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ON SOME PROPERTIES OF α-PLANES OF TYPE-2 FUZZY SETS

Zdenko Takáč

Some basic properties of α-planes of type-2 fuzzy sets are investigated and discussed in connection with the similar properties of α-cuts of type-1 fuzzy sets. It is known, that standard intersection and standard union of type-1 fuzzy sets (it means intersection and union under minimum t-norm and maximum t-conorm, respectively) are the only cutworthy operations for type-1 fuzzy sets. Recently, a similar property was declared to be true also for α-planes of type-2 fuzzy sets in a few papers. Thus, we study under which t-norms and which t-conorms are intersection and union of the type-2 fuzzy sets preserved in the α-planes. Note that understanding of the term α-plane is somewhat confusing in recent type-2 fuzzy sets literature. We discuss this problem and show how it relates to obtained results.

Keywords: type-2 fuzzy sets, α-plane, intersection of type-2 fuzzy sets, union of type-2 fuzzy sets, fuzzy sets

Classification: 03E72, 68T37

1. INTRODUCTION

Standard intersection and union operations of type-1 fuzzy sets (T1 FSs) are cutworthy operations, which means that they are preserved in α-cuts for all $\alpha \in [0, 1]$ in the classical sense (see [2]). This is a very useful property of α-cuts for computation. Furthermore, it is a 'nice' relation between the intersection and union of the fuzzy sets A, B and the intersection and union of the crisp sets A_α, B_α:

$$(A \cap B)_\alpha = A_\alpha \cap B_\alpha\,, \qquad (A \cup B)_\alpha = A_\alpha \cup B_\alpha\,. \tag{1}$$

Recall, that the symbol $\cap$ on the left side of the first equality depicts the intersection of T1 FSs, however, it depicts the intersection of crisp sets on the right side of the equality. The same holds for the second equality and the symbol $\cup$. To emphasize this fact, we will write:

$$(A \cap_{T1} B)_\alpha = A_\alpha \cap_C B_\alpha\,, \qquad (A \cup_{T1} B)_\alpha = A_\alpha \cup_C B_\alpha\,. \tag{2}$$

Recently has been published α-plane representation for type-2 fuzzy sets (T2 FSs) in [4] and [5], which is some kind of a generalization of the α-cut representation for T1 FSs. Note that understanding of the term α-plane is somewhat confusing in recent T2 FSs literature (see Remark 1 of this paper). We discuss this problem, and then we study relations between α-planes of the intersection (union) of T2 FSs and intersection (union) of α-planes of T2 FSs, which are generalizations of the formulas (2), and provide relevant proofs. Furthermore, we study these relations for the intersection and union under various t-norms and t-conorms.

The rest of this paper is organized as follows. Section 2 contains basic definitions and notations that are used in the remaining parts of the paper. We investigate properties of the α-planes in connection with the intersection and union of the T2 FSs under various t-norms and t-conorms in Section 3. The conclusions are discussed in Section 4.

2. PRELIMINARIES

2.1. Type-2 fuzzy sets and vertical slices

Let X be a crisp set. A mapping $\widetilde{A} : X \to [0,1]^{[0,1]}$ is called *type-2 fuzzy set* in a set X. For each $x \in X$, the value $\widetilde{A}(x)$, usually denoted by $f_x(u)$, is called a *membership grade* of x. Obviously, f_x is a T1 FS in $[0,1]$. Let $J_x \subset [0,1]$ denote the domain of $f_x(u)$, then T2 FS $\widetilde{A}$ can be defined as mapping $\widetilde{A} : X \to [0,1]^{J_x}$ or $\widetilde{A} : X \times [0,1] \to [0,1]$, where u indicates the *primary membership (grade)* of x and value $\widetilde{A}(x,u) = f_x(u)$ indicates the *secondary membership (grade)* of ordered pair (x,u). A mapping $\widetilde{A}(x,u)$ is called *type-2 membership function*. See [13] and [9].

The T2 FS can be depicted as a 3D graph. At each value of x, say $x = x'$, the 2D plane whose axes are u and $\widetilde{A}(x',u)$ is called a *vertical slice* of T2 FS $\widetilde{A}$ (see [7]). Then we can reinterpret a T2 FS set in a vertical-slice manner, as

$$\widetilde{A} = \{(x, \widetilde{A}(x)) \mid \forall\, x \in X\}\,. \tag{3}$$

2.2. α-planes

The two dimensional plane containing all primary memberships whose secondary grades are greater than or equal to the special value α, denoted by $\widetilde{A}_\alpha$, is called the *α-plane* of the T2 FS $\widetilde{A}$ (see [4, 5, 7]), i.e.,

$$\widetilde{A}_\alpha = \{(x,u) \mid \widetilde{A}(x,u) \ge \alpha, x \in X, u \in [0,1]\}\,. \tag{4}$$

Using the vertical slice representation of T2 FS, it is easy to see that

$$\widetilde{A}_\alpha = \{(\widetilde{A}(x))_\alpha \mid x \in X\}$$

where $(\widetilde{A}(x))_\alpha$ is the α-cut of the vertical slice $\widetilde{A}(x)$.

Note that an α-plane of a T2 FS is similar to an α-cut of a T1 FS, but in contrast to an α-cut, an α-plane is not a subset of the universe of discourse X. It is a subset of $X \times [0,1]$. Simply, the α-plane $\widetilde{A}_\alpha$ is a set of ordered pairs (x,u) with $\widetilde{A}(x,u) \ge \alpha$.

Like T1 FSs can be represented by α-cuts, T2 FSs can be represented by α-planes, i.e.,

$$\widetilde{A}(x,u) = \sup\{\alpha \mid (x,u) \in \widetilde{A}_\alpha\}\,. \tag{5}$$

This is called the α-plane representation for a T2 FS (e.g. [4, 5, 7]).

Remark 2.1. Understanding of the term α-plane is confusing in recent T2 FSs literature. Liu defined an α-plane as a set containing ordered pairs (x, u) with $\widetilde{A}(x, u) \geq \alpha$. But, Figure 8 of his paper [4] is incorrect (it does not correspond to the Definition 1 – it depicts an α-level T2 FS instead of α-plane), which may be the source of this misunderstanding.[1]

The problem is that researchers treat α-planes sometimes like crisp sets, sometimes like α-level T2 FSs, sometimes like interval-valued FSs, and sometimes like interval T2 FSs (see two examples below). This is okay in some circumstances, however it has to be done carefully: although there exists one to one correspondence between interval-valued FSs and interval T2 FSs (and α-level T2 FSs), these "objects" are not the same. In our opinion, more exact would be to say that α-plane is a crisp set (a subset of $X \times [0, 1]$)[2] and some α-level T2 FS or some interval-valued FS or some interval T2 FS can be associated with this crisp set, i. e.,

- A_α^{T2} is α-level T2 FS associated with an α-plane (Liu called it associated T2 FS of the α-plane and denoted $\widetilde{A}(\alpha)$; Wagner et al. (see [12]) called it z-slice):

$$A_\alpha^{T2}(x, u) = \begin{cases} \alpha, & (x, u) \in \tilde{A}_\alpha, \\ 0, & \text{otherwise}, \end{cases}$$

- A_α^{IV} is interval-valued FS associated with an α-plane:

$$A_\alpha^{IV}(x) = I_x \subseteq [0, 1] , \qquad \text{where} \qquad u \in I_x \Leftrightarrow (x, u) \in \tilde{A}_\alpha ,$$

- A_α^{IT2} is interval T2 FS associated with an α-plane:

$$A_\alpha^{IT2}(x, u) = \begin{cases} 1, & (x, u) \in \tilde{A}_\alpha, \\ 0, & \text{otherwise}. \end{cases}$$

See Figure 1. Recall that the associated interval-valued FS is correctly defined only for α-planes of a T2 FS with convex and continuous membership grades. Otherwise, α-cuts of vertical slices are not always intervals.

Next, we give two examples where this ambiguity in the definition of α-plane has led to errors:

1. A misnomer appeared in [8], which led to some errors that were revised by one of the authors in [6]. Authors have mistaken the α-plane $\widetilde{A}_\alpha$ and its associated T2 FS $\widetilde{A}(\alpha)$, which is the same problem as with aforementioned Figure 8 in [4].

2. After the definition of α-plane in [5], author presents following properties of α-planes:

P1: if $\alpha_1 \geq \alpha_2$, then $\widetilde{A}_{\alpha_1} \subseteq \widetilde{A}_{\alpha_2}$,

P2: $(\widetilde{A} \cup \widetilde{B})_\alpha = \widetilde{A}_\alpha \cup \widetilde{B}_\alpha$,

P3: $(\widetilde{A} \cap \widetilde{B})_\alpha = \widetilde{A}_\alpha \cap \widetilde{B}_\alpha$,

[1] Liu corrected himself and used more appropriate figure in [5] – page 2227, Figure 3.

[2] This is in accordance with the type-2 membership function $\widetilde{A}(x, u) : X \times [0, 1] \to [0, 1]$.

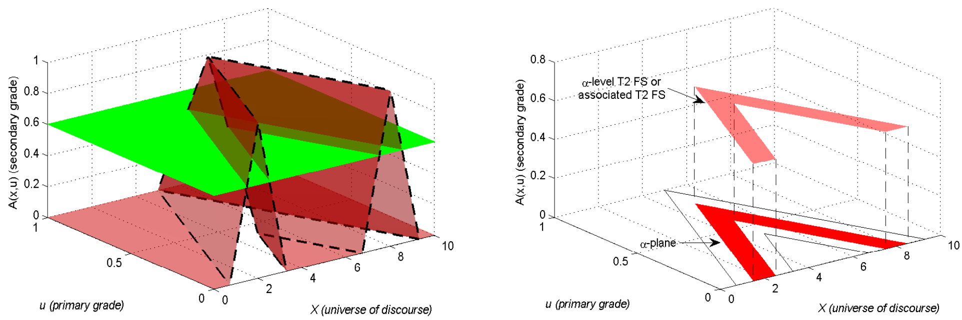

Fig. 1. 3D graph of T2 FS $\tilde{A}$ with horizontal 'cutting' plane in 0.6. An α-level T2 FS associated with an α-plane (or associated T2 FS of the α-plane) being the part of the horizontal 'cutting' plane bounded by membership function $\widetilde{A}(x,u)$; and the α-plane, which is its projection to the Xu plane.

with comment ([4, p. 11]): "Since these properties are not used in the rest of this paper, we do not provide proofs for them." In fact, property P1 holds if we consider α-planes to be crisp sets, and does not hold if we consider them to be interval-valued FSs or interval T2 FSs (see Figure 2). On the contrary, properties P2 and P3 hold if we consider α-planes to be interval-valued FSs or interval T2 FSs (see proof of the Theorem 1 in [8]), and do not hold if we consider them to be crisp sets (the main aim of this paper is to prove this claim).

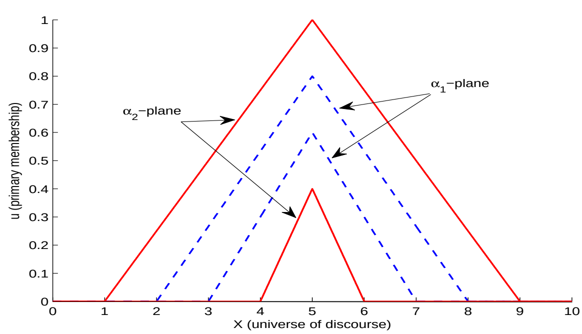

Fig. 2. Two α-planes with $\alpha_1 > \alpha_2$. Property $\widetilde{A}_{\alpha_1} \subseteq \widetilde{A}_{\alpha_2}$ holds if we consider α-planes to be crisp sets, and does not hold if we consider them to be interval-valued FSs.

2.3. Intersection and union of T2 FSs

Using Zadeh's Extension Principle (see [13]), the membership functions for intersection and union of T2 FSs $\widetilde{A}$ and $\widetilde{B}$ are defined as follows:

$$(\widetilde{A} \cap \widetilde{B})(x, w) = \sup\{\widetilde{A}(x, u) * \widetilde{B}(x, v) \,|\, u \star v = w\} \quad (6)$$

$$(\widetilde{A} \cup \widetilde{B})(x, w) = \sup\{\widetilde{A}(x, u) * \widetilde{B}(x, v) \,|\, u \circ v = w\} \quad (7)$$

where $*$ and $\star$ represent t-norms (both mostly minimum t-norm), $\circ$ represents a t-conorm (mostly maximum t-conorm) (see [1, 9]). It is not usual to distinguish between the two t-norms $*, \star$ in (6), but there is no reason not to do it (see [8, 10]). Note also, that we do not use symbols $\wedge$ and $\vee$ for t-norms and t-conorms, because we will use them hereafter for special cases – for minimum t-norm and maximum t-conorm, respectively.

Let $\widetilde{A}, \widetilde{B}$ be T2 FSs in X with convex, normal membership grades $\widetilde{A}(x')$, $\widetilde{B}(x')$ for some $x' \in X$, respectively (recall that $\widetilde{A}(x')$ is in fact a function of u). Let $v_1, v_2 \in [0, 1], v_1 \leq v_2$ and $\widetilde{A}(x', v_1) = \widetilde{B}(x', v_2) = 1$. Then, for intersection and union under minimum t-norm and maximum t-conorm following holds ([1]),

$$(\widetilde{A} \cap \widetilde{B})(x', u) = \begin{cases} \widetilde{A}(x', u) \vee \widetilde{B}(x', u), & u < v_1, \\ \widetilde{A}(x', u), & v_1 \leq u < v_2, \\ \widetilde{A}(x', u) \wedge \widetilde{B}(x', u), & u \geq v_2 \end{cases} \quad (8)$$

and

$$(\widetilde{A} \cup \widetilde{B})(x', u) = \begin{cases} \widetilde{A}(x', u) \wedge \widetilde{B}(x', u), & u < v_1, \\ \widetilde{B}(x', u), & v_1 \leq u < v_2, \\ \widetilde{A}(x', u) \vee \widetilde{B}(x', u), & u \geq v_2, \end{cases} \quad (9)$$

where $\wedge$ denotes minimum and $\vee$ maximum (see Figure 6 and Figure 9).

2.4. Two different interpretations for type-2 fuzzy sets

T2 FSs in a set X (mappings $\widetilde{A} : X \to [0, 1]^{[0,1]}$) can be understood as T1 FSs over the referential set $X \times [0, 1]$ (mappings $\widetilde{A} : X \times [0, 1] \to [0, 1]$). Then α-cuts of the later coincide with the α-planes of the former. Thus, treating α-planes of T2 FSs is equivalent to treating α-cuts of T1 FSs. This could indicate that the use of α-planes is pointless. However, it is not true because operations and relations between T2 FSs do not coincide with the corresponding operations and relations between T1 FSs – see, for example, definition of the union and intersection in Section 2.3.

3. PROPERTIES OF α-PLANES

Note that we treat α-planes like crisp sets (see Remark 1 and Remark 2 of this paper).

3.1. Inclusion

Following is a well known property of α-cuts of a T1 FS A:

$$\text{if } \alpha_1 \leq \alpha_2 \text{ then } A_{\alpha_2} \subseteq A_{\alpha_1}, \quad (10)$$

where α_1 and α_2 denote specific values of grade of T1 FS A. It shows that α-cut for greater α is included in α-cut for smaller α.

For α-planes of a T2 FS $\widetilde{A}$ similar property holds:

$$\text{if } \alpha_1 \leq \alpha_2 \text{ then } \widetilde{A}_{\alpha_2} \subseteq \widetilde{A}_{\alpha_1}, \tag{11}$$

where α_1 and α_2 denote specific values of secondary grade of T2 FS $\widetilde{A}$. Liu (see [4, p. 11], [5]) states this property of α-planes of T2 FS $\widetilde{A}$ without proof, which was not necessary with respect to his work, and which is very simple: if $(x, u) \in \widetilde{A}_{\alpha_2}$ for some $x \in X, u \in J_x, \alpha_2 \in [0, 1]$, then $\widetilde{A}(x, u) \geq \alpha_2$, and for any $\alpha_1 \leq \alpha_2$ it follows $\widetilde{A}(x, u) \geq \alpha_1$, so $(x, u) \in \widetilde{A}_{\alpha_1}$ and consequently $A_{\alpha_2} \subseteq A_{\alpha_1}$.

3.2. Intersection and union of α-cuts

We recapitulate and remind properties of intersection and union of T1 FSs with respect to α-cuts first. Following are well known properties of α-cuts of T1 FSs A, B:

$$(A \cap_{T1} B)_\alpha = A_\alpha \cap_C B_\alpha, \tag{12}$$

$$(A \cup_{T1} B)_\alpha = A_\alpha \cup_C B_\alpha, \tag{13}$$

where $\cap$ and $\cup$ on the left sides of the equalities denote intersection and union, respectively, of the T1 FSs and the same symbols on the right sides of the equalities denote intersection and union, respectively, of the crisp sets.

Note that these properties hold only for minimum t-norm and maximum t-conorm within the definition of intersection and union of T1 FSs. These standard intersection and union operations are the only cutworthy operations among the t-norms and t-conorms, which means that they are preserved in α-cuts for all $\alpha \in [0, 1]$ in the classical sense (see e. g. [2]).

Proposition 3.1. Formulas (12) and (13) hold for standard intersection and standard union of T1 FSs, i. e., for $(A \cap B)(x) = A(x) \wedge B(x)$ and $(A \cup B)(x) = A(x) \vee B(x)$.

Proposition 3.2. Formulas (12) and (13) do not hold for intersection and union of T1 FSs under t-norm, t-conorm different from minimum t-norm, maximum t-conorm, respectively, i. e., for $(A \cap B)(x) = A(x) * B(x)$ and $(A \cup B)(x) = A(x) \circ B(x)$, where t-norm $*$ is not minimum and t-conorm $\circ$ is not maximum.

Proof. It is known that minimum t-norm is the pointwise largest t-norm and maximum t-conorm is the pointwise smallest t-conorm. Let A, B be arbitrary T1 FSs in X, for which there exists $x \in X$, such that $A(x) * B(x) < A(x) \wedge B(x)$. Let $\alpha = A(x) \wedge B(x)$. Then $x \in A_\alpha \cap B_\alpha$ and $x \notin (A \cap B)_\alpha$, hence (12) fails. In a similar way, let there exists $x \in X$, such that $A(x) \circ B(x) > A(x) \vee B(x)$ and let $\alpha = A(x) \vee B(x)$. Then $x \notin A_\alpha \cup B_\alpha$ and $x \in (A \cup B)_\alpha$, hence (13) fails. □

In the proof we showed that inclusions

$$A_\alpha \cap B_\alpha \subseteq (A \cap B)_\alpha \tag{14}$$

and

$$(A \cup B)_\alpha \subseteq A_\alpha \cup B_\alpha \tag{15}$$

fail under any t-norm different from minimum and any t-conorm different from maximum. The inverse inclusions hold under each t-norm and each t-conorm, as we are going to demonstrate in the next proposition.

Proposition 3.3. Inclusions $(A \cap B)_\alpha \subseteq A_\alpha \cap B_\alpha$ and $A_\alpha \cup B_\alpha \subseteq (A \cup B)_\alpha$ hold for intersection and union of T1 FSs under each t-norm and t-conorm, respectively, i. e., for $(A \cap B)(x) = A(x) * B(x)$ and $(A \cup B)(x) = A(x) \circ B(x)$.

P r o o f. Let x be an arbitrary element of $(A \cap B)_\alpha$. Then $A(x) * B(x) \geq \alpha$, so $A(x) \geq \alpha$ and $B(x) \geq \alpha$, thus $x \in A_\alpha \cap B_\alpha$. In a similar way, let x be an arbitrary element of $A_\alpha \cup B_\alpha$. Then $A(x) \geq \alpha$ or $B(x) \geq \alpha$, thus $A(x) \circ B(x) \geq A(x) \vee B(x) \geq \alpha$, so $x \in (A \cup B)_\alpha$. □

3.3. Intersection and union of α-planes

Now, we are going to investigate similar properties of intersection and union of T2 FSs with respect to α-planes:

$$(\widetilde{A} \cap_{T2} \widetilde{B})_\alpha = \widetilde{A}_\alpha \cap_C \widetilde{B}_\alpha, \qquad (\widetilde{A} \cup_{T2} \widetilde{B})_\alpha = \widetilde{A}_\alpha \cup_C \widetilde{B}_\alpha, \tag{16}$$

where $\cap$ and $\cup$ on the left sides of the equalities denote intersection and union, respectively, of the T2 FSs and the same symbols on the right sides of the equalities denote intersection and union, respectively, of the crisp sets – see Remark 1 and Remark 2 of this paper. Hereafter, for simplicity, we will not use the subscripts. In fact, we are going to show that these properties do not hold.

3.3.1. Intersection of α-planes

We will investigate the following inclusions first:

$$\widetilde{A}_\alpha \cap \widetilde{B}_\alpha \quad \subseteq \quad (\widetilde{A} \cap \widetilde{B})_\alpha, \tag{17}$$

$$(\widetilde{A} \cap \widetilde{B})_\alpha \quad \subseteq \quad \widetilde{A}_\alpha \cap \widetilde{B}_\alpha. \tag{18}$$

Theorem 3.4. Let $\widetilde{A}$ and $\widetilde{B}$ be T2 FSs in a set X, let $*$ and $\star$ be t-norms from the definition (6) of the intersection of T2 FSs. Then

i) (17) holds, if $*$ and $\star$ are both minimum t-norm,

ii) (17) does not hold, if $*$ is a t-norm different from minimum t-norm,

iii) (17) does not hold, if $\star$ is a t-norm different from minimum t-norm.

P r o o f. i) We prove that formula (17) holds, if $*$ and $\star$ both denote minimum t-norm. Let pair (x, w), where $x \in X$ and $w \in J_x^{\widetilde{A}} \cap J_x^{\widetilde{B}}$, be an arbitrary fixed element of

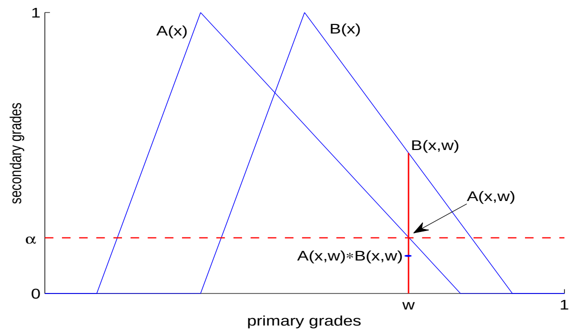

Fig. 3. Membership grades $\widetilde{A}(x), \widetilde{B}(x)$ of T2 FSs $\widetilde{A}, \widetilde{B}$ for some $x \in X$.

$\widetilde{A}_\alpha \cap \widetilde{B}_\alpha$. Then $(x,w) \in \widetilde{A}_\alpha$ and $(x,w) \in \widetilde{B}_\alpha$. Thus, $\widetilde{A}(x,w) \geq \alpha$ and $\widetilde{B}(x,w) \geq \alpha$. Hence, $\widetilde{A}(x,w) * \widetilde{B}(x,w) \geq \alpha$ (if $*$ denotes minimum t-norm!). Because of

$$(\widetilde{A} \cap \widetilde{B})(x,w) = \sup\{\widetilde{A}(x,u) * \widetilde{B}(x,v) | u \star v = w\} \tag{19}$$

and $w \star w = w$ (if $\star$ denotes minimum t-norm!), it follows $(\widetilde{A} \cap \widetilde{B})(x,w) \geq \widetilde{A}(x,w) * \widetilde{B}(x,w) \geq \alpha$, so $(x,w) \in (\widetilde{A} \cap \widetilde{B})_\alpha$.

ii) We prove failure of (17), if $*$ is a t-norm different from minimum t-norm by a counterexample. Let $\widetilde{A}, \widetilde{B}$ be T2 FSs in X, whose membership grades $\widetilde{A}(x), \widetilde{B}(x)$, respectively, for some $x \in X$ satisfy the following conditions: there exists $w \in (0,1)$ such that $\widetilde{A}(x,w) * \widetilde{B}(x,w) < \widetilde{A}(x,w) \wedge \widetilde{B}(x,w)$ and $\widetilde{A}(x), \widetilde{B}(x)$ are both decreasing in $[w,1] \cap J_x$ (see Figure 3). Let $\alpha = \widetilde{A}(x,w) \wedge \widetilde{B}(x,w)$. Then obviously $\widetilde{A}(x,w) * \widetilde{B}(x,w) < \alpha$, and because of the t-norm $*$ is monotone, it follows that for all $u, v \geq w$ holds $\widetilde{A}(x,u) * \widetilde{B}(x,v) \leq \widetilde{A}(x,w) * \widetilde{B}(x,w)$. Consequently, $\widetilde{A}(x,u) * \widetilde{B}(x,v) < \alpha$ for all $u, v \geq w$, which means that $(x,w) \notin (\widetilde{A} \cap \widetilde{B})_\alpha$. Moreover, $\alpha = \widetilde{A}(x,w) \wedge \widetilde{B}(x,w)$ entails $(x,w) \in \widetilde{A}_\alpha \cap \widetilde{B}_\alpha$, and finally $\widetilde{A}_\alpha \cap \widetilde{B}_\alpha \not\subseteq (\widetilde{A} \cap \widetilde{B})_\alpha$.

iii) We prove failure of (17), if $\star$ is a t-norm different from minimum t-norm by a counterexample. Remind, that minimum t-norm is the only t-norm, whose set of idempotents is equal to $[0,1]$, i.e., it is the only t-norm, which has not any non-idempotents (e.g. [3]). Thus, for $\star$ there exists at least one non-idempotent in $(0,1)$.

Let $\widetilde{A}, \widetilde{B}$ be T2 FSs in X, whose membership grades $\widetilde{A}(x), \widetilde{B}(x)$, respectively, for some $x \in X$ satisfy the following conditions: there exists non-idempotent $w \in (0,1)$ of the t-norm $\star$ such that $\widetilde{A}(x,w) = \widetilde{B}(x,w)$ and $\widetilde{A}(x), \widetilde{B}(x)$ are both decreasing in $[w,1] \cap J_x$ (see Figure 4). Let $\alpha = \widetilde{A}(x,w) = \widetilde{B}(x,w)$. Then $(x,w) \in \widetilde{A}_\alpha \cap \widetilde{B}_\alpha$. Furthermore,

$$(\widetilde{A} \cap \widetilde{B})(x,w) = \sup\{\widetilde{A}(x,u) * \widetilde{B}(x,v) | u \star v = w\}. \tag{20}$$

Because of w is non-idempotent of the t-norm $\star$, for all u, v with property $u \star v = w$ holds $u > w$ or $v > w$ (and obviously $u, v \geq w$), so $\widetilde{A}(x,u) < \widetilde{A}(x,w)$ or $\widetilde{B}(x,v) < \widetilde{A}(x,w)$,

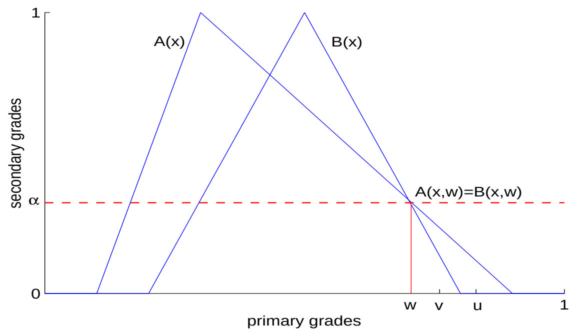

Fig. 4. Membership grades $\widetilde{A}(x), \widetilde{B}(x)$ of T2 FSs $\widetilde{A}, \widetilde{B}$ for some $x \in X$.

which means that $\widetilde{A}(x,u) \star \widetilde{B}(x,v) < \widetilde{A}(x,w) = \alpha$ and also $\sup\{\widetilde{A}(x,u) * \widetilde{B}(x,v) | u \star v = w\} < \alpha$. Consequently, $(x,w) \notin (\widetilde{A} \cap \widetilde{B})_\alpha$ and finally $\widetilde{A}_\alpha \cap \widetilde{B}_\alpha \not\subseteq (\widetilde{A} \cap \widetilde{B})_\alpha$. □

Note that the counterexamples used in the items ii and iii of the proof contain also convex, normal membership grades, so Theorem 3.4 holds in the same manner for this special and simple case of membership grades.

Theorem 3.5. Let $\widetilde{A}$ and $\widetilde{B}$ be T2 FSs in a set X, let $*$ and $\star$ be t-norms from the definition (6) of the intersection of T2 FSs. Then (18) does not hold for any t-norms $*, \star$.

P r o o f. We prove failure of (18) for any t-norms $*, \star$ by a counterexample. Let $\widetilde{A}, \widetilde{B}$ be T2 FSs in X, whose membership grades $\widetilde{A}(x), \widetilde{B}(x)$, respectively, for some $x \in X$ satisfy the following conditions: there exist $v_1, v_2, v_3 \in (0,1)$ such that $v_1 < v_2 < v_3$, $\widetilde{B}(x, v_3) = 1$ and functions $\widetilde{A}(x), \widetilde{B}(x)$ are increasing in $[v_1, v_3], [v_2, v_3]$, respectively; and moreover, there exist $u, w \in (v_2, v_3)$ such that $w = u \star v_3$ and $\widetilde{A}(x,w) > \widetilde{B}(x,w)$ (see Figure 5). Let $\alpha = \widetilde{A}(x,w)$. Then $(x,w) \notin \widetilde{B}_\alpha$, thus $(x,w) \notin \widetilde{A}_\alpha \cap \widetilde{B}_\alpha$. From

$$(\widetilde{A} \cap \widetilde{B})(x,w) = \sup\{\widetilde{A}(x,u) * \widetilde{B}(x,v) | u \star v = w\} \tag{21}$$

and $w = u \star v_3$ follows that

$$\begin{aligned} (\widetilde{A} \cap \widetilde{B})(x,w) &\geq \widetilde{A}(x,u) * \widetilde{B}(x,v_3) = \widetilde{A}(x,u) * 1 \\ &= \widetilde{A}(x,u) \geq \widetilde{A}(x,w) = \alpha\,. \end{aligned} \tag{22}$$

Thus, $(x,w) \in (\widetilde{A} \cap \widetilde{B})_\alpha$ and consequently $(\widetilde{A} \cap \widetilde{B})_\alpha \not\subseteq \widetilde{A}_\alpha \cap \widetilde{B}_\alpha$. □

Using (8) it is easy to see that (18) does not hold under minimum t-norm for convex, normal membership grades. Situation is depicted in Figure 6.

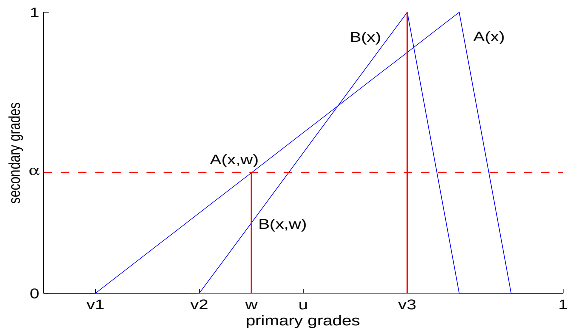

Fig. 5. Membership grades $\widetilde{A}(x), \widetilde{B}(x)$ of T2 FSs $\widetilde{A}, \widetilde{B}$ for some $x \in X$.

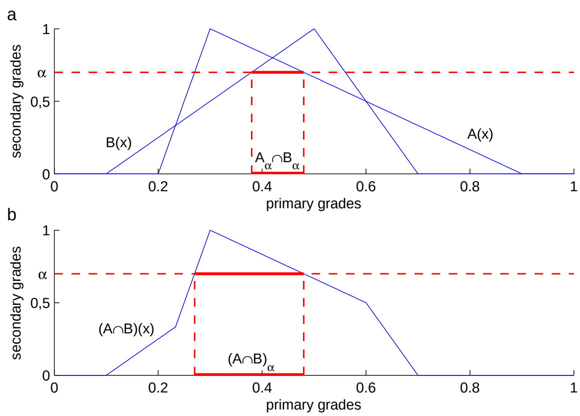

Fig. 6. a) Convex, normal membership grades $\widetilde{A}(x), \widetilde{B}(x)$ of T2 FSs $\widetilde{A}, \widetilde{B}$ for some $x \in X$. b) Membership grade $(\widetilde{A} \cap \widetilde{B})(x)$ of the intersection of the T2 FSs $\widetilde{A}, \widetilde{B}$ for the same $x \in X$ as in a).

Example 3.6. Let $\widetilde{A}, \widetilde{B}$ be T2 FSs in $X = \{x_1, x_2, x_3, x_4\}$ given as follows (we will focus just on x_1): $\widetilde{A}(x_1) = 0.2/0.3 + 0.8/0.4 + 0.3/0.5$; $\widetilde{B}(x_1) = 0.4/0.4 + 0.6/0.5 + 0.3/0.6$. Let $\alpha = 0.5$ and $u = 0.4$. Then $(x_1, u) \in \widetilde{A}_\alpha$ and $(x_1, u) \notin \widetilde{B}_\alpha$, so consequently $(x_1, u) \notin (\widetilde{A}_\alpha \cap \widetilde{B}_\alpha)$ and $(x_1, u) \in (\widetilde{A}_\alpha \cup \widetilde{B}_\alpha)$.

Using (6) we compute $\widetilde{A} \cap \widetilde{B}$ (under minimum t-norms), for which $(\widetilde{A} \cap \widetilde{B})(x_1) = 0.2/0.3 + 0.6/0.4 + 0.3/0.5$. Thus, $(x_1, u) \in (\widetilde{A} \cap \widetilde{B})_\alpha$. Hence, $(\widetilde{A} \cap \widetilde{B})_\alpha \not\subseteq \widetilde{A}_\alpha \cap \widetilde{B}_\alpha$ (see the proof of the Theorem 3.4).

Using (7) we compute $\widetilde{A} \cup \widetilde{B}$, for which $(\widetilde{A} \cup \widetilde{B})(x_1) = 0.4/0.4 + 0.6/0.5 + 0.3/0.6$. Thus, $(x_1, u) \notin (\widetilde{A} \cup \widetilde{B})_\alpha$. Hence, $\widetilde{A}_\alpha \cup \widetilde{B}_\alpha \not\subseteq (\widetilde{A} \cup \widetilde{B})_\alpha$ (see the proof of the Theorem 3.8).

Example 3.7. We show that the restriction to minimum t-norm in the first part of the Theorem 3.4 is reasonable. Let $*$ and $\star$ in the definition of the intersection of T2 FSs (6) both denote product t-norm. Let $\widetilde{A}$ and $\widetilde{B}$ be T2 FSs in X with constant secondary grades equal to 0.8. Then for all (x,u) and (x,v), where $x \in X, u \in J_x^{\widetilde{A}} \subseteq [0,1], v \in J_x^{\widetilde{B}} \subseteq [0,1]$, secondary grades satisfy $\widetilde{A}(x,u) = \widetilde{B}(x,v) = 0.8$. Then for all (x,w), where $x \in X, w = u \cdot v$, secondary grades satisfy $\widetilde{A}(x,w) = 0.8^2 = 0.64$. So, e. g. for $\alpha = 0.7$, $(\widetilde{A} \cap \widetilde{B})_\alpha$ has to be an empty set, but $\widetilde{A}_\alpha \cap \widetilde{B}_\alpha$ may be a nonempty set. Hence, (17) does not hold in general under product t-norms.

Note that this take place also if $*$ denotes product t-norm and $\star$ denotes minimum t-norm.

3.3.2. Union of α-planes

We will investigate the following inclusions next:

$$\widetilde{A}_\alpha \cup \widetilde{B}_\alpha \subseteq (\widetilde{A} \cup \widetilde{B})_\alpha, \tag{23}$$

$$(\widetilde{A} \cup \widetilde{B})_\alpha \subseteq \widetilde{A}_\alpha \cup \widetilde{B}_\alpha. \tag{24}$$

Theorem 3.8. Let $\widetilde{A}$ and $\widetilde{B}$ be T2 FSs in a set X, let $*$ and $\circ$ be a t-norm and a t-conorm, respectively, from the definition (7) of the union of T2 FSs. Then

i) (24) holds, if $\circ$ is maximum t-conorm (and $*$ is an arbitrary t-norm),

ii) (24) does not hold, if $\circ$ is a t-conorm different from maximum t-conorm (and $*$ is an arbitrary t-norm).

Proof. i) We prove that (24) holds, if $*$ denotes any t-norm and $\circ$ denotes maximum t-conorm. Let pair (x,w), where $x \in X$ and $w \in J_x^{\widetilde{A}\cup\widetilde{B}}$, be an arbitrary element of $(\widetilde{A} \cup \widetilde{B})_\alpha$. Then there exist $u \in J_x^{\widetilde{A}}$ and $v \in J_x^{\widetilde{B}}$ such that $u \vee v = w$ (where $\vee$ is maximum) and $\widetilde{A}(x,u) * \widetilde{B}(x,v) \geq \alpha$. From $u \vee v = w$ follows that $u = w$ or $v = w$. Let $u = w$ (proof for $v = w$ is similar). For any t-norm $*$, from $\widetilde{A}(x,u) * \widetilde{B}(x,v) \geq \alpha$ follows $\widetilde{A}(x,u) \geq \alpha$ (and $\widetilde{B}(x,v) \geq \alpha$). Thus, $\widetilde{A}(x,w) \geq \alpha$ and consequently $(x,w) \in \widetilde{A}_\alpha$, so $(x,w) \in \widetilde{A}_\alpha \cup \widetilde{B}_\alpha$. Finally $(\widetilde{A} \cup \widetilde{B})_\alpha \subseteq \widetilde{A}_\alpha \cup \widetilde{B}_\alpha$.

ii) We prove failure of (24), if $\circ$ is a t-conorm different from maximum t-conorm by a counterexample. Let $\widetilde{A}, \widetilde{B}$ be T2 FSs in X, whose membership grades $\widetilde{A}(x), \widetilde{B}(x)$, respectively, satisfy for some $x \in X$ the following conditions: there exist $u_1, v_1, w \in (0,1)$ such that $u_1 < v_1 < w$, $u_1 \circ v_1 = w$ (note that $\circ$ is not maximum), $\widetilde{A}(x,u_1) = 1$, $\widetilde{A}(x,w) = \widetilde{B}(x,w)$ and membership grades $\widetilde{A}(x), \widetilde{B}(x)$ are decreasing in $[u_1, w]$ (see Figure 7). Let $\alpha = \widetilde{B}(x,v_1)$. Then from

$$\begin{aligned}(\widetilde{A} \cup \widetilde{B})(x,w) &= \sup\{\widetilde{A}(x,u) * \widetilde{B}(x,v) | u \circ v = w\} \\ &\geq \widetilde{A}(x,u_1) * \widetilde{B}(x,v_1) = 1 * \widetilde{B}(x,v_1) \\ &= \widetilde{B}(x,v_1) > \widetilde{B}(x,w) = \widetilde{A}(x,w)\end{aligned} \tag{25}$$

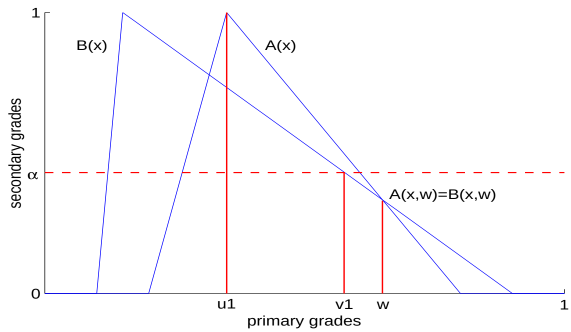

Fig. 7. Membership grades $\widetilde{A}(x), \widetilde{B}(x)$ of T2 FSs $\widetilde{A}, \widetilde{B}$ for some $x \in X$.

follows that $(x,w) \in (\widetilde{A} \cup \widetilde{B})_\alpha$, but $(x,w) \notin \widetilde{A}_\alpha$ and $(x,w) \notin \widetilde{B}_\alpha$, so consequently $(x,w) \notin \widetilde{A}_\alpha \cup \widetilde{B}_\alpha$ and finally $(\widetilde{A} \cup \widetilde{B})_\alpha \not\subseteq \widetilde{A}_\alpha \cup \widetilde{B}_\alpha$. □

Theorem 3.9. Let $\widetilde{A}$ and $\widetilde{B}$ be T2 FSs in a set X, let $*$ and $\circ$ be a t-norm and a t-conorm, respectively, from the definition (7) of the union of T2 FSs. Then (23) does not hold for any t-norm $*$ and any t-conorm $\circ$.

P r o o f. We prove failure of (23) by a counterexample. Let $\widetilde{A}, \widetilde{B}$ be T2 FSs in X, whose membership grades $\widetilde{A}(x), \widetilde{B}(x)$, respectively, satisfy for some $x \in X$ the following conditions: there exists $w \in (0,1)$ such that $\widetilde{A}(x,w) > \widetilde{B}(x,w)$ and membership grade $\widetilde{B}(x)$ is increasing in $[0,w] \cap J_x^{\widetilde{B}}$ (see Figure 8). Then

$$(\widetilde{A} \cup \widetilde{B})(x,w) = \sup\{\widetilde{A}(x,u) * \widetilde{B}(x,v) | u \circ v = w\}. \tag{26}$$

If $u \circ v = w$, then $u, v \leq w$ (holds for any t-conorm $\circ$), so we will consider only u, v smaller than or equal to w. Furthermore, for any t-norm $*$ we have

$$\widetilde{A}(x,u) * \widetilde{B}(x,v) \leq \widetilde{B}(x,v) \leq \widetilde{B}(x,w), \tag{27}$$

where the last inequality entails from the fact that $v \leq w$ and the function $\widetilde{B}(x)$ is increasing. From (26) and (27) follows

$$(\widetilde{A} \cup \widetilde{B})(x,w) \leq \widetilde{B}(x,w) < \widetilde{A}(x,w). \tag{28}$$

Let $\alpha = \widetilde{A}(x,w)$. Then $(x,w) \in \widetilde{A}_\alpha \cup \widetilde{B}_\alpha$, but $(x,w) \notin (\widetilde{A} \cup \widetilde{B})_\alpha$ and finally $\widetilde{A}_\alpha \cup \widetilde{B}_\alpha \not\subseteq (\widetilde{A} \cup \widetilde{B})_\alpha$. □

Using (9) it is easy to see that (23) does not hold under minimum t-norm and maximum t-conorm for convex, normal membership grades. Situation is depicted in Figure 9.

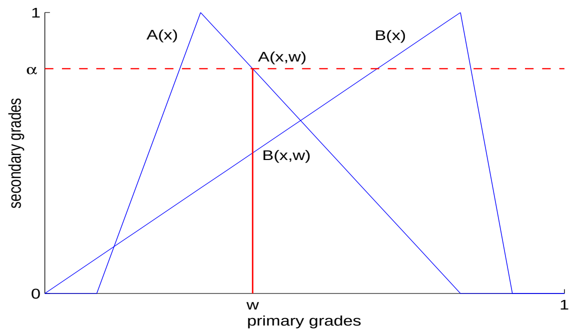

Fig. 8. Membership grades $\widetilde{A}(x), \widetilde{B}(x)$ of T2 FSs $\widetilde{A}, \widetilde{B}$ for some $x \in X$.

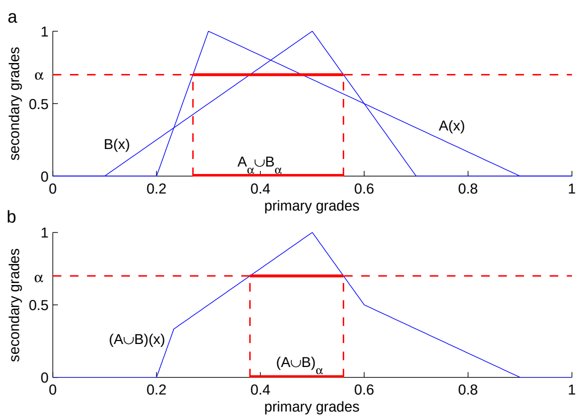

Fig. 9. a) Convex, normal membership grades $\widetilde{A}(x), \widetilde{B}(x)$ of the T2 FSs $\widetilde{A}, \widetilde{B}$ for some $x \in X$. b) Membership grade $(\widetilde{A} \cup \widetilde{B})(x)$ of the union of the T2 FSs $\widetilde{A}, \widetilde{B}$ for the same $x \in X$ as in a).

Remark 3.10. One of the reasons why α-cuts are so useful for T1 FSs mathematics is, that we can decompose a T1 FS into a collection of crisp sets. Then, the equalities (12) and (13) allow us to use the intersection and union of crisp sets (which are easy to use) instead of computing the intersection and union of fuzzy sets (which are much more complicated). Similarly, the equalities (16) would allow us to use the intersection and union of crisp sets (subsets of 2D planes) instead of computing the intersection and union of T2 FSs. But, as was shown in this paper, these equalities do not hold.[3]

Although the equalities (16) do not hold for α-planes (as crisp sets), in concordance

[3]It is possible to decompose T2 FSs into a collection of so-called (α_1, α_2)-double cuts, for which the equalities do hold (see [11]). Note that (α_1, α_2)-double cuts are crisp subsets of universe.

with Remark 1 of this paper similar equalities do hold for interval-valued FSs associated with α-planes (or α-level T2 FSs associated with an α-planes), i. e.,

$$(\widetilde{A} \cap_{T2} \widetilde{B})_\alpha^{IV} = \widetilde{A}_\alpha^{IV} \cap_{IV} \widetilde{B}_\alpha^{IV} \qquad (\widetilde{A} \cup_{T2} \widetilde{B})_\alpha^{IV} = \widetilde{A}_\alpha^{IV} \cup_{IV} \widetilde{B}_\alpha^{IV} . \tag{29}$$

This was proved in [8] – see proof of the Theorem 1.

4. CONCLUSIONS

We stated some properties of α-planes (we treated α-planes like crisp sets – see Remark 1 and Remark 2 of this paper) of T2 FSs and discussed them with respect to the similar properties of α-cuts of T1 FSs. It is known, that standard intersection and standard union of T1 FSs are the only cutworthy operations for T1 FSs. We investigated under which t-norms and which t-conorms are intersection and union of the T2 FSs preserved in the α-planes for all $\alpha \in [0, 1]$. The conclusion is that this property does not hold for the intersection under any t-norms as well as it does not hold for the union under any t-norm and t-conorm. Proposed results strongly depends on the fact that we computed the intersection and union of α-planes as the intersection and union of crisp sets.

Although we proved that no equality holds between the intersection/union of α-planes of T2 FSs and α-planes of the intersection/union of T2 FSs, some inclusions hold and we proved them. Table 1 summarizes obtained results and shows the restrictions, that are necessary to properties became true. Note, that $*, \star$ denote two t-norms within the definition of the intersection or union of T2 FSs, T denotes that inclusion holds under arbitrary t-norm T; S means that inclusion holds only under arbitrary t-conorm S; $\wedge$, $\vee$ means that inclusion holds only under minimum t-norm, maximum t-conorm, respectively; and '—' means that property does not hold under any t-norm or t-conorm.

Property	α-cuts	α-planes
$(A \cap B)_\alpha \subseteq A_\alpha \cap B_\alpha$	T	—
$A_\alpha \cap B_\alpha \subseteq (A \cap B)_\alpha$	$\wedge$	$* = \wedge, \star = \wedge$
$(A \cap B)_\alpha = A_\alpha \cap B_\alpha$	$\wedge$	—
$(A \cup B)_\alpha \subseteq A_\alpha \cup B_\alpha$	$\vee$	$* = T, \star = \vee$
$A_\alpha \cup B_\alpha \subseteq (A \cup B)_\alpha$	S	—
$(A \cup B)_\alpha = A_\alpha \cup B_\alpha$	$\vee$	—

Tab. 1. Summary of obtained results

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REFERENCES

[1] N. Karnik and J. Mendel: Operations on type-2 fuzzy sets. Fuzzy Sets and Systems *122* (2001), 327–348.

[2] G. Klir and B. Yuan: Fuzzy Sets and Fuzzy Logic: Theory and Application. Upper-Saddle River, Prentice-Hall, NJ 1995.

[3] A. Kolesárová and A. Kováčová: Fuzzy množiny a ich aplikácie. STU, Bratislava 2004.

[4] F. Liu: An Efficient Centroid Type Reduction Strategy for General Type-2 Fuzzy Logic System. IEEE Comput. Intell. Soc., Walter J. Karplus Summer Research Grant Report 2006.

[5] F. Liu: An efficient centroid type reduction strategy for general type-2 fuzzy logic system. Inform. Sci. *178* (2008), 2224–2236.

[6] J. M. Mendel: Comments on 'α-plane representation for type-2 fuzzy sets: Theory and applications'. IEEE Trans. Fuzzy Syst. *18* (2010), 229–230.

[7] J. M. Mendel and R. I. John: Type-2 fuzzy sets made simple. IEEE Trans. Fuzzy Syst. *10* (2002), 117–127.

[8] J. M. Mendel, F. Liu, and D. Zhai: α-plane representation for type-2 fuzzy sets: Theory and applications. IEEE Trans. Fuzzy Syst. *17* (2009), 1189–1207.

[9] M. Mizumoto and K. Tanaka: Some properties of fuzzy sets of type 2. Inform. Control *31* (1976), 312–340.

[10] J. Starczewski: Extended triangular norms on gaussian fuzzy sets. In: Proc. EUSFLAT-LFA 2005 (E. Montseny and P. Sobrevilla, eds.), 2005, pp.̇872–877.

[11] Z. Takáč: Intersection and union of type-2 fuzzy sets and connection to (α_1, α_2)-double cuts. In: Proc. EUSFLAT-LFA 2011 (S. Galichet, J. Montero and G. Mauris, eds.), 2011, pp. 1052–1059.

[12] C. Wagner and H. Hagras: zSlices-towards bridging the gap between interval and general type-2 fuzzy logic. In: Proc. IEEE FUZZ Conf, Hong Kong 2008, pp. 489–497.

[13] L. A. Zadeh: The concept of a linguistic variable and its application to approximate reasoning – I. Inform. Sci. *8* (1975), 199–249.

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