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GROUPOIDS ASSIGNED TO RELATIONAL SYSTEMS

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Abstract. By a relational system we mean a couple (A,R) where A is a set and R is a binary relation on A, i.e. $R \subseteq A \times A$. To every directed relational system $\mathcal{A} = (A,R)$ we assign a groupoid $\mathcal{G}(\mathcal{A}) = (A,\cdot)$ on the same base set where $xy = y$ if and only if $(x,y) \in R$. We characterize basic properties of R by means of identities satisfied by $\mathcal{G}(\mathcal{A})$ and show how homomorphisms between those groupoids are related to certain homomorphisms of relational systems.

Keywords: relational system, groupoid, directed system, g-homomorphism

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The theory of binary relations was settled by J. Riguet [8]. An algebraic approach to relational systems was developed by A. I. Mal'cev [6]. Some particular cases of relational systems and certain homomorphisms of them were treated by the first author and his co-authors in [1]–[5]. The method assigning a groupoid to a given relational system was initiated in [5] where certain concepts of this paper were introduced. We are motivated by the fact that to certain relational systems, in particular to directed posets, a certain directoid (see e.g. [3]) or semilattice can be assigned in such a way that $a \leqslant b$ if and only if $a \vee b = b$. Moreover, every homomorphism of a semilattice induces a certain homomorphism of the poset $(A, \leqslant)$ as was investigated in [3] or, in the case of (A,E) where E is an equivalence relation on A, in [2]. For quasiordered sets a similar question was solved in [4].

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We generalize these approaches in a way similar to that of [5] and produce several results which enable us to consider relational systems from a purely algebraic point of view.

Some of our results were already published in [5] but, for the reader's convenience, these results are repeated.

First, we recall the basic concepts.

Definition 1. A *relational system* is an ordered pair (A, R) consisting of a non-empty set A and a binary relation R on A. For $a, b \in A$ the (*upper*) *cone* $U_R(a, b)$ of a and b is defined by

$$U_R(a, b) := \{x \in A\colon\ (a, x), (b, x) \in R\}.$$

A relational system (A, R) is called *directed* if $U_R(a, b) \neq \emptyset$ for all $a, b \in A$.

R e m a r k 2. (i) All considerations which will follow can be dualized for the lower cone $L_R(a, b) := \{x \in A\colon\ (x, a), (x, b) \in R\}$ of a and b.

(ii) Every relational system $\mathcal{A} = (A, R)$ can be extended to a directed one by adjoining a new element $1 \notin A$ and putting $A_d := A \cup \{1\}$, $R_d := R \cup (A_d \times \{1\})$. Then $\mathcal{A}_d := (A_d, R_d)$ is directed and $\mathcal{A}$ is the restriction of $\mathcal{A}_d$ to A. Hence we will formulate our results mainly for directed relational systems and this is not an essential constraint.

Definition 3. A *groupoid* $(A, \cdot)$ *corresponds* to a directed relational system (A, R) if $ab = b$ provided $(a, b) \in R$ and $ab \in U_R(a, b)$ otherwise.

R e m a r k 4. Although a groupoid $(A, \cdot)$ corresponding to a relational system $\mathcal{A} = (A, R)$ need not be uniquely determined since for $a, b \in A$ with $(a, b) \notin R$ and $|U_R(a, b)| > 1$ there are several possibilities to define ab, we have $ab = b$ if and only if $(a, b) \in R$ in every groupoid corresponding to $\mathcal{A}$. Hence every groupoid corresponding to $\mathcal{A}$ contains complete information concerning the relation R.

R e m a r k 5. If $(A, \cdot)$ corresponds to (A, R) and $a, b \in A$ then $(a, ab) \in R$. This is clear in the case $(a, b) \notin R$, and in the case $(a, b) \in R$ it follows from $(a, ab) = (a, b) \in R$.

First we are interested in the question which groupoids may correspond to a relational system.

Theorem 6. *For a groupoid $\mathcal{G} = (G, \cdot)$ the following assertions are equivalent:*

(i) *There exists a directed relational system $\mathcal{A} = (G, R)$ with a reflexive relation R such that $\mathcal{G}$ corresponds to $\mathcal{A}$.*

(ii) *$\mathcal{G}$ satisfies the identities $xx = x$ and $x(xy) = y(xy) = xy$.*

Proof. Let $a, b \in G$.

(i) $\Rightarrow$ (ii): Since $(a, a) \in R$ we have $aa = a$. If $(a, b) \in R$ then $ab = b$ and hence $a(ab) = ab$ and $b(ab) = bb = b = ab$. If $(a, b) \notin R$ then $(a, ab), (b, ab) \in R$ and hence $a(ab) = b(ab) = ab$.

(ii) $\Rightarrow$ (i): Put $R := \{(x, y) \in G^2 \colon\ xy = y\} \cup \{(x, x) \colon\ x \in G\}$. Then R is reflexive. Since $xy \in U_R(x, y)$ for all $x, y \in G$, (A, R) is directed. Moreover, if $(a, b) \in R$ then $ab = b$ or $a = b$. In the latter case we have $ab = bb = b$. If $(a, b) \notin R$ then $ab \in U_R(a, b)$. $\square$

Remark 7. Every relational system $\mathcal{A} = (A, R)$ can be considered as a graph with vertex-set A and edge-set R. It is well-known (see e.g. [7]) that to every graph $\mathcal{A} = (A, R)$ a *graph algebra* $\mathcal{H}(\mathcal{A}) = (A^+, \circ)$ can be assigned as follows: $A^+ := A \cup \{\infty\}$, $x \circ y := x$ if $(x, y) \in R$ and $x \circ y := \infty$ if $(x, y) \notin R$ $(x, y \in A^+)$. However, there is an essential difference in applications. Contrary to a groupoid corresponding to $\mathcal{A}$, the graph algebra $\mathcal{H}(\mathcal{A})$ is determined uniquely. But if e.g. $\mathcal{A} = (A, \leqslant)$ is a join-semilattice we may put $ab := a \vee b$ for $a, b \in A$ since $a \vee b \in U_{\leqslant}(a, b)$, and the corresponding groupoid is just a join-semilattice $(A, \vee)$ which has a nice structure used in numerous applications both in algebra and beyond. On the other hand, a graph algebra can be far from a join-semilattice and need not have so nice properties. This means that our relative “vagueness” in the definition of the operation “$\cdot$” may be an essential advance in applications.

In what follows we are going to show how the properties of $\mathcal{A} = (A, R)$ can be captured by $\mathcal{G}(\mathcal{A}) = (A, \cdot)$.

Theorem 8. *If $\mathcal{A} = (A, R)$ is a directed relational system and $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ a groupoid corresponding to $\mathcal{A}$ then the following assertions hold:*

(i) *R is reflexive if and only if $\mathcal{G}(\mathcal{A})$ is idempotent.*

(ii) *R is symmetric if and only if $\mathcal{G}(\mathcal{A})$ satisfies the identity $(xy)x = x$.*

(iii) *R is transitive if and only if $\mathcal{G}(\mathcal{A})$ satisfies the identity $x((xy)z) = (xy)z$.*

(iv) *If $\mathcal{G}(\mathcal{A})$ is commutative then R is antisymmetric.*

(v) *If $\mathcal{G}(\mathcal{A})$ satisfies the identity $(xy)x = xy$ then R is antisymmetric.*

(vi) *If $\mathcal{G}(\mathcal{A})$ is a semigroup then R is transitive.*

Proof. Let $a, b, c \in A$.

(i) is evident.

(ii) "$\Rightarrow$": According to Remark 5, $(a, ab) \in R$ whence $(ab, a) \in R$, i.e. $(ab)a = a$.

"$\Leftarrow$": If $(a, b) \in R$ then $ab = b$ and hence $ba = (ab)a = a$, i.e. $(b, a) \in R$.

(iii) "$\Rightarrow$": According to Remark 5, $(a, ab), (ab, (ab)c) \in R$ and hence $(a, (ab)c) \in R$, i.e. $a((ab)c) = (ab)c$.

"$\Leftarrow$": If $(a, b), (b, c) \in R$ then $ab = b$ and $bc = c$ and hence

$$ac = a(bc) = a((ab)c) = (ab)c = bc = c,$$

i.e. $(a, c) \in R$.

(iv) If $(a, b), (b, a) \in R$ then $ab = b$ and $ba = a$ and hence $a = ba = ab = b$.

(v) If $(a, b), (b, a) \in R$ then $ab = b$ and $ba = a$ and hence $a = ba = (ab)a = ab = b$.

(vi) If $(a, b), (b, c) \in R$ then $ab = b$ and $bc = c$ and hence $ac = a(bc) = (ab)c = bc = c$, i.e. $(a, c) \in R$. $\square$

We can ask also conversely which relational systems can be induced by a given groupoid $\mathcal{G} = (G, \cdot)$.

Definition 9. Let $\mathcal{G} = (G, \cdot)$ be a groupoid. Define two corresponding relational systems $\mathcal{A}(\mathcal{G}) := (G, R(\mathcal{G}))$ and $\mathcal{A}^*(\mathcal{G}) := (G, R^*(\mathcal{G}))$ as follows:

$$R(\mathcal{G}) := \{(x, y) \in G^2 \colon\ xy = y\},$$
$$R^*(\mathcal{G}) := \bigcup_{x,y \in G} \{(x, xy), (y, xy)\}$$

Obviously, $R(\mathcal{G}) \subseteq R^*(\mathcal{G})$.

Lemma 10. *If $\mathcal{G} = (G, \cdot)$ is a groupoid then the following assertions hold:*

(i) *$\mathcal{A}^*(\mathcal{G})$ is directed.*

(ii) *If $\mathcal{G}$ satisfies the identities $x(xy) = y(xy) = xy$ then $\mathcal{A}(\mathcal{G}) = \mathcal{A}^*(\mathcal{G})$.*

Proof. (i) We have $xy \in U_{R^*(G)}(x, y)$ for all $x, y \in G$.

(ii) We have $R(\mathcal{G}) = R^*(\mathcal{G})$. $\square$

Lemma 11. *If $\mathcal{A} = (A, R)$ is a directed relational system and $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ a corresponding groupoid then $\mathcal{A}(\mathcal{G}(\mathcal{A})) = \mathcal{A}$.*

Proof. Let $a, b \in A$. If $(a, b) \in R(\mathcal{G}(\mathcal{A}))$ then $ab = b$ and since $\mathcal{G}(\mathcal{A})$ is a corresponding groupoid, we have $(a, b) \in R$. Conversely, if $(a, b) \in R$ then $ab = b$ in $\mathcal{G}(\mathcal{A})$ and hence $(a, b) \in R(\mathcal{G}(\mathcal{A}))$. $\square$

In what follows, we will study connections between homomorphisms of relational systems and homomorphisms of the corresponding groupoids.

Definition 12. Let $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ be relational systems, $h\colon A \to B$ and Θ an equivalence relation on A. The mapping h is called a *homomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if $(a, b) \in R$ implies $(h(a), h(b)) \in S$. If, moreover, for all $(c, d) \in S$ there exists $(a, b) \in R$ with $h(a) = c$ and $h(b) = d$ then h is called *strong*. Moreover, the *quotient relational system* $\mathcal{A}/\Theta := (A/\Theta, R/\Theta)$ is defined by

$$R/\Theta := \{([a]\Theta, [b]\Theta)\colon\ (a, b) \in R\}.$$

It is almost evident that if R is reflexive or symmetric then also R/Θ has this property. This need not be true for transitivity (see e.g. [1]).

We can state

Lemma 13. *If (A, R) is a relational system with transitive R and with Θ an equivalence relation on A then R/Θ is transitive if and only if $R \circ \Theta \circ R \subseteq \Theta \circ R \circ \Theta$.*

Proof. Let $a, b \in A$. Then $([a]\Theta, [b]\Theta) \in R/\Theta$ if and only if $(a, b) \in \Theta \circ R \circ \Theta$. Hence R/Θ is transitive if and only if $\Theta \circ R \circ \Theta \circ \Theta \circ R \circ \Theta \subseteq \Theta \circ R \circ \Theta$. But this is equivalent to $R \circ \Theta \circ R \subseteq \Theta \circ R \circ \Theta$. □

Lemma 14. *If (A, R) is a relational system and Θ an equivalence relation on A then the canonical mapping h from A to A/Θ is a strong homomorphism from $\mathcal{A}$ to $\mathcal{A}/\Theta$ and R/Θ is the least binary relation T on A/Θ such that h is a homomorphism from $\mathcal{A}$ to $(A/\Theta, T)$.*

Proof. This is evident. □

We can define one more modification of the notion of a homomorphism between relational systems by means of corresponding groupoids.

Definition 15. A *g-homomorphism* from a relational system $\mathcal{A} = (A, R)$ to a relational system $\mathcal{B} = (B, S)$ is a homomorphism h from $\mathcal{A}$ to $\mathcal{B}$ such that there exists a groupoid $(A, \cdot)$ corresponding to $\mathcal{A}$ such that for all $a, b, c, d \in A$ the equalities $h(a) = h(c)$ and $h(b) = h(d)$ together imply $h(ab) = h(cd)$.

Theorem 16. *If $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ are directed relational systems and $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ and $\mathcal{G}(\mathcal{B}) = (B, \circ)$ are corresponding groupoids then every homomorphism h from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{B})$ is a g-homomorphism from $\mathcal{A}$ to $\mathcal{B}$.*

Proof. Let $a, b, c, d \in A$. If $(a, b) \in R$ then $ab = b$ and hence $h(a) \circ h(b) = h(ab) = h(b)$ showing $(h(a), h(b)) \in S$. Moreover, if $a, b, c, d \in A$, $h(a) = h(c)$ and $h(b) = h(d)$ then $h(ab) = h(a) \circ h(b) = h(c) \circ h(d) = h(cd)$. □

The next theorem states that in some sense homomorphisms between relational systems are homomorphisms between corresponding groupoids.

Theorem 17. *If $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ are directed relational systems and h is a strong g-homomorphism from $\mathcal{A}$ onto $\mathcal{B}$ with the groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ then there exists a groupoid $\mathcal{G}(\mathcal{B}) = (B, \circ)$ corresponding to $\mathcal{B}$ such that h is a homomorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{B})$.*

Proof. According to Definition 15 there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that for each $a, b, c, d \in A$, if $h(a) = h(c)$ and $h(b) = h(d)$ then $h(ab) = h(cd)$. Define $h(x) \circ h(y) := h(xy)$ for all $x, y \in A$. According to Definition 15, $\circ$ is well-defined. Let $a, b \in A$. If $(h(a), h(b)) \in S$ then, since h is strong, there exists $(c, d) \in R$ with $h(c) = h(a)$ and $h(d) = h(b)$. Now $h(a) \circ h(b) = h(ab) = h(cd) = h(d) = h(b)$ according to Definition 15. If $(h(a), h(b)) \notin S$ then $(a, b) \notin R$ according to Definition 15 and hence $ab \in U_R(a, b)$, i.e. $(a, ab), (b, ab) \in R$ whence $(h(a), h(a) \circ h(b)) = (h(a), h(ab)) \in S$ and $(h(b), h(a) \circ h(b)) = (h(b), h(ab)) \in S$, i.e. $h(a) \circ h(b) \in U_S(h(a), h(b))$. This shows that $\mathcal{G}(\mathcal{B})$ corresponds to $\mathcal{B}$. Finally, h is a homomorphism from $\mathcal{G}(\mathcal{A})$ to $\mathcal{G}(\mathcal{B})$ since $h(xy) = h(x) \circ h(y)$ for all $x, y \in A$. □

Our next theorem contains some assertions concerning factor groupoids.

Theorem 18. *If $\mathcal{A} = (A, R)$ and $\mathcal{B} = (B, S)$ are directed relational groupoids then the following implications hold:*

(i) *If h is a g-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ then there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that* $\ker h \in \operatorname{Con} \mathcal{G}(\mathcal{A})$.

(ii) *If $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ is a groupoid corresponding to $\mathcal{A}$ and $\Theta \in \operatorname{Con} \mathcal{G}(\mathcal{A})$ then the canonical mapping h from A to A/Θ is a strong g-homomorphism from $\mathcal{A}$ to $\mathcal{A}/\Theta$.*

Proof. Let $a, b, c, d \in A$.

(i) Obviously, $\ker h$ is an equivalence relation on A. According to Definition 15 there exists a groupoid $\mathcal{G}(\mathcal{A}) = (A, \cdot)$ corresponding to $\mathcal{A}$ such that for each $a, b, c, d \in A$, if $h(a) = h(c)$ and $h(b) = h(d)$ then $h(ab) = h(cd)$, i.e. $(a, c), (b, d) \in \ker h$ implies $(ab, cd) \in \ker h$.

(ii) If $(a, b) \in R$ then $(h(a), h(b)) = ([a]\Theta, [b]\Theta) \in R/\Theta$. Moreover, if $h(a) = h(c)$ and $h(b) = h(d)$ then $[a]\Theta = h(a) = h(c) = [c]\Theta$ and $[b]\Theta = h(b) = h(d) = [d]\Theta$ and hence $h(ab) = [ab]\Theta = [a]\Theta \cdot [b]\Theta = [c]\Theta \cdot [d]\Theta = [cd]\Theta = h(cd)$. If, finally, $([c]\Theta, [d]\Theta) \in R/\Theta$ then there exists $(a, b) \in R$ with $([a]\Theta, [b]\Theta) = ([c]\Theta, [d]\Theta)$, i.e. with $h(a) = [a]\Theta = [c]\Theta$ and $h(b) = [b]\Theta = [d]\Theta$. □

Theorem 19. *Every homomorphism h from a groupoid $\mathcal{G} = (G, \cdot)$ to a groupoid $\mathcal{H} = (H, \circ)$ is a homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$ and from $\mathcal{A}^*(\mathcal{G})$ to $\mathcal{A}^*(\mathcal{H})$.*

Proof. Let $a, b \in G$. If $(a, b) \in R(\mathcal{G})$ then $ab = b$ and hence $h(a) \circ h(b) = h(ab) = h(b)$, i.e. $(h(a), h(b)) \in R(\mathcal{H})$. This shows that h is a homomorphism from

$\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$. If, on the other hand, $(a,b) \in R^*(\mathcal{G})$ then there exist $c, d \in G$ with $(a,b) \in \{(c, cd), (d, cd)\}$. Now $h(c), h(d) \in H$ and

$$\begin{aligned}(h(a), h(b)) &\in \{(h(c), h(cd)), (h(d), h(cd))\} \\ &= \{(h(c), h(c) \circ h(d)), (h(d), h(c) \circ h(d))\}\end{aligned}$$

and hence $(h(a), h(b)) \in R^*(\mathcal{H})$ showing that h is a homomorphism from $\mathcal{A}^*(\mathcal{G})$ to $\mathcal{A}^*(\mathcal{H})$. □

Remark 20. A homomorphism h from $\mathcal{G} = (G, \cdot)$ to $\mathcal{H} = (H, \circ)$ need not be a g-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$ as can be seen from the following example:

Example 21. Put $\mathcal{G} := (\{-1, 0, 1\}, \cdot)$ and $\mathcal{H} := (\{0, 1\}, \cdot)$ where $\cdot$ denotes the multiplication of integers, and let h denote the mapping $x \mapsto |x|$ from $\{-1, 0, 1\}$ to $\{0, 1\}$. Then h is a homomorphism from $\mathcal{G}$ to $\mathcal{H}$ and $R(\mathcal{G}) = \{(-1, 0), (0, 0), (1, -1), (1, 0), (1, 1)\}$. Let $(\{-1, 0, 1\}, *)$ be a groupoid corresponding to $\mathcal{A}(\mathcal{G})$. Then for $x, y \in \{-1, 0, 1\}$ we have $1 * x = x$ and $x * y = 0$ otherwise. Now $h(-1) = h(1)$ but $h((-1) * (-1)) = h(0) = 0 \neq 1 = h(1) = h(1 * 1)$ and hence h is not a g-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$.

The next theorem gives the final answer to the question whether a homomorphism between groupoids is a g-homomorphism between corresponding relational systems. The groupoids have to satisfy the identity natural for corresponding groupoids of relational systems.

Theorem 22. *If $\mathcal{G} = (G, \cdot)$ is a groupoid satisfying the identities $xx = x$ and $x(xy) = y(xy) = xy$ then every homomorphism h from $\mathcal{G}$ to a groupoid $\mathcal{H} = (H, \circ)$ is a g-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$.*

Proof. Obviously, $\mathcal{G}$ corresponds to $\mathcal{A}(\mathcal{G})$ and h is a homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$. If $a, b, c, d \in G$, $h(a) = h(c)$ and $h(b) = h(d)$ then $h(ab) = h(a) \circ h(b) = h(c) \circ h(d) = h(cd)$ and hence h is a g-homomorphism from $\mathcal{A}(\mathcal{G})$ to $\mathcal{A}(\mathcal{H})$. □

In the remaining part of the paper we point out a relationship between relation preserving functions and corresponding groupoids. This has an application in the theory of clones since both the set of functions preserving a given relation and the set of functions commuting with a given operation are clones.

Definition 23. An m-ary *operation* f on A is said to *preserve* a binary relation R on A if $(a_1, b_1), \ldots, (a_m, b_m) \in R$ implies

$$(f(a_1, \ldots, a_m), f(b_1, \ldots, b_m)) \in R.$$

An m-ary *operation* f and an n-ary *operation* g on A are said to *commute with each other* if

$$f(g(x_{11},\dots,x_{1n}),\dots,g(x_{m1},\dots,x_{mn})) = g(f(x_{11},\dots,x_{m1}),\dots,f(x_{1n},\dots,x_{mn}))$$

for all $x_{11},\dots,x_{1n},\dots,x_{m1},\dots,x_{mn} \in A$.

Remark 24. If $m = 2$ and $n = 1$ then f and g commute with each other if and only if g is an endomorphism of the groupoid (A, f).

Lemma 25. *If (A, R) is a directed relational system and $(A, \cdot)$ a corresponding groupoid then every m-ary operation f on A commuting with $\cdot$ preserves R.*

Proof. If $(a_1, b_1),\dots,(a_m, b_m) \in R$ then $a_i b_i = b_i$ for $i = 1,\dots,m$ and hence

$$f(a_1,\dots,a_m)f(b_1,\dots,b_m) = f(a_1b_1,\dots,a_mb_m) = f(b_1,\dots,b_m)$$

whence $(f(a_1,\dots,a_m), f(b_1,\dots,b_m)) \in R$. □

Example 26. If $\mathcal{A} = (A, \leqslant)$ is a poset which is a join-semilattice $(A, \vee)$ then $(A, \vee)$ is a groupoid corresponding to $(A, \leqslant)$ and Lemma 25 witnesses the fact that every join-preserving operation (i.e. every operation commuting with $\vee$) is order-preserving.

However, the assumption that f commutes with the operation of a corresponding groupoid is only sufficient but not necessary. It is e.g. elementary to show that an order preserving function on a join-semilattice need not commute with $\vee$. Hence we ask for a necessary and sufficient condition formulated in terms of a corresponding groupoid which ensures that a given operation preserves R. The answer is as follows:

Theorem 27. *If (A, R) is a directed relational system, $(A, \cdot)$ a corresponding groupoid and f an m-ary operation on A then the following conditions are equivalent:*
(i) *f preserves R.*
(ii) *f satisfies the identity*

$$f(x_1,\dots,x_m)f(x_1y_1,\dots,x_my_m) = f(x_1y_1,\dots,x_my_m).$$

Proof. Let $a_1,\dots,a_m,b_1,\dots,b_m \in A$.
(i) ⇒ (ii): Since $(a_1, a_1b_1),\dots,(a_m, a_mb_m) \in R$ according to Remark 5, we have

$$(f(a_1,\dots,a_m), f(a_1b_1,\dots,a_mb_m)) \in R$$

whence $f(a_1,\dots,a_m)f(a_1b_1,\dots,a_mb_m) = f(a_1b_1,\dots,a_mb_m)$.

(ii) ⇒ (i): If $(a_1, b_1), \ldots, (a_m, b_m) \in R$ then $a_1 b_1 = b_1, \ldots, a_m b_m = b_m$ and hence

$$\begin{aligned} f(a_1, \ldots, a_m) f(b_1, \ldots, b_m) &= f(a_1, \ldots, a_m) f(a_1 b_1, \ldots, a_m b_m) \\ &= f(a_1 b_1, \ldots, a_m b_m) = f(b_1, \ldots, b_m) \end{aligned}$$

whence $(f(a_1, \ldots, a_m), f(b_1, \ldots, b_m)) \in R$. □

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