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EXISTENCE AND GLOBAL ATTRACTIVITY OF POSITIVE PERIODIC SOLUTIONS FOR A DELAYED COMPETITIVE SYSTEM WITH THE EFFECT OF TOXIC SUBSTANCES AND IMPULSES

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Abstract. In this paper, a class of non-autonomous delayed competitive systems with the effect of toxic substances and impulses is considered. By using the continuation theorem of coincidence degree theory, we derive a set of easily verifiable sufficient conditions that guarantees the existence of at least one positive periodic solution, and by constructing a suitable Lyapunov functional, the uniqueness and global attractivity of the positive periodic solution are established.

Keywords: competitive system, toxic substance, periodic solution, impulse, coincidence degree theory

MSC 2010: 34K13, 34K25

1. Introduction

In recent years, the dynamical behavior of a competitive system has been one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Most widely studied competitive systems are mainly continuous or discrete [2], [3], [5], [7], [10], [11], [13], [14], [15]. Recently there has been a new category of competitive systems, which are neither purely continuous-time nor purely discrete-time ones; these are called impulsive competitive system. This category of impulsive competitive systems displays a combination of characteristics of both the continuous-time and discrete-time systems [4], [8].

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In 2003, Song and Chen [14] proposed a delay two-species competitive system in which two species have toxic inhibitory effects on each other:

$$
(1.1)\qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(t)[K_1(t) - \alpha_1(t)x(t) - \beta_1(t)y(t) - \gamma_1(t)x(t)y(t-\tau_1(t))], \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(t)[K_2(t) - \alpha_2(t)y(t) - \beta_2(t)x(t) - \gamma_2(t)x(t-\tau_2(t))y(t)], \end{cases}
$$

where $x(t), y(t)$ stand for the population densities of two competing species, respectively. $K_i(t)$ $(i = 1,2)$ are the intrinsic growth rates of the two competing species; $\alpha_i(t)$ $(i = 1,2)$ denote the coefficients of interspecific competition; $K_i(t)/\alpha_i(t)$ $(i = 1,2)$ are the environmental carrying capacities of two competing species; γ_1 and γ_2 stand for, respectively, the rates of toxic inhibition of the species x by the species y and vice versa. For more details about the model, one can see [12]. By applying the coincidence degree theory, Song and Chen [12] established the existence of a positive periodic solution for system (1.1).

Considering the impulsive effects and periodic perturbations, Liu et al. [9] investigated the periodic impulsive delay competitive system with the effect of toxic substances

$$
(1.2)\qquad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(t)[K_1(t) - \alpha_1(t)x(t) - \beta_1(t)y(t) - \gamma_1(t)x(t)y(t-\tau_1(t))], & t \neq t_k, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(t)[K_2(t) - \alpha_2(t)y(t) - \beta_2(t)x(t) - \gamma_2(t)x(t-\tau_2(t))y(t)], & t \neq t_k, \\ x(t_k^+) = x(t_k) + p, & t = t_k, \\ y(t_k^+) = (1+b_k)y(t_k), & t = t_k \end{cases}
$$

with an initial condition $(x(s), y(s)) = \varphi(s) = (\varphi_1(s), \varphi_2(s))$ for $-\tau \leqslant s \leqslant 0, \varphi(0) > 0, \varphi \in PC([-\tau, 0], \mathbb{R}_+^2)$, where $\tau = \max\limits_{1\leqslant i\leqslant 2}\max\limits_{t\in[0,\omega]}\{\tau_i(t)\}$; $K_i(t)$, $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$, $\tau_i(t)$ $(i = 1,2)$ are continuous ω-periodic functions, and $\alpha_i(t)$, $\beta(t)$, $\gamma_i(t)$ $(i = 1,2)$ are positive and $\tau_i(t)$ $(i = 1,2)$ are nonnegative. The intrinsic growth rates $K_i(t)$ $(i = 1,2)$ are not necessarily positive and may be negative. Also $k \in \mathbb{N}$ and $\mathbb{N}$ is the set of positive integers. The jump conditions reflect the possibility of impulsive effects on the species x and y. $p > 0$ is the impulsive stocking amount of the species x at time $t = t_k$, which implies that the populations are subject to impulsive stocking at a constant rate p. The term $b_k y(t_k) < 0$ $(k \in \mathbb{N})$ represents the impulsive harvesting amount of the species y at time $t = t_k$, while $b_k y(t_k) > 0$ which represents the perturbations may stand for the impulsive stocking amount of the species y at time $t = t_k$. By applying the theory of impulsive differential equations and some analysis techniques, Liu et al. [9] obtained a set of sufficient conditions for the permanence and partial extinction of system (1.2).

Considering that the harvest of many populations is not continuous, the harvest can be viewed as an annual harvest pulse. To describe a system more accurately, we should consider using impulsive differential equations. Then system (1.1) is revised into the following form:

$$(1.3)\quad \begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(t)[K_1(t) - \alpha_1(t)x(t) - \beta_1(t)y(t) - \gamma_1(t)x(t)y(t-\tau_1(t))], & t \neq t_k, \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y(t)[K_2(t) - \alpha_2(t)y(t) - \beta_2(t)x(t) - \gamma_2(t)x(t-\tau_2(t))y(t)], & t \neq t_k, \\ \Delta x(t_k) = x(t_k^+) - x(t_k^-) = \varrho_{1k}x(t_k), & t = t_k, \\ \Delta y(t_k) = y(t_k^+) - y(t_k^-) = \varrho_{2k}y(t_k), & t = t_k, \end{cases}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$ are the impulses at moments t_k and $t_1 < t_2 < \ldots$ is a strictly increasing sequence such that $\lim\limits_{k\to\infty} t_k = +\infty$.

The principal object of this article is by using Mawhin's continuation theorem of coincidence degree theory and by constructing Lyapunov functions to investigate the stability and existence of periodic solutions of (1.3). To the best of the authors' knowledge, it is the first time one deals with the existence and stability of periodic solutions of (1.3).

In order to obtain our main results, throughout the paper we always assume that the following conditions are fulfilled:

(H1) $K_i(t),\ \alpha_i(t),\ \beta_i(t),\ \gamma_i(t)$ $(i = 1, 2)$, are all continuous ω periodic, i.e., $K_i(t+\omega) = K_i(t),\ \alpha_i(t+\omega) = \alpha_i(t),\ \beta_i(t+\omega) = \beta_i(t)$ $(i = 1, 2)$, $\gamma_i(t+\omega) = \gamma_i(t)$ for any $t \in \mathbb{R}$.

(H2) $K_i(t),\ \alpha_i(t),\ \beta_i(t),\ \gamma_i(t)$ $(i = 1, 2)$ are all positive.

(H3) $\varrho_{ik} \geqslant 0$ for all $k \in \mathbb{N}$ and there exists a positive integer q such that $t_{k+q} = t_k + \omega,\ \varrho_{ik+q} = \varrho_{ik}$ $(i = 1, 2;\ k = 1, 2, 3, \ldots)$.

(H4) At least one of the following four conditions $\overline{\alpha}_1\overline{\alpha}_2 \neq \overline{\beta}_1\overline{\beta}_2, \overline{\alpha}_1\overline{\gamma}_2 \neq \overline{\beta}_2\overline{\gamma}_1, \overline{\alpha}_2\overline{\gamma}_2 \neq \overline{\beta}_1\overline{\gamma}_2, \overline{\gamma}_2{}^2 \neq \overline{\gamma}_1\overline{\gamma}_2$ holds.

The organization of the paper is as follows. In Section 2, we introduce some notation and definitions, and state some preliminary results needed in later sections. We then establish, in Section 3, some simple criteria for the existence of positive periodic solutions of system (1.3) by using the continuation theorem of the coincidence degree theory proposed by Gains and Mawhin [6]. In Section 4, the uniqueness and global attractivity of the positive periodic solution are presented. In Section 5, an illustrative example is given to demonstrate the correctness of the results obtained.

2. Preliminaries

We shall introduce some notation and definitions, and state some preliminary results. Consider the impulsive system

$$\text{(2.1)} \qquad \begin{cases} \dot{x}(t) = f(t,x), \ t \neq t_k, \ k = 1,2,\dots, \\ \Delta x(t)|_{t=t_k} = I_k(x(t_k^-)), \end{cases}$$

where $x \in \mathbb{R}^n$, $f\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and $f(t+\omega,x) = f(t,x)$; $I_k\colon \mathbb{R}^n \to \mathbb{R}^n$ are continuous, and there exists a positive integer q such that $t_{k+q} = t_k+\omega, I_{k+q}(x) = I_k(x)$ with $t_k \in \mathbb{R}, t_{k+1} > t_k, \lim\limits_{k\to\infty} t_k = \infty, \Delta x(t)|_{t=t_k} = x(t_k^+) - x(t_k^-)$. For $t_k \neq 0$ $(k = 1,2,\dots)$, $[0,\omega]\cap\{t_k\} = \{t_1,t_2,\dots,t_q\}$. As we know, $\{t_k\}$ are called the points of jump.

Let us recall some definitions for the Cauchy problem

$$\text{(2.2)} \qquad \begin{cases} \dot{x}(t) = f(t,x), \quad t \in [0,\omega], \ t \neq t_k, \\ \Delta x(t)|_{t=t_k} = I_k(x(t_k^-)), x(0) = x_0. \end{cases}$$

Definition 1.1. A map $x\colon [0,\omega] \to \mathbb{R}^n$ is said to be a solution of (2.2), if it satisfies the following conditions:

(i) $x(t)$ is a piecewise continuous map with first-class discontinuity points at $t_k \cap [0,\omega]$, and at each discontinuity point it is continuous from the left;
(ii) $x(t)$ satisfies (2.2).

Definition 1.2. A map $x\colon [0,\omega] \to \mathbb{R}^n$ is said to be an ω periodic solution of (2.1), if

(i) $x(t)$ satisfies (i) and (ii) of Definition 1 in the interval $[0,\omega]$;
(ii) $x(t)$ satisfies $x(t+\omega-0) = x(t-0), t \in \mathbb{R}$.

Obviously, if $x(t)$ is a solution of (2.2) defined on $[0,\omega]$ such that $x(0) = x(\omega)$, then by the periodicity of (2.2) in t, the function $x^*(t)$ defined by

$$x^*(t) = \begin{cases} x(t-j\omega), \quad t \in [j\omega,(j+1)\omega] \setminus \{t_k\}, \\ x^*(t) \text{ is left continuous at } t = t_k \end{cases}$$

is an ω periodic solution of (2.1).

For system (1.3), finding the periodic solutions is equivalent to finding solutions of the following boundary value problem:

$$
\begin{cases}
\dfrac{\mathrm{d}x}{\mathrm{d}t} = x(t)[K_1(t) - \alpha_1(t)x(t) - \beta_1(t)y(t) - \gamma_1(t)x(t)y(t-\tau_1(t))], \\
\qquad t \neq t_k, t \in [0,\omega],\ k = 1,2,\dots,q, \\
\dfrac{\mathrm{d}y}{\mathrm{d}t} = y(t)[K_2(t) - \alpha_2(t)y(t) - \beta_2(t)x(t) - \gamma_2(t)x(t-\tau_2(t))y(t)], \\
\qquad t \neq t_k, t \in [0,\omega],\ k = 1,2,\dots,q, \\
\Delta x(t_k) = x(t_k^+) - x(t_k^-) = \varrho_{1k}x(t_k),\ t = t_k, x(0) = x(\omega),\ k = 1,2,\dots,q, \\
\Delta y(t_k) = y(t_k^+) - y(t_k^-) = \varrho_{2k}y(t_k),\ t = t_k, y(0) = y(\omega),\ k = 1,2,\dots,q.
\end{cases}
\tag{2.3}
$$

3. Existence of positive periodic solutions

In this section, based on Mawhin's continuation theorem, we shall study the existence of at least one periodic solution of (1.3). To do so, we shall make some preparations.

Let X, Y be normed vector spaces, $L\colon \operatorname{Dom} L \subset X \to Y$ a linear mapping, $N\colon X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projections $P\colon X \to X$ and $Q\colon Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$, it follows that $L|\operatorname{Dom} L \cap \operatorname{Ker} P\colon (I-P)X \to \operatorname{Im} L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N\colon \overline{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exist isomorphisms $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$.

Now we introduce Mawhin's continuation theorem [6] as follows.

Lemma 3.1 ([6] Continuation Theorem). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\overline{\Omega}$. Suppose*

(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*
(b) *$QNx \neq 0$ for each $x \in \operatorname{Ker} L \cap \partial\Omega$, and $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \overline{\Omega}$.

For convenience and simplicity of the following discussion, we use the notation below throughout the paper:

$$
\overline{f} = \frac{1}{\omega}\int_0^\omega f(t)\,\mathrm{d}t,\ f^L = \min_{t\in[0,\omega]} f(t),\ f^M = \max_{t\in[0,\omega]} f(t),\ \overline{|f|} = \frac{1}{\omega}\int_0^\omega |f(t)|\,\mathrm{d}t,
$$

where $f(t)$ is a ω continuous periodic function. For any non-negative integer p, let $C^{(p)}[0,\omega;t_1,t_2,\dots,t_q] = \{x\colon\ [0,\omega]\to\mathbb{R}^m | x^{(p)}(t)$ exist for $t\neq t_1,\dots,t_q$; $x^{(p)}(t^+)$ and $x^{(p)}(t^-)$ exist at $t_1,t_2,\dots,t_q$; and $x^{(j)}(t_k) = x^{(j)}(t_k^-), k=1,\dots,m, j=0,1,2,\dots,p\}$ with the norm $\|x\|_p = \max\{\sup\limits_{t\in[0,\omega]}\|x^{(j)}(t)\|\}_{j=1}^p$, where $\|.\|$ is any norm in $\mathbb{R}^m$. It is easy to see that $C^{(p)}[0,\omega;t_1,t_2,\dots,t_q]$ is a Banach space.

Now we are in a position to state and prove the existence of periodic solutions of (2.3).

Theorem 3.1. *Let B_3 and B_9 be defined by* (3.12) *and* (3.20), *respectively. In addition to* (H1)–(H4), *assume further that*

$$\text{(H5)}\qquad \overline{K}_2\omega + \sum_{k=1}^{q}\ln(1+\varrho_{2k}) > \max\{\overline{\gamma}_2\omega\exp(B_3), \overline{\gamma}_2\omega\exp(B_9)\},$$

then the system (1.2) *has at least one ω periodic solution.*

Proof. According to the discussion above in Section 2, we only need to prove that the boundary value problem (2.3) has a solution. Since the solutions of (2.3) remain positive for all $t\geqslant 0$, we let $u_1(t)=\ln[x(t)]$, $u_2(t)=\ln[y(t)]$. Then system (2.3) can be transformed to

$$(3.1)\qquad \begin{cases} \dot u_1(t) = K_1(t) - \alpha_1(t)\exp(u_1(t)) - \beta_1(t)\exp(u_2(t)) \\ \qquad\qquad - \gamma_1(t)\exp(u_1(t))\exp(u_2(t-\tau_1(t))), \\ \qquad\qquad t\neq t_k,\ t\in[0,\omega],\ k=1,2,\dots,q, \\ \dot u_2(t) = K_2(t) - \alpha_2(t)\exp(u_2(t)) - \beta_2(t)\exp(u_1(t)) \\ \qquad\qquad - \gamma_2(t)\exp(u_1(t-\tau_2(t)))\exp(u_2(t)), \\ \qquad\qquad t\neq t_k,\ t\in[0,\omega],\ k=1,2,\dots,q, \\ \Delta u_i(t_k) = \ln(1+\varrho_{ik}), t=t_k, k=1,2,\dots,q; \\ \qquad\qquad u_1(0)=u_1(\omega), u_2(0)=u_2(\omega). \end{cases}$$

It is easy to see that if system (3.1) has an ω periodic solution $(u_1^*(t),u_2^*(t))^{\mathrm T}$, then $(x^*(t),y^*(t))^{\mathrm T} = (\mathrm{e}^{u_1^*(t)},\mathrm{e}^{u_2^*(t)})^{\mathrm T}$ is a positive solution of system (1.3). Therefore, to complete the proof, it suffices to show that system (3.1) has at least one ω periodic solution.

In order to use the continuation theorem of coincidence degree theory to establish the existence of a solution of (3.1), we take

$$X = \{u\in C[0,\omega;t_1,t_2,\dots,t_q]\},\ Y = X\times\mathbb{R}^{2\times(q+1)}.$$

Then X is a Banach space with the norm $\|.\|_0$, and Y is also a Banach space with the norm $\|z\| = \|x\|_0 + \|y\|,\ x \in X,\ y \in \mathbb{R}^{2\times(q+1)}$.

Let

$$
\begin{aligned}
&\operatorname{Dom} L = \{u = (u_1, u_2)^{\mathrm{T}} \in C[0,\omega];\ t_1, t_2, \dots, t_q\},\\
&L\colon \operatorname{Dom} L \subset X \to Y, x \to (x', \Delta u(t_1), \Delta u(t_2), \dots, \Delta u(t_q), 0),\\
&N\colon X \to Y,\\
&Nu = \left(\begin{pmatrix} K_1(t) - \alpha_1(t)\exp(u_1(t)) - \beta_1(t)\exp(u_2(t)) - \gamma_1(t)\exp(u_1(t))\exp(u_2(t-\tau_1(t))) \\ K_2(t) - \alpha_2(t)\exp(u_2(t)) - \beta_2(t)\exp(u_1(t)) - \gamma_2(t)\exp(u_1(t-\tau_2(t)))\exp(u_2(t)) \end{pmatrix},\right.\\
&\qquad \left.\begin{pmatrix} \ln(1+\varrho_{11}) \\ \ln(1+\varrho_{21}) \end{pmatrix}, \begin{pmatrix} \ln(1+\varrho_{21}) \\ \ln(1+\varrho_{22}) \end{pmatrix}, \ \dots,\ \begin{pmatrix} \ln(1+\varrho_{31}) \\ \ln(1+\varrho_{32}) \end{pmatrix},\ 0\right).
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
\operatorname{Ker} L &= \{u\colon\ u(t) = h \in \mathbb{R}^2, t \in [0,\omega]\},\\
\operatorname{Im} L &= \left\{ z = (f, a_1, a_2, \dots, a_q, d) \in Y\colon\ \int_0^\omega f(s)\,\mathrm{d}s + \sum_{k=1}^q a_k + d = 0 \right\}\\
&= X \times \mathbb{R}^{2\times q} \times \{0\},\\
\dim \operatorname{Ker} L &= 2 = \operatorname{codim} \operatorname{Im} L.
\end{aligned}
$$

So, $\operatorname{Im} L$ is closed in Y, L is a Fredholm mapping of index zero. Define two projections

$$
\begin{aligned}
Px &= \frac{1}{\omega}\int_0^\omega x(t)\,\mathrm{d}t,\\
Qz &= Q(f, a_1, a_2, \dots, a_q, d) = \left(\frac{1}{\omega}\left[\int_0^\omega f(s)\,\mathrm{d}s + \sum_{k=1}^q a_k + d,\right], 0, 0, \dots, 0\right).
\end{aligned}
$$

It is easy to show that P and Q are continuous and satisfy $\operatorname{Im} P = \operatorname{Ker} L, \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$.

Further, through an easy computation, we can find that the inverse K_P of L, $K_P\colon\ \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{Dom} L$ has the following form:

$$
K_P(z) = \int_0^t f(s)\,\mathrm{d}s + \sum_{t_k < t} a_k - \frac{1}{\omega}\int_0^\omega\int_0^t f(s)\,\mathrm{d}s\,\mathrm{d}t - \sum_{k=1}^q a_k.
$$

Moreover, it is easy to check that

$$QNu = \left(\begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^t F_1(s)\,\mathrm{d}s + \dfrac{1}{\omega}\sum_{k=1}^{q}\ln(1+\varrho_{1k}) \\ \dfrac{1}{\omega}\displaystyle\int_0^t F_2(s)\,\mathrm{d}s + \dfrac{1}{\omega}\sum_{k=1}^{q}\ln(1+\varrho_{2k}) \end{pmatrix}, 0, 0, \ldots, 0 \right),$$

$$\begin{aligned} K_P(I-Q)Nu = {} & \begin{pmatrix} \displaystyle\int_0^t F_1(s)\,\mathrm{d}s + \sum_{t>t_k}\ln(1+\varrho_{1k}) \\ \displaystyle\int_0^t F_2(s)\,\mathrm{d}s + \sum_{t>t_k}\ln(1+\varrho_{2k}) \end{pmatrix} \\ & - \begin{pmatrix} \dfrac{1}{\omega}\displaystyle\int_0^\omega\int_0^t F_1(s)\,\mathrm{d}s\,\mathrm{d}t + \sum_{k=1}^{q}\ln(1+\varrho_{1k}) \\ \dfrac{1}{\omega}\displaystyle\int_0^\omega\int_0^t F_2(s)\,\mathrm{d}s\,\mathrm{d}t + \sum_{k=1}^{q}\ln(1+\varrho_{2k}) \end{pmatrix} \\ & - \begin{pmatrix} \Big(\dfrac{t}{\omega} - \dfrac{1}{2}\Big)\displaystyle\int_0^\omega F_1(s)\,\mathrm{d}s + \sum_{k=1}^{q}\ln(1+\varrho_{1k}) \\ \Big(\dfrac{t}{\omega} - \dfrac{1}{2}\Big)\displaystyle\int_0^\omega F_2(s)\,\mathrm{d}s + \sum_{k=1}^{q}\ln(1+\varrho_{2k}) \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} F_1(s) &= K_1(s) - \alpha_1(s)\exp(u_1(s)) - \beta_1(s)\exp(u_2(s)) \\ &\quad - \gamma_1(s)\exp(u_1(s))\exp(u_2(s-\tau_1(s))), \\ F_2(s) &= K_2(s) - \alpha_2(s)\exp(u_2(s)) - \beta_2(s)\exp(u_1(s)) \\ &\quad - \gamma_2(s)\exp(u_1(s-\tau_2(s)))\exp(u_2(s)). \end{aligned}$$

Obviously, QN and $K_P(I-Q)N$ are continuous. Since X is a finite-dimensional Banach space, using the Ascoli-Arzela theorem, it is not difficult to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

Now we are at the point to search for an appropriate open, bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation

$Lu = \lambda Nu$, $\lambda \in (0,1)$, we have

$$(3.2)\qquad \begin{cases} \dot u_1(t) = \lambda[K_1(t) - \alpha_1(t)\exp(u_1(t)) - \beta_1(t)\exp(u_2(t)) \\ \qquad\qquad - \gamma_1(t)\exp(u_1(t))\exp(u_2(t-\tau_1(t)))], \\ \qquad\quad t \neq t_k,\ t \in [0,\omega],\ k = 1,2,\dots,q, \\ \dot u_2(t) = \lambda[K_2(t) - \alpha_2(t)\exp(u_2(t)) - \beta_2(t)\exp(u_1(t)) \\ \qquad\qquad - \gamma_2(t)\exp(u_1(t-\tau_2(t)))\exp(u_2(t))], \\ \qquad\quad t \neq t_k,\ t \in [0,\omega],\ k = 1,2,\dots,q, \\ \Delta u_i(t_k) = \lambda \ln(1+\varrho_{ik}), i = 1,2; \\ \qquad\quad k = 1,2,\dots,q;\ u_1(0) = u_1(\omega), u_2(0) = u_2(\omega). \end{cases}$$

Suppose that $u(t) = (u_1(t), u_2(t))^{\mathrm T} \in X$ is an arbitrary solution of system (3.2) for a certain $\lambda \in (0,1)$. Integrating both sides of (3.2) over the interval $[0,\omega]$ with respect to t, we obtain

$$(3.3)\qquad \begin{cases} \int_0^\omega f_1(t)\,\mathrm dt = \sum\limits_{k=1}^{q} \ln(1+\varrho_{1k}) + \int_0^\omega K_1(t)\,\mathrm dt, \\ \int_0^\omega f_2(t)\,\mathrm dt = \sum\limits_{k=1}^{q} \ln(1+\varrho_{2k}) + \int_0^\omega K_2(t)\,\mathrm dt, \end{cases}$$

where

$$f_1(t) = \alpha_1(t)\exp(u_1(t)) + \beta_1(t)\exp(u_2(t)) + \gamma_1(t)\exp(u_1(t))\exp(u_2(t-\tau_1(t))),$$
$$f_2(t) = \alpha_2(t)\exp(u_2(t)) + \beta_2(t)\exp(u_1(t)) + \gamma_2(t)\exp(u_1(t-\tau_2(t)))\exp(u_2(t)).$$

From (3.2) and (3.3), we can obtain

$$(3.4)\qquad \int_0^\omega |\dot u_1(t)|\,\mathrm dt < 2\overline{K}_1\omega + \sum_{k=1}^{q} \ln(1+\varrho_{1k}),$$

$$(3.5)\qquad \int_0^\omega |\dot u_2(t)|\,\mathrm dt < 2\overline{K}_2\omega + \sum_{k=1}^{q} \ln(1+\varrho_{2k}).$$

Let

$$(3.6)\qquad u_i(\xi_i) = \min_{t\in[0,\omega]} u_i(t),\ u_i(\eta_i) = \max_{t\in[0,\omega]} u_i(t),\ i = 1,2.$$

Then, by (3.3), we get

$$\int_0^\omega f_1(t)\,\mathrm dt < 2\overline{K}_1\omega + \sum_{k=1}^{q} \ln(1+\varrho_{1k}),$$

which leads to

$$\int_0^\omega \alpha_1(t)\exp(u_1(\xi_1))\,\mathrm{d}t < 2\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k}),$$
$$\int_0^\omega \beta_1(t)\exp(u_2(\xi_2))\,\mathrm{d}t < 2\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k}).$$

Thus

$$u_1(\xi_1) < \ln\left[\frac{2\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k})}{\overline{\alpha}_1\omega}\right], \tag{3.7}$$
$$u_2(\xi_2) < \ln\left[\frac{2\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k})}{\overline{\beta}_1\omega}\right]. \tag{3.8}$$

In the sequel, we consider two cases.

(a) If $u_1(\eta_1) \geqslant u_2(\eta_2)$, then it follows from (3.3) that

$$\overline{(\alpha_1+\beta_1)}\omega\exp(u_1(\eta_1)) + \overline{\gamma}_1\omega\exp(2u_1(\eta_1)) \geqslant \overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k}),$$

which leads to

$$u_1(\eta_1) > \ln\left[\frac{-\overline{(\alpha_1+\beta_1)}\omega + \sqrt{[\overline{(\alpha_1+\beta_1)}\omega]^2 + 4\overline{\gamma}_1\omega(\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k}))}}{2\overline{\gamma}_1\omega}\right]. \tag{3.9}$$

It follows from (3.7) and (3.9) that

$$\begin{aligned} u_1(t) &\leqslant u_1(\xi_1) + \int_0^\omega |\dot{u}_1(t)|\,\mathrm{d}t \\ &\leqslant \ln\left[\frac{2\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k})}{\overline{\alpha}_1\omega}\right] + 2\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k}) := B_1, \end{aligned} \tag{3.10}$$

$$\begin{aligned} u_1(t) &\geqslant u_1(\eta_1) - \int_0^\omega |\dot{u}_1(t)|\,\mathrm{d}t \\ &\geqslant \ln\left[\frac{-\overline{(\alpha_1+\beta_1)}\omega + \sqrt{[\overline{(\alpha_1+\beta_1)}\omega]^2 + 4\overline{\gamma}_1\omega(\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k}))}}{2\overline{\gamma}_1\omega}\right] \\ &\quad - 2\overline{K}_1\omega - \sum_{k=1}^q \ln(1+\varrho_{1k}) := B_2. \end{aligned} \tag{3.11}$$

From (3.10) and (3.11) we have

$$\sup_{t\in[0,\omega]} |u_1(t)| < \max\{|B_1|, |B_2|\} := B_3. \tag{3.12}$$

From (3.3) we obtain

$$\overline{\alpha}_2\omega \exp(u_2(\eta_2)) + \overline{\beta}_2\omega \exp(B_3) + \overline{\gamma}_2\omega \exp(B_3)\exp(u_2(\eta_2)) \geqslant \overline{K}_2\omega + \sum_{k=1}^{q} \ln(1+\varrho_{2k}).$$

Then

$$u_2(\eta_2) \geqslant \ln\left[\frac{\overline{K}_2\omega + \sum_{k=1}^{q}\ln(1+\varrho_{2k}) - \overline{\gamma}_2\omega\exp(B_3)}{\overline{\alpha}_2\omega + \overline{\gamma}_2\omega\exp(B_3)}\right]. \tag{3.13}$$

Thus

$$\begin{aligned} u_2(t) &\leqslant u_2(\xi_2) + \int_0^{\omega} |\dot{u}_2(t)|\,\mathrm{d}t \\ &\leqslant \ln\left[\frac{2\overline{K}_1\omega + \sum_{k=1}^{q}\ln(1+\varrho_{1k})}{\overline{\beta}_1\omega}\right] + 2\overline{K}_2\omega \\ &\quad + \sum_{k=1}^{q}\ln(1+\varrho_{1k}) := B_4, \end{aligned} \tag{3.14}$$

$$\begin{aligned} u_2(t) &\geqslant u_2(\eta_2) - \int_0^{\omega} |\dot{u}_2(t)|\,\mathrm{d}t \\ &\geqslant \ln\left[\frac{\overline{K}_2\omega + \sum_{k=1}^{q}\ln(1+\varrho_{2k}) - \overline{\gamma}_2\omega\exp(B_3)}{\overline{\alpha}_2\omega + \overline{\gamma}_2\omega\exp(B_3)}\right] \\ &\quad - 2\overline{K}_2\omega - \sum_{k=1}^{q}\ln(1+\varrho_{2k}) := B_5. \end{aligned} \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\sup_{t\in[0,\omega]} |u_2(t)| < \max\{|B_4|, |B_5|\} := B_6. \tag{3.16}$$

(b) If $u_1(\eta_1) < u_2(\eta_2)$, then it follows from (3.3) that

$$\overline{(\alpha_1+\beta_1)}\omega\exp(u_2(\eta_2)) + \overline{\gamma}_1\omega\exp(2u_2(\eta_2)) \geqslant \overline{K}_1\omega + \sum_{k=1}^{q}\ln(1+\varrho_{1k}),$$

which leads to

$$u_2(\eta_2) > \ln\left[\frac{-\overline{(\alpha_1+\beta_1)}\omega + \sqrt{[\overline{(\alpha_1+\beta_1)}\omega]^2 + 4\overline{\gamma}_1\omega(\overline{K}_1\omega + \sum_{k=1}^{q}\ln(1+\varrho_{1k}))}}{2\overline{\gamma}_1\omega}\right]. \tag{3.17}$$

It follows from (3.7) and (3.9) that

(3.18)
$$\begin{aligned} u_2(t) &\leqslant u_2(\xi_1) + \int_0^\omega |\dot u_2(t)|\,\mathrm{d}t \\ &\leqslant \ln\left[\frac{2\overline{K}_2\omega + \sum_{k=1}^q \ln(1+\varrho_{2k})}{\overline{\alpha}_2\omega}\right] + 2\overline{K}_2\omega + \sum_{k=1}^q \ln(1+\varrho_{2k}) := B_7, \end{aligned}$$

(3.19)
$$\begin{aligned} u_2(t) &\geqslant u_2(\eta_2) - \int_0^\omega |\dot u_2(t)|\,\mathrm{d}t \\ &\geqslant \ln\left[\frac{-\overline{(\alpha_1+\beta_1)}\omega + \sqrt{[\overline{(\alpha_1+\beta_1)}\omega]^2 + 4\overline{\gamma}_1\omega(\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k}))}}{2\overline{\gamma}_1\omega}\right] \\ &\quad - 2\overline{K}_2\omega - \sum_{k=1}^q \ln(1+\varrho_{2k}) := B_8. \end{aligned}$$

From (3.10) and (3.11) we derive

$$\sup_{t\in[0,\omega]} |u_2(t)| < \max\{|B_7|,|B_8|\} := B_9. \tag{3.20}$$

From (3.3) we have

$$\overline{\alpha}_2\omega\exp(B_9) + \overline{\beta}_2\omega\exp(u_1(\eta_1)) + \overline{\gamma}_2\omega\exp(B_9)\exp(u_1(\eta_1)) \geqslant \overline{K}_2\omega + \sum_{k=1}^q \ln(1+\varrho_{2k}).$$

Then

$$u_1(\eta_1) \geqslant \ln\left[\frac{\overline{K}_2\omega + \sum_{k=1}^q \ln(1+\varrho_{2k}) - \overline{\gamma}_2\omega\exp(B_9)}{\overline{\alpha}_2\omega + \overline{\gamma}_2\omega\exp(B_9)}\right]. \tag{3.21}$$

Thus

$$\begin{aligned} u_1(t) &\leqslant u_1(\xi_1) + \int_0^\omega |\dot u_1(t)|\,\mathrm{d}t \\ &\leqslant \ln\left[\frac{2\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k})}{\overline{\alpha}_1\omega}\right] + 2\overline{K}_1\omega + \sum_{k=1}^q \ln(1+\varrho_{1k}) := B_{10}, \end{aligned} \tag{3.22}$$

$$\begin{aligned} u_1(t) &\geqslant u_1(\eta_1) - \int_0^\omega |\dot u_1(t)|\,\mathrm{d}t \\ &\geqslant \ln\left[\frac{\overline{K}_2\omega + \sum_{k=1}^q \ln(1+\varrho_{2k}) - \overline{\gamma}_2\omega\exp(B_9)}{\overline{\alpha}_2\omega + \overline{\gamma}_2\omega\exp(B_9)}\right] \\ &\quad - 2\overline{K}_1\omega - \sum_{k=1}^q \ln(1+\varrho_{1k}) := B_{11}. \end{aligned} \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\sup_{t\in[0,\omega]} |u_1(t)| < \max\{|B_{10}|, |B_{11}|\} := B_{12}. \tag{3.24}$$

Obviously, B_i $(i = 3, 6, 9, 12)$ are independent of $\lambda \in (0, 1)$. Similarly to the proof of Theorem 2.1 of [16], we can easily find a sufficiently large $M > 0$ so that if we denote

$$\Omega = \{u(t) = (u_1(t), u_2(t))^{\mathrm{T}} \in x\colon\ \|u\| < M,\ u(t_k^+) \in \Omega,\ k = 1, 2, \ldots, q\},$$

it is clear that Ω satisfies the requirement (a) in Lemma 3.1.

When $(u_1(t), u_2(t))^{\mathrm{T}} \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega\cap\mathbb{R}^2, u = \{(u_1, u_2)^{\mathrm{T}}\}$ is a constant vector in $\mathbb{R}^2$ with $\|u\| = \|(u_1(t), u_2(t))^{\mathrm{T}}\| = M$, then we have

$$QNu = \left(\begin{pmatrix} \overline{K}_1 - \overline{\alpha}_1 \exp(u_1) - \overline{\beta}_1 \exp(u_2) - \overline{\gamma}_1 \exp(u_1)\exp(u_2) \\ \quad + \dfrac{1}{\omega}\displaystyle\sum_{k=1}^{q} \ln(1+\varrho_{1k}) \\ \overline{K}_2 - \overline{\alpha}_2 \exp(u_2) - \overline{\beta}_2 \exp(u_1) - \overline{\gamma}_2 \exp(u_1)\exp(u_2) \\ \quad + \dfrac{1}{\omega}\displaystyle\sum_{k=1}^{q} \ln(1+\varrho_{2k}) \end{pmatrix}, 0, \ldots, 0 \right) \neq 0.$$

Letting $J\colon \operatorname{Im} Q \to \operatorname{Ker} L, (r, 0, \ldots, 0, 0) \to r$, by direct calculation we get

$$\begin{aligned} &\deg\{JQN(u_1, u_2)^{\mathrm{T}};\ \Omega \cap \ker L; 0\} \\ &\quad = \operatorname{sign}\det \begin{pmatrix} -(\overline{\alpha}_1 + \overline{\gamma}_2 e^{u_2})e^{u_1} & -(\overline{\beta}_1 + \overline{\gamma}_1 e^{u_1})e^{u_2} \\ -(\overline{\beta}_2 + \overline{\gamma}_2 e^{u_2})e^{u_1} & -(\overline{\alpha}_2 + \overline{\gamma}_2 e^{u_1})e^{u_2} \end{pmatrix} \\ &\quad = \operatorname{sign}\{(\overline{\alpha}_1\overline{\alpha}_2 - \overline{\beta}_1\overline{\beta}_2) + (\overline{\alpha}_1\overline{\gamma}_2 - \overline{\beta}_2\overline{\gamma}_1)e^{u_1} + (\overline{\alpha}_2\overline{\gamma}_2 - \overline{\beta}_1\overline{\gamma}_2)e^{u_2} \\ &\qquad + (\overline{\gamma}_2{}^2 - \overline{\gamma}_1\overline{\gamma}_2)e^{u_1+u_2}\} \neq 0. \end{aligned}$$

This proves that condition (b) in Lemma 3.1 is satisfied. By now, we have proved that Ω verifies all requirements of Lemma 3.1, hence it follows that $Lu = Nu$ has at least one solution $(u_1(t), u_2(t))^{\mathrm{T}}$ in $\operatorname{Dom} L \cap \overline{\Omega}$, that is to say, (3.1) has at least one ω periodic solution in $\operatorname{Dom} L \cap \overline{\Omega}$. Then we know that $(x(t), y(t))^{\mathrm{T}} = (e^{u_1(t)}, e^{u_2(t)})^{\mathrm{T}}$ is an ω periodic solution of system (2.3) with strictly positive components. This completes the proof. □

Under the hypotheses (H1), (H2), (H3), we consider the following ordinary differential equation without impulses:

$$
(4.1)\quad
\begin{cases}
\dot z_1(t) = z_1(t)\Big[K_1(t) - \alpha_1(t) \prod\limits_{0<t_k<t} (1+\varrho_{1k}) z_1(t) - \beta_1(t) \prod\limits_{0<t_k<t} (1+\varrho_{2k}) z_2(t) \\
\qquad - \gamma_1(t) \prod\limits_{0<t_k<t} (1+\varrho_{1k}) z_1(t) \prod\limits_{0<t_k<t} (1+\varrho_{2k}) z_2(t-\tau_1(t))\Big], \\
\dot z_2(t) = z_2(t)\Big[K_2(t) - \alpha_2(t) \prod\limits_{0<t_k<t} (1+\varrho_{2k}) z_2(t) - \beta_2(t) \prod\limits_{0<t_k<t} (1+\varrho_{1k}) z_1(t) \\
\qquad - \gamma_2(t) \prod\limits_{0<t_k<t} (1+\varrho_{1k}) z_1(t-\tau_2(t)) \prod\limits_{0<t_k<t} (1+\varrho_{2k}) z_2(t)\Big]
\end{cases}
$$

with the initial conditions $z_i(0) > 0,\ i = 1, 2$.

Let $\tau = \max\limits_{1\leqslant i\leqslant 2}\{\max\limits_{t\in[0,\omega]} \tau_i(t)\}$. The following lemmas will be helpful in the proofs of our results. The proof of Lemma 4.1 is similar to that of Theorem 1 in [17], and will be omitted.

Lemma 4.1. *Assume that* (H1), (H2), (H3) *hold. Then*

(i) *if* $z(t) = (z_1(t), z_2(t))^{\mathrm{T}}$ *is a solution of* (4.1) *on* $[0, +\infty)$, *then* $x_i(t) = \prod\limits_{0<t_k<t} (1+\varrho_{ik}) z_i(t)$ $(i = 1, 2)$ *is a solution of* (2.3) *on* $[-\tau, +\infty)$*;*

(ii) *if* $x(t) = (x_1(t), x_2(t))^{\mathrm{T}}$ *is a solution of* (2.3) *on* $[0, +\infty)$, *then* $z_i(t) = \prod\limits_{0<t_k<t} (1+\varrho_{ik})^{-1} x_i(t)$ $(i = 1, 2)$ *is a solution of* (4.1) *on* $[-\tau, +\infty)$.

Lemma 4.2. *Let* $z(t) = (z_1(t), z_2(t))^{\mathrm{T}}$ *denote any positive solution of system* (4.1) *with initial conditions* $z_i(0) > 0$ $(i = 1, 2)$*. Then there exists a* $T_3 > 0$ *such that*

$$0 < z_i(t) \leqslant M_i \quad (i = 1, 2) \text{ for } t \geqslant T_3,$$

where

$$M_1 > M_1^* = \frac{K_1^M}{\alpha_1^L \prod\limits_{0<t_k<t} (1+\varrho_{1k})},$$

$$M_2 > M_2^* = \frac{K_2^M}{\alpha_2^L \prod\limits_{0<t_k<t} (1+\varrho_{2k})},$$

P r o o f. From the first equation of (4.1), we can obtain

$$\dot{z}_1(t) \leqslant z_1(t)\Big[K_1(t) - \alpha_1(t)\prod_{0<t_k<t}(1+\varrho_{1k})z_1(t)\Big], \tag{4.2}$$
$$\leqslant z_1(t)\Big[K_1^M - \alpha_1^L\prod_{0<t_k<t}(1+\varrho_{1k})z_1(t)\Big].$$

By (4.2), we can derive

(A1) If $z_1(0) \leqslant M_1$, then $z_1(t) \leqslant M_1,\ t \geqslant 0$.

(A2) If $z_1(0) > M_1$, let $-\theta_1 = M_1\big[K_1^M - \alpha_1^L\prod_{0<t_k<t}(1+\varrho_{1k})M_1\big]$ $(\theta_1 > 0)$. Then there exists $\varepsilon_1 > 0$ such that $t \in [0,\varepsilon_1)$, then $z_1(t) > M_1$, and also we have

$$\dot{z}_1(t) < -\theta_1 < 0.$$

From what has been discussed above, we can easily conclude that if $z_1(0) > M_1$, then $z_1(t)$ is strictly monotone decreasing with speed at least θ_1. Therefore there exists a $T_1 > 0$ such that if $t > T_1$, then $z_1(t) \leqslant M_1$.

From the second equation of (4.1), we can obtain

$$\dot{z}_2(t) \leqslant z_2(t)\Big[K_2(t) - \alpha_2(t)\prod_{0<t_k<t}(1+\varrho_{1k})z_2(t)\Big] \tag{4.3}$$
$$\leqslant z_2(t)\Big[K_2^M - \alpha_2^L\prod_{0<t_k<t}(1+\varrho_{2k})z_2(t)\Big].$$

By (4.3), we can derive

(B1) If $z_2(0) \leqslant M_2$, then $z_2(t) \leqslant M_2,\ t \geqslant 0$.

(B2) If $z_2(0) > M_2$, let $-\theta_2 = M_2\big[K_2^M - \alpha_2^L\prod_{0<t_k<t}(1+\varrho_{2k})M_2\big]$ $(\theta_2 > 0)$. Then there exists $\varepsilon_2 > 0$ such that $t \in [0,\varepsilon_2)$, then $z_2(t) > M_2$, and also we have

$$\dot{z}_2(t) < -\theta_2 < 0.$$

From what has been discussed above, we can easily conclude that if $z_2(0) > M_2$, then $z_2(t)$ is strictly monotone decreasing with speed at least θ_2. Therefore there exists a $T_2 > 0$ such that if $t > T_2$, then $z_2(t) \leqslant M_2$. The proof is complete. □

Lemma 4.3. *Let* (H1), (H2), (H3) *hold. Assume that the following condition holds.*

(H6) $$K_1^L > \beta_1^M \prod_{0<t_k<t} (1+\varrho_{2k}) M_2,\ K_2^L - \beta_2^M \prod_{0<t_k<t} (1+\varrho_{2k}) M_1.$$

Then there exist positive constants $T > 0$ *and* m_i $(i = 1,2)$ *such that for all* $t > T$,

$$m_i < z_i(t) \quad (i = 1,2) \quad \text{for } t \geqslant T,$$

where

$$m_1 < m_1^* = \frac{K_1^L - \beta_1^M \prod\limits_{0<t_k<t} (1+\varrho_{2k}) M_2}{\alpha_1^M \prod\limits_{0<t_k<t} (1+\varrho_{1k}) + \gamma_1^M \prod\limits_{0<t_k<t} (1+\varrho_{1k}) \prod\limits_{0<t_k<t} (1+\varrho_{2k}) M_2},$$

$$m_2 < m_2^* = \frac{K_2^L - \beta_2^M \prod\limits_{0<t_k<t} (1+\varrho_{2k}) M_1}{\alpha_2^M \prod\limits_{0<t_k<t} (1+\varrho_{2k}) + \gamma_2^M \prod\limits_{0<t_k<t} (1+\varrho_{1k}) \prod\limits_{0<t_k<t} (1+\varrho_{2k}) M_1}.$$

P r o o f. By the first equation of (4.1), It is easy to obtain that for $t \geqslant T_3$,

$$\dot z_1(t) \geqslant z_1(t)\Big[K_1^L - \alpha_1^M \prod_{0<t_k<t} (1+\varrho_{1k}) z_1(t) - \beta_1^M \prod_{0<t_k<t} (1+\varrho_{2k}) M_2 - \gamma_1^M \prod_{0<t_k<t} (1+\varrho_{1k}) z_1(t) \prod_{0<t_k<t} (1+\varrho_{2k}) M_2\Big],$$

where T_3 is defined in Lemma 4.2.

(C1) If $z_1(T_3) \geqslant m_1$, then $z_1(t) \geqslant m_1,\ t \geqslant T_3$.

(C2) If $z_1(T_3) < m_1$, let us denote

$$\mu_1 = z_1(T_3)\Big[K_1^L - \alpha_1^M \prod_{0<t_k<t} (1+\varrho_{1k}) m_1 - \beta_1^M \prod_{0<t_k<t} (1+\varrho_{2k}) M_2 - \gamma_1^M \prod_{0<t_k<t} (1+\varrho_{1k}) m_1 \prod_{0<t_k<t} (1+\varrho_{2k}) M_2\Big].$$

Then there exists $\varepsilon_3 > 0$ such that if $t \in [T_3, T_3 + \varepsilon_3)$, then $z_1(t) > m_1$, and also we have

$$\dot z_2(t) > \mu_1 > 0.$$

Then we know that if $z_1(T_3) < m_1$, $z_1(t)$ will strictly monotonically increase with speed μ_1. Thus there exists $T_4 > T_3$ such that if $t \geqslant T_4$, then $z_1(t) \geqslant m_1$.

By the second equation of (4.1), It is easy to obtain that for $t \geqslant T_3$,

$$\dot{z}_2(t) \geqslant z_2(t)\Big[K_2^L - \alpha_2^M \prod_{0<t_k<t}(1+\varrho_{2k})z_2(t) - \beta_2^M \prod_{0<t_k<t}(1+\varrho_{2k})M_1 - \gamma_2^M \prod_{0<t_k<t}(1+\varrho_{1k})M_1 \prod_{0<t_k<t}(1+\varrho_{2k})z_2(t)\Big],$$

where T_3 is defined in Lemma 4.2.

(D1) If $z_2(T_3) \geqslant m_2$, then $z_2(t) \geqslant m_2,\ t \geqslant T_3$.

(D2) If $z_2(T_3) < m_2$, let us denote

$$\mu_2 = z_2(T_3)\Big[K_2^L - \alpha_2^M \prod_{0<t_k<t}(1+\varrho_{2k})m_2 - \beta_2^M \prod_{0<t_k<t}(1+\varrho_{2k})M_1 - \gamma_2^M \prod_{0<t_k<t}(1+\varrho_{1k})M_1 \prod_{0<t_k<t}(1+\varrho_{2k})m_2\Big].$$

Then there exists $\varepsilon_4 > 0$ such that if $t \in [T_3, T_3 + \varepsilon_4)$, then $z_2(t) > m_1$, and also we have

$$\dot{z}_2(t) > \mu_2 > 0.$$

Then we know that if $z_2(T_3) < m_2$, $z_2(t)$ will strictly monotonically increase with speed μ_2. Thus there exists $T_5 > T_3$ such that if $t \geqslant T_5$, then $z_2(t) \geqslant m_2$.

Set $T = \max\{T_4, T_5\}$, then we have

$$z_i(t) > m_i \quad (i = 1, 2) \quad \text{for } t \geqslant T.$$

□

In the sequel, we formulate the uniqueness and global stability of the ω periodic solution $x^*(t)$ in Theorem 4.1. It is immediate that if $x^*(t)$ is globally asymptotically stable, then $x^*(t)$ is in fact unique.

Theorem 4.1. *In addition to* (H1)–(H6), *assume further that*

(H7) $$\lim_{t\to\infty} \inf A_i(t) > 0,$$

where

$$A_1 = -\alpha_1(t)\prod_{0<t_k<t}(1+\varrho_{1k}) - 2m_2\gamma_1(t)\prod_{0<t_k<t}(1+\varrho_{1k})\prod_{0<t_k<t}(1+\varrho_{2k}) + \beta_2(t)\prod_{0<t_k<t}(1+\varrho_{1k}),$$

$$A_2 = -\alpha_2(t)\prod_{0<t_k<t}(1+\varrho_{2k}) - 2m_1\gamma_2(t)\prod_{0<t_k<t}(1+\varrho_{1k})\prod_{0<t_k<t}(1+\varrho_{2k}) + \beta_1(t)\prod_{0<t_k<t}(1+\varrho_{2k}).$$

Then system (2.3) has a unique positive ω periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))^{\rm T}$ which is globally asymptotically stable.

P r o o f. According to the conclusion of Theorem 3.1, we only need to show the global asymptotic stability of the positive periodic solution of (2.3). Let $x^*(t) = (x_1^*(t), x_2^*(t))^{\rm T}$ be a positive ω periodic solution of system (2.3), let $x(t) = (x_1(t), x_2(t))^{\rm T}$ be any positive solution of system (2.3). Then $z^*(t) = (z_1^*(t), z_2^*(t))^{\rm T}$, where $(z_1^*(t) = \prod\limits_{0<t_k<t}(1+\varrho_{1k})x_1^*(t), z_2^*(t) = \prod\limits_{0<t_k<t}(1+\varrho_{2k})x_2^*(t)$, is the positive ω periodic solution of (4.1), and $z(t)$ is the positive solution of (4.1). It follows from Lemma 4.2 and Lemma 4.3 that there exist positive constants $T > 0$, M_i and m_i (defined by Lemma 4.2 and Lemma 4.2, respectively) such that for all $t > T$,

$$m_i < z_i^*(t) \leqslant M_i, \quad m_i < z_i(t) \leqslant M_i, \quad i = 1, 2.$$

Define

$$V(t) = |\ln z_1^*(t) - \ln z_1(t)| + |\ln z_2^*(t) - \ln z_2(t)|. \tag{4.4}$$

Calculating the upper-right derivative of $V(t)$ along the solution of (4.1), it follows for $t \geqslant T$ that

$$\begin{aligned}
D^+V(t) &= \sum_{i=1}^{2}\Big(\frac{z_i^{*\prime}(t)}{z_i^*(t)} - \frac{z_i{}'(t)}{z_i(t)}\Big)\operatorname{sgn}(z_i^*(t) - z_i(t))\\
&= \operatorname{sgn}(z_1^*(t) - z_1(t))\Big[-\alpha_1(t)\prod_{0<t_k<t}(1+\varrho_{1k})(z_1^*(t) - z_1(t))\\
&\quad - \beta_1(t)\prod_{0<t_k<t}(1+\varrho_{2k})(z_2^*(t) - z_2(t)) - \gamma_1(t)\prod_{0<t_k<t}(1+\varrho_{1k})(z_1^*(t) - z_1(t))\\
&\quad \times \prod_{0<t_k<t}(1+\varrho_{2k})(z_2^*(t-\tau_1(t)) - z_2(t-\tau_1(t)))\Big]\\
&\quad + \operatorname{sgn}(z_2^*(t) - z_2(t))\Big[-\alpha_2(t)\prod_{0<t_k<t}(1+\varrho_{2k})(z_2^*(t) - z_2(t))\\
&\quad - \beta_2(t)\prod_{0<t_k<t}(1+\varrho_{1k})(z_1^*(t) - z_1(t)) - \gamma_2(t)\prod_{0<t_k<t}(1+\varrho_{1k})\\
&\quad \times (z_1^*(t-\tau_1(t)) - z_1(t-\tau_2(t)))\prod_{0<t_k<t}(1+\varrho_{2k})(z_2^*(t) - z_2(t))\Big]\\
&\leqslant \sum_{i=1}^{2} A_i|z_i^*(t) - z_i(t)| \quad (i = 1, 2),
\end{aligned}$$

where

$$A_1 = -\alpha_1(t)\prod_{0<t_k<t}(1+\varrho_{1k}) - 2m_2\gamma_1(t)\prod_{0<t_k<t}(1+\varrho_{1k})\prod_{0<t_k<t}(1+\varrho_{2k}) + \beta_2(t)\prod_{0<t_k<t}(1+\varrho_{1k}),$$
$$A_2 = -\alpha_2(t)\prod_{0<t_k<t}(1+\varrho_{2k}) - 2m_1\gamma_2(t)\prod_{0<t_k<t}(1+\varrho_{1k})\prod_{0<t_k<t}(1+\varrho_{2k}) + \beta_1(t)\prod_{0<t_k<t}(1+\varrho_{2k}).$$

By hypothesis (H7) there exist constants α_i $(i=1,2)$ and $T^* > T$ such that

$$A_i(t) \geqslant \alpha_i > 0 \quad (i=1,2) \quad \text{for } t \geqslant T^*. \tag{4.5}$$

Integrating both sides of (4.11) over the interval $[T^*, t]$ yields

$$V(t) + \sum_{i=1}^{2}\int_{T^*}^{t} A_i(t)|z_i^*(t) - z_i(t)|\,\mathrm{d}s \leqslant V(T^*). \tag{4.6}$$

It follows from (4.12) and (4.13) that

$$\sum_{i=1}^{2}\int_{T^*}^{t} A_i(t)|z_i^*(t) - z_i(t)|\,\mathrm{d}s \leqslant V(T^*) < \infty \quad \text{for } t \geqslant T^*. \tag{4.7}$$

Since $z_i^*(t)$ and $z_i(t)$ $(i=1,2)$ are bounded for $t \geqslant T^*$, so $|z_i^*(t) - z_i(t)|$ $(i=1,2)$ are uniformly continuous on $[T^*, \infty)$. By Barbalat's Lemma [1] we have

$$\lim_{t\to\infty}|z_i^*(t) - z_i(t)| = \lim_{t\to\infty}\Big[\prod_{0<t_k<t}(1+\varrho_{ik})^{-1}|x_i^*(t) - x_i(t)|\Big] = 0 \quad (i=1,2).$$

Thus

$$\lim_{t\to\infty}|x_i^*(t) - x_i(t)| = 0 \quad (i=1,2). \tag{4.8}$$

By Theorems 7.4 and 8.2 in [18] we know that the positive periodic solution $x^*(t) = (x_1^*(t), x_2^*(t))^{\mathrm{T}}$ of equation (2.3) is uniformly asymptotically stable. The proof of Theorem 4.1 is complete. □

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