velo}}^{\mathrm{in}} & \text{on } \Gamma_{\mathrm{in}}^{\mathrm{3D}}, \\ h_{\mathrm{velo}}^{\mathrm{out}} & \text{on } \Gamma_{\mathrm{out}}^{\mathrm{3D}}, \end{cases} \tag{1.5}$$

$h^{\mathrm{in}}_{\mathrm{velo}}$ is the normal velocity at the entrance $\Gamma^{\mathrm{3D}}_{\mathrm{in}}$ ($h^{\mathrm{in}}_{\mathrm{velo}} < 0$) and $h^{\mathrm{out}}_{\mathrm{velo}}$ is the normal velocity at the exit $\Gamma^{\mathrm{3D}}_{\mathrm{out}}$. Further, we assume

$$\int_{\Gamma^{\mathrm{3D}}_{\mathrm{in}} \cup \Gamma^{\mathrm{3D}}_{\mathrm{out}}} g \, \mathrm{d}S = 0. \tag{1.6}$$

The variational formulation for the potential function has the following form: We look for the function $\Phi \in H^1(G^e_{\mathrm{Ca}})$ such that

$$\int_{G^e_{\mathrm{Ca}}} \frac{\partial \Phi}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}V = \int_{\Gamma^{\mathrm{3D}}_{\mathrm{in}} \cup \Gamma^{\mathrm{3D}}_{\mathrm{out}}} g\varphi \, \mathrm{d}S \quad \forall \varphi \in H^1(G^e_{\mathrm{Ca}}). \tag{1.7}$$

The velocity field of the flowing water $\mathbf{u} = (u_1, u_2, u_3)$ in the cavity G^e_{Ca} is given as

$$\mathbf{u} = \mathrm{grad}\Phi. \tag{1.8}$$

Theorem 1.1 (Existence and uniqueness of the velocity field). *Under the assumption* (1.6) *there exists a unique velocity field of the form* (1.8) *satisfying the estimate of the Euclid norm in the form*

$$\| \, |\mathbf{u}| \, \|_{L^2(G^e_{\mathrm{Ca}})} \leqslant c(\|h^{\mathrm{in}}_{\mathrm{velo}}\|_{L^2(\Gamma^{\mathrm{3D}}_{\mathrm{in}})} + \|h^{\mathrm{out}}_{\mathrm{velo}}\|_{L^2(\Gamma^{\mathrm{3D}}_{\mathrm{out}})}). \tag{1.9}$$

Proof. According to Theorem 35.1 (see [3] page 423) there exists a unique weak solution $\Phi \in H^1(G^e_{\mathrm{Ca}})$ of the Neumann problem (1.7), which satisfies the condition

$$\int_{G^e_{\mathrm{Ca}}} \Phi \, \mathrm{d}V = 0 \tag{1.10}$$

and

$$\|\Phi\|_{H^1(G^e_{\mathrm{Ca}})} \leqslant c\|g\|_{L^2(\partial G^e_{\mathrm{Ca}})}. \tag{1.11}$$

Further, from (1.8) we get

$$\left\| \sqrt{u_1^2 + u_2^2 + u_3^2} \right\|_{L^2(G^e_{\mathrm{Ca}})} \leqslant \|\Phi\|_{H^1(G^e_{\mathrm{Ca}})}, \tag{1.12}$$

which together with

$$\|g\|_{L^2(\partial G^e_{\mathrm{Ca}})} = \|h^{\mathrm{in}}_{\mathrm{velo}}\|_{L^2(\Gamma^{\mathrm{3D}}_{\mathrm{in}})} + \|h^{\mathrm{out}}_{\mathrm{velo}}\|_{L^2(\Gamma^{\mathrm{3D}}_{\mathrm{out}})} \tag{1.13}$$

gives (1.9). □

The energy equation for the stationary flow $\mathbf{u}$ with steady temperature in three dimensions has the form

$$c_v \operatorname{grad} \vartheta \cdot \mathbf{u} - \frac{k}{\varrho} \Delta \vartheta = \frac{1}{\varrho} 2\mu |D(\mathbf{u})|^2 + q, \tag{1.14}$$

where c_v is the specific heat upon constant volume, ϑ the absolute temperature, k the coefficient of thermal conductivity, ϱ the density of the flowing liquid, μ the dynamic viscosity,

$$D(\mathbf{u}) = (d_{ij})_{i,j=1}^{3}, \quad d_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \tag{1.15}$$

the strain velocity tensor and q the density of the heat sources. We assume that the cooling medium is water, which has dynamic viscosity $0.833 \cdot 10^{-3} < \mu < 1.231 \cdot 10^{-3}\,[\mathrm{Nsm}^{-2}]$ at temperatures considered. It allows us to neglect the term representing the energy of the inner friction of water. So we assume the energy equation in the form

$$c_v \operatorname{grad} \vartheta \cdot \mathbf{u} - \frac{k}{\varrho} \Delta \vartheta = q. \tag{1.16}$$

We put $\mathbf{u} = 0$ in G_{Pl}^e, G_{Gl} and G_{Mo} (the regions created by rotation of Ω_{Pl}^e, Ω_{Gl} and Ω_{Mo} around the x axis) because there is no flowing liquid inside. Further we consider $q = 0$ in G_{Pl}^e, G_{Ca}^e and G_{Mo} (there are no heat sources inside). Denote $G = G_{\mathrm{Pl}}^e \cup G_{\mathrm{Gl}} \cup G_{\mathrm{Ca}}^e \cup G_{\mathrm{Mo}}$. We divide the searched function ϑ representing the distribution of temperature in the system into the sum of four functions as

$$\vartheta = \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3,$$

where

$$\vartheta_i = \begin{cases} \vartheta|_{G_i} & \text{in } G_i \\ 0 & \text{in } G \setminus G_i \end{cases} \qquad \text{for } i = 0,1,2,3 \tag{1.17}$$

($G_0 \equiv G_{\mathrm{Pl}}^e$, $G_1 \equiv G_{\mathrm{Gl}}$, $G_2 \equiv G_{\mathrm{Ca}}^e$, $G_3 \equiv G_{\mathrm{Mo}}$).

Further, we denote by $\vartheta_i|_{\Gamma_j^{3\mathrm{D}}}$ the trace of solution ϑ_i on the boundary $\Gamma_j^{3\mathrm{D}}$ for i, j if $\Gamma_j^{3\mathrm{D}}$ is a boundary of G_i.

We assume the following *boundary conditions*:

At the entrance the cooling water has constant temperature 15°C, i.e. 288 K, thus

$$\vartheta_2|_{\Gamma_{\mathrm{in}}^{3\mathrm{D}}} = 288 \quad \text{on } \Gamma_{\mathrm{in}}^{3\mathrm{D}}.$$

The output distribution of temperature is given by the function $h^e_{\text{out}} \in C(\Gamma^{3D}_{\text{out}})$, thus

$$\vartheta_2|_{\Gamma^{3D}_{\text{out}}} = h^e_{\text{out}} \quad \text{on } \Gamma^{3D}_{\text{out}}.$$

We assume that the supply tube is isolated, thus

$$\frac{\partial \vartheta_2}{\partial n} = 0 \quad \text{on } \Gamma^{3D}_5.$$

The boundary condition on Γ^{3D}_3 is given as

$$\vartheta_i|_{\Gamma^{3D}_3} = h_3 \quad \text{on } \Gamma^{3D}_3,\ i = 0, 1, 3,$$

where $h_3 \in C(\Gamma^{3D}_3)$ is the steady-state temperature at the place of connection with the glass press.

The heat-transfer through Γ^{3D}_2 (i.e. between the plunger and water) is modeled as the boundary condition for contact of two bodies, where "the body" representing water has a convective term (see [4]), thus

$$\Big(-k_0\frac{\partial \vartheta_0}{\partial n}\Big)^- = \Big(-k_2\frac{\partial \vartheta_2}{\partial n}\Big)^+ \quad \text{on } \Gamma^{3D}_2, \tag{1.18}$$

where $\partial/\partial n$ denotes the derivative with respect to the outward normal with respect to the region G^e_{Pl}, or G^e_{Ca}, "+" standing for the limit in the direction of the normal to the boundary from outside and "−" from inside of G^e_{Pl}.

The heat-transfer through the boundary Γ^{3D}_7 (i.e. between the mould and environment) is modeled as a boundary condition of the third kind for contact between body and environment (see [4]), thus

$$\Big(-k_3\frac{\partial \vartheta_3}{\partial n}\Big)^- = \alpha(\vartheta_3|_{\Gamma^{3D}_7} - \vartheta_4) \quad \text{on } \Gamma^{3D}_7, \tag{1.19}$$

where $\partial/\partial n$ denotes the derivative with respect to the outward normal with respect to the region G_{Mo}, "−" the limit in the direction of the normal to the boundary from inside of G_{Mo}, $\alpha > 0$ denotes the coefficient of heat-transfer between the mould and environment, $\vartheta_3|_{\Gamma^{3D}_7}$ the trace of ϑ_3 on the boundary of the region G_{Mo} and $\vartheta_4 > 0$ the temperature of environment. We use the transit condition for contact between two bodies, where one of them changes its state of matter because of the influence of solidification (see [4]), to describe the heat-transfer through the boundary Γ^{3D}_1 between the glass piece and the plunger. Thus

$$\Big(k_1\frac{\partial \vartheta_1}{\partial n}\Big)^+ - \Big(k_0\frac{\partial \vartheta_0}{\partial n}\Big)^- = \beta_1 \quad \text{on } \Gamma^{3D}_1, \tag{1.20}$$

where $\beta_1 > 0$, $\beta_1 \in C^{(0),1}(\Gamma^{3D}_1)$ represents the flux density of the modified mass of the body, $\partial/\partial n$ denotes the derivative with respect to the outward normal with respect to the region G^e_{Pl}, or G_{Gl}, "+" the limit in the direction of the normal to the boundary from outside and "−" from inside of G^e_{Pl}.

Analogously we describe the heat-transfer through the boundary Γ_6^{3D} between the glass and the mould. Thus

$$\Big(k_1\frac{\partial\vartheta_1}{\partial n}\Big)^+ - \Big(k_3\frac{\partial\vartheta_3}{\partial n}\Big)^- = \beta_6 \quad \text{on } \Gamma_6^{3D}, \tag{1.21}$$

where $\beta_6 > 0$, $\beta_6 \in C^{(0),1}(\Gamma_6^{3D})$ represents the flux density of the modified mass of the body, $\partial/\partial n$ denotes the derivative with respect to the outward normal with respect to the region G_{Mo}, or G_{Gl}, "+" the limit in the direction of the normal to the boundary from outside and "−" from inside of the region G_{Mo}.

We start from the variational formulation of the energy equation in three dimensions. Due to rotational symmetry we transform the problem to cylindrical coordinates and use dimensional reduction to x, r coordinates.

In this way we obtain a two dimensional velocity field of flowing water $\mathbf{w}^e = (w_1, w_2)$ where

$$w_1 = u_1, \tag{1.22}$$
$$w_2 = \sqrt{(u_2)^2 + (u_3)^2}, \tag{1.23}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is defined in (1.8).

Dimensional reduction leads to one more boundary condition on the axis of the system Γ_4, which means that there is no heat flow in the normal direction to the axis, thus

$$\frac{\partial\vartheta_i}{\partial n} = 0 \quad \text{on } \Gamma_4,\ i = 0,1,2,3.$$

To define the state problem based on the variational formulation of the energy equation in two dimensions we define operators

$$\mathrm{Energy}_\Omega^{\mathrm{velo}}(\vartheta, \mathbf{w}, \psi) = c_v\varrho_2 \int_{\Omega^e_{\mathrm{Ca}}} \Big(\frac{\partial\vartheta_2}{\partial x}w_1 + \frac{\partial\vartheta_2}{\partial r}w_2\Big)\psi r\,\mathrm{d}\Omega, \tag{1.24}$$

$$\begin{aligned}\mathrm{Energy}_\Omega^{\mathrm{cond}}(\vartheta, \psi) = k_0 &\int_{\Omega^e_{\mathrm{Pl}}} \Big(\frac{\partial\vartheta_0}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\vartheta_0}{\partial r}\frac{\partial\psi}{\partial r}\Big) r\,\mathrm{d}\Omega \\ + k_1 &\int_{\Omega_{\mathrm{Gl}}} \Big(\frac{\partial\vartheta_1}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\vartheta_1}{\partial r}\frac{\partial\psi}{\partial r}\Big) r\,\mathrm{d}\Omega \\ + k_2 &\int_{\Omega^e_{\mathrm{Ca}}} \Big(\frac{\partial\vartheta_2}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\vartheta_2}{\partial r}\frac{\partial\psi}{\partial r}\Big) r\,\mathrm{d}\Omega \\ + k_3 &\int_{\Omega_{\mathrm{Mo}}} \Big(\frac{\partial\vartheta_3}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\vartheta_3}{\partial r}\frac{\partial\psi}{\partial r}\Big) r\,\mathrm{d}\Omega,\end{aligned} \tag{1.25}$$

$$\mathrm{Environment}_\Omega(\vartheta, \psi) = \int_{\Gamma_7} \alpha\vartheta_3|_{\Gamma_7}\psi r\,\mathrm{d}\Gamma, \tag{1.26}$$

$$\text{(1.27)} \qquad \mathrm{Source}_{\Omega}(\psi) = \varrho_1 \int_{\Omega_{\mathrm{Gl}}} q\psi r \,\mathrm{d}\Omega,$$

$$\text{(1.28)} \qquad \mathrm{Coeff}_{\Omega}(\psi) = \int_{\Gamma_1} \beta_1 \psi r \,\mathrm{d}\Gamma + \int_{\Gamma_6} \beta_6 \psi r \,\mathrm{d}\Gamma + \int_{\Gamma_7} \alpha\vartheta_4 \psi r \,\mathrm{d}\Gamma.$$

Further, we denote

$$\text{(1.29)} \qquad \begin{aligned} A_{\Omega}(\vartheta, \mathbf{w}, \psi) = {} & \mathrm{Energy}_{\Omega}^{\mathrm{velo}}(\vartheta, \mathbf{w}, \psi) + \mathrm{Energy}_{\Omega}^{\mathrm{cond}}(\vartheta, \psi) \\ & + \mathrm{Environment}_{\Omega}(\vartheta, \psi) \end{aligned}$$

and

$$\text{(1.30)} \qquad F_{\Omega}(\psi) = \mathrm{Source}_{\Omega}(\psi) + \mathrm{Coeff}_{\Omega}(\psi).$$

We introduce the weighted Sobolev space $H_r^1(\Omega_i)$ (see [2]) with the norm

$$\text{(1.31)} \qquad \|v\|_{1,r,\Omega_i} = \left(\int_{\Omega_i} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial r} \right)^2 + v^2 \right] r \,\mathrm{d}\Omega \right)^{1/2}, \quad i = 0, 1, 2, 3,$$

($\Omega_0 \equiv \Omega_{\mathrm{Pl}}^e$, $\Omega_1 \equiv \Omega_{\mathrm{Gl}}$, $\Omega_2 \equiv \Omega_{\mathrm{Ca}}^e$, $\Omega_3 \equiv \Omega_{\mathrm{Mo}}$).

Further, we denote

$$\mathbf{H}(\Omega) = \{\vartheta;\ \vartheta \text{ defined in (1.17)},\ \vartheta_i \in H_r^1(\Omega_i) \text{ for any } i = 0, 1, 2, 3\}.$$

We define the norm in $\mathbf{H}(\Omega)$ as

$$\text{(1.32)} \qquad \|\vartheta\|_{\mathbf{H}} = (\|\vartheta_0\|_{1,r,\Omega_0}^2 + \|\vartheta_1\|_{1,r,\Omega_1}^2 + \|\vartheta_2\|_{1,r,\Omega_2}^2 + \|\vartheta_3\|_{1,r,\Omega_3}^2)^{1/2}.$$

Theorem 1.2. *The set* $\mathbf{H}(\Omega)$ *with the norm* (1.32) *is a Hilbert space.*

We denote by $\mathbf{H}^*(\Omega)$ the dual space to the space $\mathbf{H}(\Omega)$ with the norm

$$\|F_{\Omega}\|_{\mathbf{H}^*} = \sup_{\psi \neq 0} \frac{F_{\Omega}(\psi)}{\|\psi\|_{\mathbf{H}}}.$$

Denote

$$\Omega_H = \Omega \cup \Gamma_3 \cup \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$$

and

$${}^e\mathcal{H}^{2\mathrm{D}} = \{v \in C^{\infty}(\Omega_H);\ v|_{\Gamma_3 \cup \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}} = 0\}.$$

Let $\mathbf{H_0}(\Omega)$ be the closure of the set ${}^e\mathcal{H}^{2D}$ with respect to the norm $\mathbf{H}(\Omega)$.

We assume the existence of a function $\vartheta^e_\Gamma \in \mathbf{H}(\Omega)$ such that

$$\vartheta^e_\Gamma|_{\Gamma_{\rm in}} = 288 \quad \text{on } \Gamma_{\rm in}, \tag{1.33}$$

$$\vartheta^e_\Gamma|_{\Gamma_{\rm out}} = h^e_{\rm out} \quad \text{on } \Gamma_{\rm out}, \tag{1.34}$$

$$\vartheta^e_\Gamma|_{\Gamma_3} = h_3 \quad \text{on } \Gamma_3, \tag{1.35}$$

where $h_3 \in C(\Gamma_3)$ is a given function representing the stagnation temperature on the boundary Γ_3 with the presser and $h^e_{\rm out} \in C(\Gamma_{\rm out})$ is a given function representing the distribution of temperature at the output from the cavity of the plunger $\Gamma_{\rm out}$. We use the variational formulation of the energy equation to formulate

The State Problem:

We look for the function $\vartheta \equiv \vartheta(F^e_2) \in \mathbf{H}(\Omega)$ such that

$$A_\Omega(\vartheta, \mathbf{w}^e, \psi) = F_\Omega(\psi) \quad \forall \psi \in \mathbf{H}_0(\Omega), \tag{1.36}$$

$$\vartheta - \vartheta^e_\Gamma \in \mathbf{H}_0(\Omega), \tag{1.37}$$

where $F^e_2 \in U^e_{\rm ad}$ and $\mathbf{w}^e$ is the corresponding flow pattern given as the gradient of the solution (1.7).

The physical assumption of cooling:

A1: The average temperature of water coming into the plunger cavity is less than the average temperature of the leaving water.

Theorem 1.3. *The bilinear form* (1.24) *satisfies the condition*

$$Energy^{\rm velo}_\Omega(\vartheta, \mathbf{w}^e, \vartheta) > 0 \tag{1.38}$$

for $\vartheta, \mathbf{w}^e$ *satisfying the physical assumption of cooling* **A1**.

P r o o f. The volume of water flowing into the region $G^e_{\rm Ca}$, or flowing out of the region $G^e_{\rm Ca}$, during one second is

$$P = -\int_{\Gamma^{\rm 3D}_{\rm in}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = -\int_{\Gamma^{\rm 3D}_{\rm in}} h^{\rm in}_{\rm velo}\,\mathrm{d}S = \int_{\Gamma^{\rm 3D}_{\rm out}} h^{\rm out}_{\rm velo}\,\mathrm{d}S = \int_{\Gamma^{\rm 3D}_{\rm out}} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S,$$

because of assumption (1.6).

Further we assume that water flows into the region $G^e_{\rm Ca}$ through the boundary $\Gamma^{\rm 3D}_{\rm in}$, so

$$\mathbf{u}\cdot\mathbf{n} < 0 \quad \text{on } \Gamma^{\rm 3D}_{\rm in}$$

and flows out of the region $G^e_{\rm Ca}$ through the boundary $\Gamma^{\rm 3D}_{\rm out}$, so

$$\mathbf{u}\cdot\mathbf{n} > 0 \quad \text{on } \Gamma^{\rm 3D}_{\rm out}.$$

Then the expression

$$-\frac{1}{P}\int_{\Gamma_{\text{in}}^{3\text{D}}} \vartheta_2 \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = -\frac{1}{P}\int_{\Gamma_{\text{in}}^{3\text{D}}} 288 h_{\text{velo}}^{\text{in}}\,\mathrm{d}S$$

means the average temperature of water flowing into G_{Ca}^{e} during one second (recall $h_{\text{velo}}^{\text{in}} < 0$) and

$$\frac{1}{P}\int_{\Gamma_{\text{out}}^{3\text{D}}} \vartheta_2 \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = \frac{1}{P}\int_{\Gamma_{\text{out}}^{3\text{D}}} h_{\text{out}}^{e} h_{\text{velo}}^{\text{out}}\,\mathrm{d}S$$

means the average temperature of water flowing out of G_{Ca}^{e} during one second.

We assume cooling process, that means the average temperature of water flowing into is less than the average temperature of water flowing out (assumption **A1**), so

$$-\frac{1}{P}\int_{\Gamma_{\text{in}}^{3\text{D}}} 288 h_{\text{velo}}^{\text{in}}\,\mathrm{d}S < \frac{1}{P}\int_{\Gamma_{\text{out}}^{3\text{D}}} h_{\text{out}}^{e} h_{\text{velo}}^{\text{out}}\,\mathrm{d}S. \tag{1.39}$$

Now we have

$$\begin{aligned}
\text{Energy}_{G}^{\text{velo}}(\vartheta,\mathbf{u},\vartheta) &= c_v\varrho_2\int_{G_{\text{Ca}}^{e}}\Big(\frac{\partial\vartheta_2}{\partial x}\vartheta_2 u_1 + \frac{\partial\vartheta_2}{\partial y}\vartheta_2 u_2 + \frac{\partial\vartheta_2}{\partial z}\vartheta_2 u_3\Big)\,\mathrm{d}V\\
&= \frac{1}{2}c_v\varrho_2\int_{\partial G_{\text{Ca}}^{e}}(\vartheta_2^2 u_1\nu_x + \vartheta_2^2 u_2\nu_y + \vartheta_2^2 u_3\nu_z)\,\mathrm{d}S\\
&= \frac{1}{2}c_v\varrho_2\int_{\partial G_{\text{Ca}}^{e}}\vartheta_2^2\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S\\
&= \frac{1}{2}c_v\varrho_2\Big[\int_{\Gamma_{\text{out}}^{3\text{D}}}\vartheta_2^2\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S + \int_{\Gamma_{\text{in}}^{3\text{D}}}\vartheta_2^2\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S\Big]\\
&= \frac{1}{2}c_v\varrho_2\Big[\int_{\Gamma_{\text{out}}^{3\text{D}}}\vartheta_2^2 h_{\text{velo}}^{\text{out}}\,\mathrm{d}S + \int_{\Gamma_{\text{in}}^{3\text{D}}}\vartheta_2^2 h_{\text{velo}}^{\text{in}}\,\mathrm{d}S\Big] \geqslant\\
&\geqslant \frac{1}{2}c_v\varrho_2\Big[\min h_{\text{out}}^{e}\int_{\Gamma_{\text{out}}^{3\text{D}}} h_{\text{out}}^{e} h_{\text{velo}}^{\text{out}}\,\mathrm{d}S + 288\int_{\Gamma_{\text{in}}^{3\text{D}}} 288 h_{\text{velo}}^{\text{in}}\,\mathrm{d}S\Big]\\
&= \frac{1}{2}c_v\varrho_2\Big[(\min h_{\text{out}}^{e} - 288)\int_{\Gamma_{\text{out}}^{3\text{D}}} h_{\text{out}}^{e} h_{\text{velo}}^{\text{out}}\,\mathrm{d}S\\
&\qquad + 288\Big(\int_{\Gamma_{\text{out}}^{3\text{D}}} h_{\text{out}}^{e} h_{\text{velo}}^{\text{out}}\,\mathrm{d}S + \int_{\Gamma_{\text{in}}^{3\text{D}}} 288 h_{\text{velo}}^{\text{in}}\,\mathrm{d}S\Big)\Big] > 0,
\end{aligned}$$

where we used Green's formula, the fact that $\min h_{\text{out}}^{e} > 288$ and (1.39). Transformation to cylindrical coordinates does not change the inequality. □

Theorem 1.4 (Existence and uniqueness of solution of the state problem)**.** *The state problem* (1.36), (1.37) *has a unique solution* $\vartheta(F_2^e)$ *for each* $F_2^e \in U_{\rm ad}^e$ *and the associated flow pattern* $\mathbf{w}^e$ *obtained as the gradient of the unique solution of* (1.7) *and*

$$\|\vartheta(F_2^e)\|_{\mathbf{H}} \leqslant \frac{1}{\min(c_0, c_1, c_2, c_3)} \|F_\Omega\|_{\mathbf{H}^*}. \tag{1.40}$$

Proof. It is sufficient to verify the assumptions of the Lax-Milgram Theorem (see [1] page 12). We denote $V = \mathbf{H}(\Omega)$. According to Theorem 1.2 V is a Hilbert space.

We denote the seminorms of the space $\mathbf{H}(\Omega)$ as

$$\begin{aligned}
\|u\|_{0,2,r} &= \Big(\int_\Omega u^2 r \,\mathrm{d}\Omega\Big)^{1/2},\\
\|u_x\|_{0,2,r} &= \Big(\int_\Omega \Big(\frac{\partial u}{\partial x}\Big)^2 r \,\mathrm{d}\Omega\Big)^{1/2},\\
\|u_r\|_{0,2,r} &= \Big(\int_\Omega \Big(\frac{\partial u}{\partial r}\Big)^2 r \,\mathrm{d}\Omega\Big)^{1/2}.
\end{aligned}$$

Then

$$\|u\|_{\mathbf{H}} = (\|u\|_{0,2,r}^2 + \|u_x\|_{0,2,r}^2 + \|u_r\|_{0,2,r}^2)^{1/2}.$$

According to Theorem 1.1 there exists a unique flow pattern $\mathbf{w}^e$ corresponding to $F_2 \in U_{\rm ad}^e$. We substitute this vector function $\mathbf{w}^e$ into the bilinear form (1.24):

$$\begin{aligned}
|\mathrm{Energy}_\Omega^{\rm velo}(\vartheta, \mathbf{w}^e, \psi)| &= c_v \varrho_2 \Big| \int_{\Omega_{\rm Ca}^e} \Big(\frac{\partial \vartheta_2}{\partial x} w_1 + \frac{\partial \vartheta_2}{\partial r} w_2\Big) \psi r \,\mathrm{d}\Omega \Big|\\
&\leqslant c_v \varrho_2 \max(|w_1|, |w_2|, 1)(\|\vartheta_{2x}\|_{0,2,r} \|\psi\|_{0,2,r}\\
&\qquad + \|\vartheta_{2r}\|_{0,2,r} \|\psi\|_{0,2,r})\\
&\leqslant 2 c_v \varrho_2 \max(|w_1|, |w_2|, 1) \|\vartheta\|_{\mathbf{H}} \|\psi\|_{\mathbf{H}},
\end{aligned}$$

because

$$\begin{aligned}
\|u\|_{\mathbf{H}}^2 \|v\|_{\mathbf{H}}^2 &= (\|u\|_{0,2,r}^2 + \|u_x\|_{0,2,r}^2 + \|u_r\|_{0,2,r}^2)\,(\|v\|_{0,2,r}^2 + \|v_x\|_{0,2,r}^2 + \|v_r\|_{0,2,r}^2)\\
&= \|u\|_{0,2,r}^2 \|v\|_{0,2,r}^2 + \|u\|_{0,2,r}^2 \|v_x\|_{0,2,r}^2 + \|u\|_{0,2,r}^2 \|v_r\|_{0,2,r}^2\\
&\quad + \|u_x\|_{0,2,r}^2 \|v\|_{0,2,r}^2 + \|u_x\|_{0,2,r}^2 \|v_x\|_{0,2,r}^2 + \|u_x\|_{0,2,r}^2 \|v_r\|_{0,2,r}^2\\
&\quad + \|u_r\|_{0,2,r}^2 \|v\|_{0,2,r}^2 + \|u_r\|_{0,2,r}^2 \|v_x\|_{0,2,r}^2 + \|u_r\|_{0,2,r}^2 \|v_r\|_{0,2,r}^2.
\end{aligned}$$

$$
\begin{aligned}
|\mathrm{Energy}^{\mathrm{cond}}_{\Omega}(\vartheta,\psi)| &= k_0\int_{\Omega^e_{\mathrm{Pl}}}\Big(\frac{\partial\vartheta_0}{\partial x}\,\frac{\partial\psi}{\partial x}+\frac{\partial\vartheta_0}{\partial r}\,\frac{\partial\psi}{\partial r}\Big)r\,\mathrm{d}\Omega\\
&\quad + k_1\int_{\Omega_{\mathrm{Gl}}}\Big(\frac{\partial\vartheta_1}{\partial x}\,\frac{\partial\psi}{\partial x}+\frac{\partial\vartheta_1}{\partial r}\,\frac{\partial\psi}{\partial r}\Big)r\,\mathrm{d}\Omega\\
&\quad + k_2\int_{\Omega^e_{\mathrm{Ca}}}\Big(\frac{\partial\vartheta_2}{\partial x}\,\frac{\partial\psi}{\partial x}+\frac{\partial\vartheta_2}{\partial r}\,\frac{\partial\psi}{\partial r}\Big)r\,\mathrm{d}\Omega\\
&\quad + k_3\int_{\Omega_{\mathrm{Mo}}}\Big(\frac{\partial\vartheta_3}{\partial x}\,\frac{\partial\psi}{\partial x}+\frac{\partial\vartheta_3}{\partial r}\,\frac{\partial\psi}{\partial r}\Big)r\,\mathrm{d}\Omega\\
&\leqslant k_0(\|\vartheta_{0x}\|_{0,2,r}\|\psi_x\|_{0,2,r}+\|\vartheta_{0r}\|_{0,2,r}\|\psi_r\|_{0,2,r})\\
&\quad + k_1(\|\vartheta_{1x}\|_{0,2,r}\|\psi_x\|_{0,2,r}+\|\vartheta_{1r}\|_{0,2,r}\|\psi_r\|_{0,2,r})\\
&\quad + k_2(\|\vartheta_{2x}\|_{0,2,r}\|\psi_x\|_{0,2,r}+\|\vartheta_{2r}\|_{0,2,r}\|\psi_r\|_{0,2,r})\\
&\quad + k_3(\|\vartheta_{3x}\|_{0,2,r}\|\psi_x\|_{0,2,r}+\|\vartheta_{3r}\|_{0,2,r}\|\psi_r\|_{0,2,r})\\
&\leqslant 2\max(k_0,k_1,k_2,k_3)\|\vartheta\|_{\mathbf{H}}\|\psi\|_{\mathbf{H}},\\
|\mathrm{Environment}_{\Omega}(\vartheta,\psi)| &= \Big|\int_{\Gamma_7}\alpha(\vartheta_3|_{\Gamma_7})\psi r\,\mathrm{d}\Gamma\Big| \leqslant \int_{\Gamma_7}\alpha|(\vartheta_3|_{\Gamma_7})\psi r|\,\mathrm{d}\Gamma\\
&\leqslant \alpha\Big(\int_{\Gamma_7}(\vartheta_3|_{\Gamma_7})^2 r\,\mathrm{d}\Gamma\Big)^{1/2}\Big(\int_{\Gamma_7}\psi^2 r\,\mathrm{d}\Gamma\Big)^{1/2}\\
&\leqslant \alpha C\|\vartheta_3\|_{\mathbf{H}}\|\psi\|_{\mathbf{H}} \leqslant \alpha C_1\|\vartheta\|_{\mathbf{H}}\|\psi\|_{\mathbf{H}},
\end{aligned}
$$

where we have used the Hölder inequality and the Trace Theorem [1] page 9.

Together we get

$$
\begin{aligned}
&|A_{\Omega}(\vartheta,\mathbf{w}^e,\psi)|\\
&\qquad\leqslant [2c_v\varrho_2\max(|w_1|,|w_2|,1)+2\max(k_0,k_1,k_2,k_3)+\alpha C_1]\|\vartheta\|_{\mathbf{H}}\|\psi\|_{\mathbf{H}},
\end{aligned}
$$

which proves continuity of the left hand side.

Further,

$$
\begin{aligned}
&\mathrm{Energy}^{\mathrm{cond}}_{\Omega}(\vartheta,\vartheta)+\mathrm{Environment}_{\Omega}(\vartheta,\vartheta)\\
&\quad = k_0\int_{\Omega^e_{\mathrm{Pl}}}\Big[\Big(\frac{\partial\vartheta_0}{\partial x}\Big)^2+\Big(\frac{\partial\vartheta_0}{\partial r}\Big)^2\Big]r\,\mathrm{d}\Omega + k_1\int_{\Omega_{\mathrm{Gl}}}\Big[\Big(\frac{\partial\vartheta_1}{\partial x}\Big)^2+\Big(\frac{\partial\vartheta_1}{\partial r}\Big)^2\Big]r\,\mathrm{d}\Omega\\
&\qquad + k_2\int_{\Omega^e_{\mathrm{Ca}}}\Big[\Big(\frac{\partial\vartheta_2}{\partial x}\Big)^2+\Big(\frac{\partial\vartheta_2}{\partial r}\Big)^2\Big]r\,\mathrm{d}\Omega + k_3\int_{\Omega_{\mathrm{Mo}}}\Big[\Big(\frac{\partial\vartheta_3}{\partial x}\Big)^2+\Big(\frac{\partial\vartheta_3}{\partial r}\Big)^2\Big]r\,\mathrm{d}\Omega\\
&\qquad + \int_{\Gamma_7}\alpha(\vartheta_3|_{\Gamma_7})^2 r\,\mathrm{d}\Gamma \geqslant c_0\|\vartheta_0\|^2_{\mathbf{H}}+c_1\|\vartheta_1\|^2_{\mathbf{H}}+c_2\|\vartheta_2\|^2_{\mathbf{H}}+c_3\|\vartheta_3\|^2_{\mathbf{H}}\\
&\quad \geqslant \min(c_0,c_1,c_2,c_3)\|\vartheta\|^2_{\mathbf{H}},
\end{aligned}
$$

where we have used Friedrichs' inequality (see [1] page 10).

Together with Theorem 1.3 we get

$$|A_\Omega(\vartheta, \mathbf{w}^e, \vartheta)| \geqslant \min(c_0, c_1, c_2, c_3)\|\vartheta\|^2_{\mathbf{H}}. \tag{1.41}$$

This proves **H**-ellipticity.

Further we have

$$\begin{aligned} |\mathrm{Source}_\Omega(\psi)| &\leqslant \varrho_1 \int_{\Omega_{\mathrm{Gl}}} |q\psi r| \,\mathrm{d}\Omega \leqslant \varrho_1 \Big(\int_{\Omega_{\mathrm{Gl}}} q^2 r \,\mathrm{d}\Omega\Big)^{1/2} \Big(\int_{\Omega_{\mathrm{Gl}}} \psi^2 r \,\mathrm{d}\Omega\Big)^{1/2} \\ &\leqslant \varrho_1 \|q\|_{L^2_r(\Omega_{\mathrm{Gl}})} \|\psi\|_{L^2_r(\Omega_{\mathrm{Gl}})} \leqslant \varrho_1 \|q\|_{L^2_r(\Omega_{\mathrm{Gl}})} \|\psi\|_{\mathbf{H}} \end{aligned} \tag{1.42}$$

and

$$\begin{aligned} |\mathrm{Coeff}_\Omega(\psi)| &\leqslant \int_{\Gamma_1} \beta_1 |\psi r| \,\mathrm{d}\Gamma + \int_{\Gamma_6} \beta_6 |\psi r| \,\mathrm{d}\Gamma + \int_{\Gamma_7} \alpha\vartheta_4 |\psi r| \,\mathrm{d}\Gamma \\ &\leqslant \beta_1 \|1\|_{L^2_r(\partial\Omega_{\mathrm{Gl}})} \|\psi\|_{L^2_r(\partial\Omega_{\mathrm{Gl}})} + \beta_6 \|1\|_{L^2_r(\partial\Omega_{\mathrm{Gl}})} \|\psi\|_{L^2_r(\partial\Omega_{\mathrm{Gl}})} \\ &\quad + \alpha\vartheta_4 \|1\|_{L^2_r(\partial\Omega_{\mathrm{Gl}})} \|\psi\|_{L^2_r(\partial\Omega_{\mathrm{Gl}})} \\ &\leqslant (\beta_1 + \beta_6 + \alpha\vartheta_4) \|1\|_{L^2_r(\partial\Omega_{\mathrm{Gl}})} \|\psi\|_{\mathbf{H}}, \end{aligned} \tag{1.43}$$

where we have used the Hölder inequality and the Trace Theorem (see [1] page 9).

The linearity of the right hand side of (1.36) together with (1.42) and (1.43) gives its continuity. According to the Lax-Milgram theorem there exists a unique solution of problem (1.36), (1.37). □

Remark. The problem includes both the pure conduction of heat in the regions $\Omega^e_{\mathrm{Pl}} \cup \Omega_{\mathrm{Gl}} \cup \Omega_{\mathrm{Mo}}$ (flow pattern is equal to zero) and the combination of heat convection with conduction of heat in region Ω^e_{Ca}.

We will solve the **problem of optimal design for the plunger cavity shape**: We define the *cost functional* as

$$\mathcal{J}^S(F^e_2) = \|\vartheta(F^e_2)\,|_{\Gamma_1} - T_{\Gamma_1}\|^2_{0,r,\Gamma_1}, \tag{1.44}$$

where $\vartheta(F^e_2)\,|_{\Gamma_1}$ is the trace of the solution $\vartheta(F^e_2)$ of the state problem (1.36), (1.37) in the region Ω^e_{Pl} on the boundary Γ_1, T_{Γ_1} is a chosen fixed constant corresponding to the optimal surface plunger temperature. We look for the *optimal design* $F_{\mathrm{Opt}} \in U^e_{\mathrm{ad}}$ such that

$$\mathcal{J}^S(F_{\mathrm{Opt}}) \leqslant \mathcal{J}^S(F^e_2) \quad \forall F^e_2 \in U^e_{\mathrm{ad}}. \tag{1.45}$$

Theorem 1.5 (Existence of solution of the problem of optimal design for plunger cavity shape)**.** *The optimal design problem* (1.45) *has at least one solution.*

P r o o f. We use Theorem 2.1 published in [1] page 29. We denote $\widetilde{U} = C([0,1])$, $U^\circ = \{f \in \widetilde{U};\ 0 \leqslant f(x) \leqslant f_1(x)\ \forall x \in [0,1]\}$, where $f_1 \in C([0,1])$ is a fixed given increasing function.

The set $U^e_{\rm ad}$ is bounded and closed in $C([0,1])$ and, moreover, consists of uniformly continuous functions. The theorem of Arzela-Ascoli implies the compactness of $U^e_{\rm ad}$ in $C([0,1])$.

We denote $\Omega^n = \Omega^n_{\rm Pl} \cup \Omega_{\rm Gl} \cup \Omega^n_{\rm Ca} \cup \Omega_{\rm Mo}$. Let $\vartheta^n = \vartheta^n_0 + \vartheta^n_1 + \vartheta^n_2 + \vartheta^n_3$ be the solution of the state problem (1.36), (1.37) in the region Ω^n (see (1.17)). Further we denote by $\mathbf{w}^n = (w^n_1, w^n_2)$ the associated velocity field derived from the unique solution of the problem (1.7) in the region $\Omega^n_{\rm Ca}$.

Let $F^e_n \in U^e_{\rm ad}$ be a sequence of functions, then there exists a subsequence $F^e_{n_k} \to F^e \in U^e_{\rm ad}$ such that $F^e_{n_k} \rightrightarrows F^e$ uniformly on $[0,1]$ so then $\Omega^{n_k}_{\rm Pl} \to \Omega_{\rm Pl}$ and thus $\Omega^{n_k} \to \Omega$ on the set Θ.

The variational formulation (1.7) of the problem for finding the potential function in the region $\Omega^e_{\rm Ca}$ has the form

$$\int_{\Omega^e_{\rm Ca}} \Big[\frac{\partial \Phi}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \Phi}{\partial r}\frac{\partial \varphi}{\partial r}\Big] r \,{\rm d}\Omega = \int_{\Gamma_{\rm in}\cup\Gamma_{\rm out}} g\varphi r \,{\rm d}\Gamma \quad \forall \varphi \in H^1_r(\Omega^e_{\rm Ca} \cup \Omega^e_{\rm Pl}) \tag{1.46}$$

and the variational formulation of the analogous problem in the region $\Omega^{n_k}_{\rm Ca}$ has the form

$$\int_{\Omega^{n_k}_{\rm Ca}} \Big[\frac{\partial \Phi^{n_k}}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \Phi^{n_k}}{\partial r}\frac{\partial \varphi}{\partial r}\Big] r \,{\rm d}\Omega = \int_{\Gamma_{\rm in}\cup\Gamma_{\rm out}} g\varphi r \,{\rm d}\Gamma \quad \forall \varphi \in H^1_r(\Omega^{n_k}_{\rm Ca} \cup \Omega^{n_k}_{\rm Pl}). \tag{1.47}$$

We subtract (1.47) from (1.46) and obtain

$$\begin{aligned}&\int_{\Omega^e_{\rm Ca}\cap\Omega^{n_k}_{\rm Ca}} \Big[\frac{\partial(\Phi-\Phi^{n_k})}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial(\Phi-\Phi^{n_k})}{\partial r}\frac{\partial \varphi}{\partial r}\Big] r \,{\rm d}\Omega \\ &+ \int_{\Omega^e_{\rm Ca}\setminus\Omega^{n_k}_{\rm Ca}} \Big[\frac{\partial \Phi}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \Phi}{\partial r}\frac{\partial \varphi}{\partial r}\Big] r \,{\rm d}\Omega - \int_{\Omega^{n_k}_{\rm Ca}\setminus\Omega^e_{\rm Ca}} \Big[\frac{\partial \Phi^{n_k}}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial \Phi^{n_k}}{\partial r}\frac{\partial \varphi}{\partial r}\Big] r \,{\rm d}\Omega = 0.\end{aligned}$$

We substitute $\varphi = \Phi - \Phi^{n_k}$ and get

$$\begin{aligned}&\int_{\Omega^e_{\rm Ca}\cap\Omega^{n_k}_{\rm Ca}} \Big[\Big(\frac{\partial(\Phi-\Phi^{n_k})}{\partial x}\Big)^2 + \Big(\frac{\partial(\Phi-\Phi^{n_k})}{\partial r}\Big)^2\Big] r \,{\rm d}\Omega \\ &\quad + \int_{\Omega^e_{\rm Ca}\setminus\Omega^{n_k}_{\rm Ca}} \Big[\frac{\partial \Phi}{\partial x}\frac{\partial(\Phi-\Phi^{n_k})}{\partial x} + \frac{\partial \Phi}{\partial r}\frac{\partial(\Phi-\Phi^{n_k})}{\partial r}\Big] r \,{\rm d}\Omega \\ &\quad - \int_{\Omega^{n_k}_{\rm Ca}\setminus\Omega^e_{\rm Ca}} \Big[\frac{\partial \Phi^{n_k}}{\partial x}\frac{\partial(\Phi-\Phi^{n_k})}{\partial x} + \frac{\partial \Phi^{n_k}}{\partial r}\frac{\partial(\Phi-\Phi^{n_k})}{\partial r}\Big] r \,{\rm d}\Omega = 0.\end{aligned}$$

The last two integrals on the left hand side have zero limit for $\Omega^{n_k} \to \Omega$ because we integrate bounded functions $\Phi \in H_r^1(\Omega_{\mathrm{Ca}}^e)$ and $\Phi^{n_k} \in H_r^1(\Omega_{\mathrm{Ca}}^{n_k})$ over the regions with $\operatorname{meas}(\Omega_{\mathrm{Ca}}^e \setminus \Omega_{\mathrm{Ca}}^{n_k}) \to 0$ and $\operatorname{meas}(\Omega_{\mathrm{Ca}}^{n_k} \setminus \Omega_{\mathrm{Ca}}^e) \to 0$. In the first integral we integrate a nonnegative function and thus

$$\int_{\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k}} \Big[\Big(\frac{\partial(\Phi - \Phi^{n_k})}{\partial x}\Big)^2 + \Big(\frac{\partial(\Phi - \Phi^{n_k})}{\partial r}\Big)^2\Big] r \,\mathrm{d}\Omega \\ = \int_{\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k}} \Big[\Big(w_1 - w_1^{n_k}\Big)^2 + \Big(w_2 - w_2^{n_k}\Big)^2\Big] r \,\mathrm{d}\Omega \to 0.$$

From the Hölder inequality we get

$$\int_{\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k}} (w_i - w_i^{n_k}) r \,\mathrm{d}\Omega \leqslant \int_{\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k}} |w_i - w_i^{n_k}| r \,\mathrm{d}\Omega \\ \leqslant \Big(\int_{\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k}} (w_i - w_i^{n_k})^2 r \,\mathrm{d}\Omega\Big)^{1/2} \operatorname{meas}(\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k}) \to 0 \tag{1.48}$$

for $i = 1, 2$ and thus $w_i^{n_k} \to w_i$ in $L_r^2(\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k})$.

The variational formulation of the state problem in the region Ω^{n_k} has the form

$$A_{\Omega^{n_k}}(\vartheta^{n_k}, \mathbf{w}^{\mathbf{n_k}}, \psi) = F_{\Omega^{n_k}}(\psi) \quad \forall \psi \in \mathbf{H}_0(\Omega^{n_k}). \tag{1.49}$$

We subtract (1.49) from (1.36) and obtain

$$c_v \varrho_2 \int_{\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k}} \Big[\frac{\partial \vartheta_2}{\partial x} w_1 \psi - \frac{\partial \vartheta_2^{n_k}}{\partial x} w_1^{n_k} \psi + \frac{\partial \vartheta_2}{\partial r} w_2 \psi - \frac{\partial \vartheta_2^{n_k}}{\partial r} w_2^{n_k} \psi\Big] r \,\mathrm{d}\Omega \\ + k_0 \int_{\Omega_{\mathrm{Pl}}^e \cap \Omega_{\mathrm{Pl}}^{n_k}} \Big[\frac{\partial(\vartheta_0 - \vartheta_0^{n_k})}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial(\vartheta_0 - \vartheta_0^{n_k})}{\partial r} \frac{\partial \psi}{\partial r}\Big] r \,\mathrm{d}\Omega \\ + k_1 \int_{\Omega_{\mathrm{Gl}}} \Big[\frac{\partial(\vartheta_1 - \vartheta_1^{n_k})}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial(\vartheta_1 - \vartheta_1^{n_k})}{\partial r} \frac{\partial \psi}{\partial r}\Big] r \,\mathrm{d}\Omega \\ + k_2 \int_{\Omega_{\mathrm{Ca}}^e \cap \Omega_{\mathrm{Ca}}^{n_k}} \Big[\frac{\partial(\vartheta_2 - \vartheta_2^{n_k})}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial(\vartheta_2 - \vartheta_2^{n_k})}{\partial r} \frac{\partial \psi}{\partial r}\Big] r \,\mathrm{d}\Omega \\ + k_3 \int_{\Omega_{\mathrm{Mo}}} \Big[\frac{\partial(\vartheta_3 - \vartheta_3^{n_k})}{\partial x} \frac{\partial \psi}{\partial x} + \frac{\partial(\vartheta_3 - \vartheta_3^{n_k})}{\partial r} \frac{\partial \psi}{\partial r}\Big] r \,\mathrm{d}\Omega$$

$$
\begin{aligned}
&+ \alpha \int_{\Gamma_7} (\vartheta_3|_{\Gamma_7} - \vartheta_3^{n_k}|_{\Gamma_7}) \psi r \,\mathrm{d}\Gamma \\
&+ \int_{\Omega^e_{\mathrm{Ca}} \cap \Omega^{n_k}_{\mathrm{Pl}}} \Big[\Big(k_2 \frac{\partial \vartheta_2}{\partial x} - k_0 \frac{\partial \vartheta_0^{n_k}}{\partial x} \Big) \frac{\partial \psi}{\partial x} + \Big(k_2 \frac{\partial \vartheta_2}{\partial r} - k_0 \frac{\partial \vartheta_0^{n_k}}{\partial r} \Big) \frac{\partial \psi}{\partial r} \\
&+ c_v \varrho_2 \Big(\frac{\partial \vartheta_2}{\partial x} w_1 + \frac{\partial \vartheta_2}{\partial r} w_2 \Big) \psi \Big] r \,\mathrm{d}\Omega \\
&+ \int_{\Omega^e_{\mathrm{Pl}} \cap \Omega^{n_k}_{\mathrm{Ca}}} \Big[\Big(k_0 \frac{\partial \vartheta_0}{\partial x} - k_2 \frac{\partial \vartheta_2^{n_k}}{\partial x} \Big) \frac{\partial \psi}{\partial x} + \Big(k_0 \frac{\partial \vartheta_0}{\partial r} - k_2 \frac{\partial \vartheta_2^{n_k}}{\partial r} \Big) \frac{\partial \psi}{\partial r} \\
&- c_v \varrho_2 \Big(\frac{\partial \vartheta_2^{n_k}}{\partial x} w_1^{n_k} + \frac{\partial \vartheta_2^{n_k}}{\partial r} w_2^{n_k} \Big) \psi \Big] r \,\mathrm{d}\Omega = 0.
\end{aligned}
$$

We add and subtract the terms $\partial \vartheta_2^{n_k} / \partial x w_1 \psi$ and $\partial \vartheta_2^{n_k} / \partial r w_2 \psi$ in the first integral. Then we substitute $\psi = \vartheta - \vartheta^{n_k}$ and get

$$
\begin{aligned}
&c_v \varrho_2 \int_{\Omega^e_{\mathrm{Ca}} \cap \Omega^{n_k}_{\mathrm{Ca}}} \Big[\frac{\partial (\vartheta_2 - \vartheta_2^{n_k})}{\partial x} w_1 (\vartheta_2 - \vartheta_2^{n_k}) + \frac{\partial (\vartheta_2 - \vartheta_2^{n_k})}{\partial r} w_2 (\vartheta_2 - \vartheta_2^{n_k}) \Big] r \,\mathrm{d}\Omega \\
&+ k_0 \int_{\Omega^e_{\mathrm{Pl}} \cap \Omega^{n_k}_{\mathrm{Pl}}} \Big[\Big(\frac{\partial (\vartheta_0 - \vartheta_0^{n_k})}{\partial x} \Big)^2 + \Big(\frac{\partial (\vartheta_0 - \vartheta_0^{n_k})}{\partial r} \Big)^2 \Big] r \,\mathrm{d}\Omega \\
&+ k_1 \int_{\Omega_{\mathrm{Gl}}} \Big[\Big(\frac{\partial (\vartheta_1 - \vartheta_1^{n_k})}{\partial x} \Big)^2 + \Big(\frac{\partial (\vartheta_1 - \vartheta_1^{n_k})}{\partial r} \Big)^2 \Big] r \,\mathrm{d}\Omega \\
&+ k_2 \int_{\Omega^e_{\mathrm{Ca}} \cap \Omega^{n_k}_{\mathrm{Ca}}} \Big[\Big(\frac{\partial (\vartheta_2 - \vartheta_2^{n_k})}{\partial x} \Big)^2 + \Big(\frac{\partial (\vartheta_2 - \vartheta_2^{n_k})}{\partial r} \Big)^2 \Big] r \,\mathrm{d}\Omega \\
&+ k_3 \int_{\Omega_{\mathrm{Mo}}} \Big[\Big(\frac{\partial (\vartheta_3 - \vartheta_3^{n_k})}{\partial x} \Big)^2 + \Big(\frac{\partial (\vartheta_3 - \vartheta_3^{n_k})}{\partial r} \Big)^2 \Big] r \,\mathrm{d}\Omega \\
&+ \alpha \int_{\Gamma_7} (\vartheta_3|_{\Gamma_7} - \vartheta_3^{n_k}|_{\Gamma_7})^2 r \,\mathrm{d}\Gamma \\
&+ c_v \varrho_2 \int_{\Omega^e_{\mathrm{Ca}} \cap \Omega^{n_k}_{\mathrm{Ca}}} \Big[\frac{\partial \vartheta_2^{n_k}}{\partial x} (w_1 - w_1^{n_k}) (\vartheta_2 - \vartheta_2^{n_k}) + \frac{\partial \vartheta_2^{n_k}}{\partial r} (w_2 - w_2^{n_k}) (\vartheta_2 - \vartheta_2^{n_k}) \Big] r \,\mathrm{d}\Omega \\
&+ \int_{\Omega^e_{\mathrm{Ca}} \cap \Omega^{n_k}_{\mathrm{Pl}}} \Big[\Big(k_2 \frac{\partial \vartheta_2}{\partial x} - k_0 \frac{\partial \vartheta_0^{n_k}}{\partial x} \Big) \frac{\partial (\vartheta_2 - \vartheta_0^{n_k})}{\partial x} + \Big(k_2 \frac{\partial \vartheta_2}{\partial r} - k_0 \frac{\partial \vartheta_0^{n_k}}{\partial r} \Big) \frac{\partial (\vartheta_2 - \vartheta_0^{n_k})}{\partial r} \\
&+ c_v \varrho_2 \Big(\frac{\partial \vartheta_2}{\partial x} w_1 + \frac{\partial \vartheta_2}{\partial r} w_2 \Big) (\vartheta_2 - \vartheta_0^{n_k}) \Big] r \,\mathrm{d}\Omega \\
&+ \int_{\Omega^e_{\mathrm{Pl}} \cap \Omega^{n_k}_{\mathrm{Ca}}} \Big[\Big(k_0 \frac{\partial \vartheta_0}{\partial x} - k_2 \frac{\partial \vartheta_2^{n_k}}{\partial x} \Big) \frac{\partial (\vartheta_0 - \vartheta_2^{n_k})}{\partial x} + \Big(k_0 \frac{\partial \vartheta_0}{\partial r} - k_2 \frac{\partial \vartheta_2^{n_k}}{\partial r} \Big) \frac{\partial (\vartheta_0 - \vartheta_2^{n_k})}{\partial r} \\
&- c_v \varrho_2 \Big(\frac{\partial \vartheta_2^{n_k}}{\partial x} w_1^{n_k} + \frac{\partial \vartheta_2^{n_k}}{\partial r} w_2^{n_k} \Big) (\vartheta_0 - \vartheta_2^{n_k}) \Big] r \,\mathrm{d}\Omega = 0.
\end{aligned}
$$

The last two integrals on the left hand side have the zero limit for $\Omega^{n_k} \to \Omega$ because we integrate bounded functions over regions with $\mathrm{meas}(\Omega^e_{\mathrm{Ca}} \cap \Omega^{n_k}_{\mathrm{Pl}}) \to 0$ and $\mathrm{meas}(\Omega^e_{\mathrm{Pl}} \cap \Omega^{n_k}_{\mathrm{Ca}}) \to 0$. The last but two integral has the zero limit because $\partial \vartheta_2^{n_k} / \partial x$,

$\partial\vartheta_2^{n_k}/\partial r$, ϑ_2, $\vartheta_2^{n_k}$ are bounded functions and $\mathbf{w}^{n_k} \to \mathbf{w}$ (see (1.48)). The first six integrals are positive and converge to $A_\Omega(\vartheta - \vartheta^{n_k}, \mathbf{w}, \vartheta - \vartheta^{n_k})$.

From the **H**-ellipticity of $A_\Omega(\vartheta, \mathbf{w}, \psi)$ (see (1.41)) we get

$$\|\vartheta - \vartheta^{n_k}\|_{\mathbf{H}}^2 \leqslant C A_\Omega(\vartheta - \vartheta^{n_k}, \mathbf{w}, \vartheta - \vartheta^{n_k}) \to 0 \tag{1.50}$$

and thus $\vartheta^{n_k} \to \vartheta$ in $\mathbf{H}(\Omega)$. We have to verify that

$$\mathcal{J}^S(F^e) \leqslant \liminf_{n\to\infty} \mathcal{J}^S(F^e_{n_k}),$$

but this is true because the square of the norm $\|w|_{\Gamma_1}\|_{0,r,\Gamma_1}$ is a weak lower semicontinuous functional. $\square$

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