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DISTRIBUTIONAL CHAOS FOR FLOWS

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Abstract. Schweizer and Smítal introduced the distributional chaos for continuous maps of the interval in B. Schweizer, J. Smítal, Measures of chaos and a spectral decomposition of dynamical systems on the interval. Trans. Amer. Math. Soc. 344 (1994), 737–854. In this paper, we discuss the distributional chaos DC1–DC3 for flows on compact metric spaces. We prove that both the distributional chaos DC1 and DC2 of a flow are equivalent to the time-1 maps and so some properties of DC1 and DC2 for discrete systems also hold for flows. However, we prove that DC2 and DC3 are not invariants of equivalent flows although DC2 is a topological conjugacy invariant in discrete case.

Keywords: distributional chaos, flow, invariant

MSC 2010: 37B99, 37B05, 37E25

1. Introduction

Let (M,d) be a compact metric space and $\varphi_t\colon\ (M,d)\to(M,d)$ $(t\in\mathbb{R})$ a continuous flow. $\{\varphi_t\}_{t\in\mathbb{R}}$ is also called a continuous dynamical system and is denoted briefly by φ. For any two points $x,y\in M$ and $T\in\mathbb{R}$, we define the distribution function

$$\Phi_{xy}^T\colon\ \mathbb{R}^+\cup\{0\}\to[0,1],$$
$$\varepsilon\mapsto\frac{1}{T}\cdot(m\{t\in[0,T]\colon\ d(\varphi_t(x),\varphi_t(y))<\varepsilon\}),$$

where m is the Lebesgue measure on $\mathbb{R}$. We also set

$$\Phi_{xy}(\varepsilon)=\liminf_{T\to\infty}\Phi_{xy}^T(\varepsilon)\ \text{and}\ \Phi_{xy}^*(\varepsilon)=\limsup_{T\to\infty}\Phi_{xy}^T(\varepsilon).$$

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It is obvious that

$$0 \leqslant \Phi_{xy}(\varepsilon) \leqslant \Phi^*_{xy}(\varepsilon) \leqslant 1, \ \forall \varepsilon \geqslant 0. \tag{1.1}$$

Now we give the definitions of distributional chaos for the flow φ_t.

Definition 1.1. Let $\varphi_t\colon (M,d) \to (M,d)$ $(t \in \mathbb{R})$ be a continuous flow of compact metric space. Then

(1) φ exhibits distributional chaos of type 1, briefly denoted by DC1, if there are two points $x, y \in M$ such that $\Phi^*_{xy}(\varepsilon) = 1$, $\forall \varepsilon > 0$ and $\Phi_{xy}(\varepsilon_0) = 0$ for some $\varepsilon_0 > 0$;
(2) φ exhibits distributional chaos of type 2, briefly denoted by DC2, if there are two points $x, y \in M$ such that $\Phi^*_{xy}(\varepsilon) = 1$, $\forall \varepsilon > 0$ and $\Phi_{xy}(\varepsilon_0) < \Phi^*_{xy}(\varepsilon_0)$ for some $\varepsilon_0 > 0$;
(3) φ exhibits distributional chaos of type 3, briefly denoted by DC3, if there are two points $x, y \in M$ such that $\Phi_{xy}(\varepsilon) < \Phi^*_{xy}(\varepsilon)$ for all ε in an interval.

From the definitions, it is obvious that DC1 $\Rightarrow$ DC2 $\Rightarrow$ DC3.

The definitions of distributional chaos of flow follow some basic ideas of discrete systems (e.g. see [1], [4], [5]). In fact, we will prove, in Section 2 that both the distributional chaos DC1 and DC2 of a flow are equivalent to the time-1 maps and so some properties of DC1 and DC2 for discrete systems also hold for flows.

For the discrete dynamical systems, both DC1 and DC2 are topological conjugacy invariants [5], but DC3 is not an invariant [1]. However, we will show that neither DC2 nor DC3 is an invariant for equivalent flows in Section 3.

2. DC1 and DC2 of a flow are equivalent to the time-1 maps

Given a continuous flow $\varphi_t\colon (M,d) \to (M,d)$ $(t \in \mathbb{R})$, it is well known that the time-1 map φ_1 is a homeomorphism of M and $\varphi_1^i = \varphi_i$ for all integer i.

Theorem 2.1. *For $i = 1, 2$, φ is DCi if and only if φ_1 is DCi.*

Proof. We only prove the case $i = 1$; the other case $(i = 2)$ can be proved similarly.

For the discrete system φ_1, let the distributional function be

$$\Psi^n_{xy}(\varepsilon) = \frac{\#\{0 \leqslant i \leqslant n-1\colon\ d(\varphi_1^i(x), \varphi_1^i(y)) < \varepsilon\}}{n},$$

where $n \in \mathbb{Z}$, $\varepsilon > 0$ and $x, y \in M$.

We denote

$$\Psi_{xy}(\varepsilon) = \liminf_{n\to\infty} \Psi_{xy}^n(\varepsilon) \quad \text{and} \quad \Psi_{xy}^*(\varepsilon) = \limsup_{n\to\infty} \Psi_{xy}^n(\varepsilon).$$

Recall that (e.g. see [1]) φ_1 is DC1 if and only if there are two points $x, y \in M$ such that $\Psi_{xy}^*(\varepsilon) = 1$, $\forall \varepsilon > 0$ and $\Psi_{xy}(\varepsilon_0) = 0$ for some $\varepsilon_0 > 0$.

We first prove that for any $\varepsilon > 0$ there is a positive number ε_1 such that

$$\Psi_{xy}^n(\varepsilon_1) \leqslant \Phi_{xy}^n(\varepsilon), \Phi_{xy}^n(\varepsilon_1) \leqslant \frac{n+1}{n}\Psi_{xy}^{n+1}(\varepsilon), \quad \forall x, y \in M \text{ and } \forall n \in \mathbb{N}. \tag{2.1}$$

In fact, for given $\varepsilon > 0$, we select ε_1 satisfying that, for any $u, v \in M$,

$$d(u, v) < \varepsilon_1 \Rightarrow d(\varphi_t(u), \varphi_t(v)) < \varepsilon, \quad \forall t \in [0, 1].$$

For $x, y \in M$ and $i \in \mathbb{N}$ we denote

$$A_i(\varepsilon) = \{0 \leqslant j \leqslant i - 1\colon\ d(\varphi_1^j(x), \varphi_1^j(y)) < \varepsilon\}$$

and

$$a_i(\varepsilon) = m(\{t \in [i, i+1]\colon\ d(\varphi_t(x), \varphi_t(y)) < \varepsilon\}).$$

Then

$$\Phi_{xy}^n(\varepsilon) = \frac{1}{n}\sum_{i=0}^{n-1} a_i(\varepsilon), \quad \Psi_{xy}^n(\varepsilon) = \frac{\#A_n(\varepsilon)}{n}.$$

If there is an $i \in A_n(\varepsilon_1)$, then $d(\varphi_1^i(x), \varphi_1^i(y)) < \varepsilon_1$. By the selection of ε_1, we have

$$d(\varphi_{i+t}(x), \varphi_{i+t}(y)) < \varepsilon, \quad \forall t \in [0, 1].$$

This implies that $a_i(\varepsilon) = 1$. That is to say, $\#A_n(\varepsilon_1) \leqslant \sum_{i=0}^{n-1} a_i(\varepsilon)$ and hence

$$\Psi_{xy}^n(\varepsilon_1) \leqslant \Phi_{xy}^n(\varepsilon).$$

On the other hand, if there is an $i \in \{0, 1, \ldots, n-1\}$ such that $a_i(\varepsilon_1) > 0$, then there must be a $t \in [i, i+1]$ satisfying that $d(\varphi_t(x), \varphi_t(y)) < \varepsilon_1$. By the selection of ε_1 again, we can conclude $i + 1 \in A_{n+1}(\varepsilon)$. That is to say, $\sum_{i=0}^{n-1} a_i(\varepsilon_1) \leqslant \#A_{n+1}(\varepsilon)$ and hence

$$\Phi_{xy}^n(\varepsilon_1) \leqslant \frac{n+1}{n}\Psi_{xy}^{n+1}(\varepsilon).$$

So we proved (2.1).

Now we prove that φ is DC1 if and only if φ_1 is DC1. We only prove that the DC1 of φ implies that φ_1 is DC1. The other implication can be proved similarly and we omit its proof.

Supposing that φ is DC1, by definition there are two points $x, y \in M$ such that $\Phi^*_{xy}(\varepsilon) = 1$, $\forall \varepsilon > 0$ and $\Phi_{xy}(\varepsilon_0) = 0$ for some $\varepsilon_0 > 0$. For any $\varepsilon' > 0$, by (2.1) there is ε_1 such that $\Phi^n_{xy}(\varepsilon_1) \leqslant (n+1)n^{-1}\Psi^{n+1}_{xy}(\varepsilon')$. Noting (1.1) and (2.1), we have

$$1 = \Phi^*_{xy}(\varepsilon_1) \leqslant \Psi^*_{xy}(\varepsilon') \leqslant 1, \quad \forall \varepsilon' > 0. \tag{2.2}$$

On the other hand, for the given ε_0 there is another ε_1 such that $\Psi^n_{xy}(\varepsilon_1) \leqslant \Phi^n_{xy}(\varepsilon_0)$, $\forall n \in \mathbb{N}$. Taking the inferior limit as n tends to infinity and noting the inequalities (1.1) we get

$$\Psi_{xy}(\varepsilon_1) = 0. \tag{2.3}$$

By (2.2) and (2.3), one can conclude that φ_1 is DC1. Then we complete the proof. □

For a flow φ, the *topological entropy* $h(\varphi)$ of φ is the topological entropy of the time-1 map of φ. That is to say, $h(\varphi) = h(\varphi_1)$. As a corollary of Theorem 2.1, we can extend the main result (Theorem 1.1) of [3] to the flow case.

Corollary 2.1. *If a flow φ has positive topological entropy, then φ is* DC2.

3. DC2 and DC3 are not invariants of equivalent flows

Two flows $\varphi_t\colon M \to M$ and $\psi_t\colon N \to N$ defined on metric spaces are *equivalent* if there exists a homeomorphism $\pi\colon M \to N$ that sends each orbit of φ onto an orbit of ψ while preserving the time orientation. In this section, we will prove that DC2 and DC3 are not invariants for two equivalent flows.

Theorem 3.1. *There exist two equivalent flows ψ and φ on a compact space M such that ψ is* DC2 *(and hence is* DC3*) but φ is not* DC3 *(and hence is not* DC2*).*

P r o o f. We will construct two equivalent flows ψ and φ satisfying the theorem. The essential construction is included in [6] and we mainly prove that this is suitable for our goal.

By [6], there are two (smooth) flows ψ and φ on a compact space M (in fact, M is a smooth manifold) such that ψ is a time-changed flow of φ (and hence ψ is equivalent to φ) and both flows have only the same singular point p. Furthermore,

$h(\psi) > 0$ and the Dirac measure μ_p is the only invariant measure of φ. Then, by Corollary 2.1, ψ is DC2 since $h(\psi) > 0$.

Let us note that if

$$\Phi_{xy}(\varepsilon) = 1, \quad \forall x, y \in M \text{ and } \forall \varepsilon > 0, \tag{3.1}$$

then there are no $x, y \in M$ and $\varepsilon_0 > 0$ such that $\Phi_{xy}(\varepsilon_0) < \Phi^*_{xy}(\varepsilon_0)$. So, to prove φ is not DC3, we only need to prove that (3.1) holds.

For proving (3.1), we need the following lemma.

Lemma 3.1. *Let ζ be a continuous function on M and $\lambda \in \mathbb{R}$. If $\int_M \zeta(x)\,\mathrm{d}\mu > \lambda$ for every φ-invariant measure μ, then for any $x \in M$ there is $L(x)$ such that*

$$\frac{1}{T}\int_0^T \zeta(\varphi_t(x))\,\mathrm{d}t > \lambda, \quad \forall T \geqslant L(x).$$

This lemma is essentially the flow version of Lemma 2 of [2] and we omit its easy proof.

Given $\varepsilon > 0$, we define a continuous function $\xi\colon M \to [0,1]$ such that $\xi(v) = 1$ if $d(v,p) \leqslant \frac{1}{4}\varepsilon$ and $\xi(v) = 0$ if $d(v,p) \geqslant \frac{1}{2}\varepsilon$.

Since the Dirac measure μ_p is the unique φ-invariant measure and $\int_M \xi(x)\,\mathrm{d}\mu_p = 1$, by Lemma 3.1, for any $x \in M$ and any $\delta > 0$ small enough, there is $L(x)$ such that

$$\frac{1}{T}\int_0^T \xi(\varphi_t(x))\,\mathrm{d}t > 1 - \delta, \quad \forall T \geqslant L(x).$$

Then we have

$$\frac{1}{T}m\Big(\Big\{t \in [0,T]\colon\ d(\varphi_t(x),p) < \frac{\varepsilon}{2}\Big\}\Big) \geqslant \frac{1}{T}\int_0^T \xi(\varphi_t(x))\,\mathrm{d}t > 1 - \delta, \quad \forall T \geqslant L(x).$$

Similarly, for any $y \in M$, there is $L(y)$ such that

$$\frac{1}{T}m\Big(\Big\{t \in [0,T]\colon\ d(\varphi_t(y),p) < \frac{\varepsilon}{2}\Big\}\Big) > 1 - \delta,\ \forall T \geqslant L(y).$$

So, for $L_{xy} = \max\{L(x), L(y)\}$ and any $T \geqslant L_{xy}$, we have

$$m\Big(\Big\{t \in [0,T]\colon\ d(\varphi_t(x),p) < \frac{\varepsilon}{2}\Big\} \cap \Big\{t \in [0,T]\colon\ d(\varphi_t(y),p) < \frac{\varepsilon}{2}\Big\}\Big) > (1-2\delta)T.$$

Noting that $d(\varphi_t(x),\varphi_t(y)) \leqslant d(\varphi_t(x),p) + d(\varphi_t(y),p)$ and using the above inequality, one can conclude

$$m(\{t \in [0,T]\colon\ d(\varphi_t(x),\varphi_t(y)) < \varepsilon\}) > (1-2\delta)T, \quad \forall T \geqslant L_{xy}.$$

So,

$$\Phi_{xy}(\varepsilon) = \liminf_{T\to\infty} \frac{1}{T} m(\{t \in [0,T]\colon\ d(\varphi_t(x), \varphi_t(y)) < \varepsilon\}) > 1 - 2\delta.$$

By the arbitrariness of δ and (1.1), we have

$$\Phi_{xy}(\varepsilon) = 1.$$

This is exactly (3.1) and we have completed the proof. □

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