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m^*-FUZZY BASICALLY DISCONNECTED SPACES IN SMOOTH FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, the concepts of m^*r-fuzzy $\tilde{g}$-open F_σ sets and m^*-fuzzy basically disconnected spaces are introduced in the sense of Šostak and Ramadan. Some interesting properties and characterizations are studied. Tietze extension theorem for m^*-fuzzy basically disconnected spaces is discussed.

Keywords: m^*r-fuzzy $\tilde{g}$-open F_σ set, m^*-fuzzy basically disconnected space, m^*r-fuzzy open function

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [13] in his classical paper. Fuzzy sets have applications in many fields such as information [9] and control [10]. In 1985, Šostak [11] introduced a new form of topological structure. In 1992, Ramadan [8] studied the concept of smooth fuzzy topological spaces. The concept of $\tilde{g}$-open sets was discussed by Rajesh and Erdal Ekici [7]. The concept of fuzzy basically disconnected spaces was introduced and studied in [12]. The notions of m-structures, m-spaces and m-continuity were introduced by Popa and Noiri [5], [6]. The concepts of r-fuzzy G_δ sets and r-fuzzy F_σ sets were introduced in [3]. In this paper, the concepts of m^*r-fuzzy $\tilde{g}$-open F_σ sets and m^*-fuzzy basically disconnected spaces are introduced in the sense of Šostak [11] and Ramadan [8]. Some interesting properties and characterizations are studied. Tietze extension theorem for m^*-fuzzy basically disconnected spaces is discussed as in [1].

Throughout this paper, let X be a nonempty set, $I = [0,1]$ and $I_0 = (0,1]$. For $\langle \in I$, $\zeta(x) = \langle$ for all $x \in X$. The characteristics function of $\lambda \in I^X$ is the function $1_\lambda\colon X \to I^X$ defined by $1_\lambda(x) = \lambda(x)$, $x \in X$, $r \in I_0$.

Definition 1.1 [11]. A function $T\colon I^X \to I$ is called a smooth fuzzy topology on X if it satisfies the following conditions:

(1) $T(\bar{0}) = T(\bar{1}) = 1$.

(2) $T(\mu_1 \wedge \mu_2) \geqslant T(\mu_1) \wedge T(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.

(3) $T\Big(\bigvee\limits_{i\in\Gamma} \mu_j\Big) \geqslant \bigwedge\limits_{j\in\Gamma} T(\mu_j)$ for any $\{\mu_j\}_{j\in\Gamma} \subset I^X$.

The pair (X, T) is called a smooth fuzzy topological space.

Remark 1.1. Let (X, T) be a smooth fuzzy topological space. Then, for each $r \in I_0$, $T_r = \{\mu \in I^X\colon\ T(\mu) \geqslant r\}$ is Chang's fuzzy topology on X.

Definition 1.2 [2]. Let (X, T) be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $C_T\colon I^X \times I_0 \to I^X$ is defined as follows: $C_T(\lambda, r) = \bigwedge\{\mu\colon\ \mu \geqslant \lambda, T(\bar{1} - \mu) \geqslant r\}$. For each $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(1) $C_T(\bar{0}, r) = \bar{0}$.

(2) $\lambda \leqslant C_T(\lambda, r)$.

(3) $C_T(\lambda, r) \vee C_T(\mu, r) = C_T(\lambda \vee \mu, r)$.

(4) $C_T(\lambda, r) \leqslant C_T(\lambda, s)$, if $r \leqslant s$.

(5) $C_T(C_T(\lambda, r), r) = C_T(\lambda, r)$.

Proposition 1.1 [2]. *Let (X, T) be a smooth fuzzy topological space. For each $\lambda \in I^X$, $r \in I_0$, an operator $I_T\colon I^X \times I_0 \to I^X$ is defined as follows: $I_T(\lambda, r) = \bigvee\{\mu\colon\ \mu \leqslant \lambda, T(\mu) \geqslant r\}$. For $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:*

(1) $I_T(\bar{1} - \lambda, r) = \bar{1} - C_r(\lambda, r)$.

(2) $I_T(\bar{1}, r) = \bar{1}$.

(3) $\lambda \geqslant I_T(\lambda, r)$.

(4) $I_T(\lambda, r) \wedge I_T(\mu, r) = I_T(\lambda \wedge \mu, r)$.

(5) $I_T(\lambda, r) \geqslant I_T(\lambda, s)$, *if* $r \leqslant s$.

(6) $I_T(I_T(\lambda, r), r) = I_T(\lambda, r)$.

Definition 1.3 [3]. Let (X, T) be a smooth fuzzy topological space, $r \in I_0$. For any $\lambda \in I^X$ and $r \in I^0$, λ is called

(1) an r-fuzzy G_δ set if $\lambda = \bigwedge\limits_{i=1}^{\infty} \lambda_i$ where each λ_i is such that $T(\lambda_i) \geqslant r$;

(2) an r-fuzzy F_σ set if $\lambda = \bigvee\limits_{i=1}^{\infty} \lambda_i$ where each $\bar{1} - \lambda_i$ is such that $T(\bar{1} - \lambda_i) \geqslant r$.

Definition 1.4 [8]. Let (X,T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$,

(1) λ is called r-fuzzy semi-closed (briefly, r-fsc) if $\lambda \geqslant I_T(C_T(\lambda,r),r)$;
(2) λ is called r-fuzzy semi-open (briefly, r-fso) if $\lambda \leqslant C_T(I_T(\lambda,r),r)$.

Definition 1.5 [8]. Let (X,T) be a smooth fuzzy topological space. For $\lambda \in I^X$ and $r \in I_0$,

(1) $SC_T(\lambda,r) = \bigwedge\{\mu \in I^X\colon\ \mu \geqslant \lambda, \mu$ is r-fuzzy semi-closed$\}$ is called the r-fuzzy semi-closure of λ;
(2) $SI_T(\lambda,r) = \bigvee\{\mu \in I^X\colon\ \mu \leqslant \lambda, \mu$ is r-fuzzy semi-open$\}$ is called the r-fuzzy semi-interior of λ.

Definition 1.6 [1]. Let (X,T) be a smooth fuzzy topological space. For any $\lambda \in I^X$ and $r \in I_0$, λ is called

(1) r-fuzzy g-closed if $C_T(\lambda,r) \leqslant \mu$ whenever $\lambda \leqslant \mu$ and μ is r-fuzzy semi-open. The complement of an r-fuzzy g-closed set is said to be an r-fuzzy g-open set;
(2) r-fuzzy $^*g$-closed if $C_T(\lambda,r) \leqslant \mu$ whenever $\lambda \leqslant \mu$ and μ is r-fuzzy $\hat{g}$-open. The complement of an r-fuzzy $^*g$-closed set is said to be an r-fuzzy $^*g$-open set;
(3) r-fuzzy $^{\#}g$-semiclosed (briefly r-f$^{\#}$gs-closed) if $SC_T(\lambda,r) \leqslant \mu$ whenever $\lambda \leqslant \mu$ and μ is r-fuzzy $^*g$-open. The complement of an r-fuzzy $^{\#}g$-semiclosed set is said to be an r-fuzzy $^{\#}g$-semiopen set (briefly r-$^{\#}$fgs-open set);
(4) r-fuzzy $\tilde{g}$-closed if $C_T(\lambda,r) \leqslant \mu$ whenever $\lambda \leqslant \mu$ and μ is r-fuzzy $^{\#}$fgs-open. The complement of an r-fuzzy $\tilde{g}$-closed set is said to be an r-fuzzy $\tilde{g}$-open set;

Definition 1.7 [5], [6]. A subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a minimal structure (briefly, m-structure) on X if $\varphi \in m_X$ and $X \in m_X$. By (X, m_X) we denote a non-empty subset X with a minimal structure m_X on X and call it an m-space. Each member of m_X is said to be m_X-open (or briefly, m-open) and the complement of an m_X-open set is said to be m_X-closed (or briefly, m-closed).

Notation 1.1. Let (X,T) be a smooth fuzzy topological space, $r \in I_0$.

(1) The family of r-fuzzy $\tilde{g}$ open sets in (X,T) is denoted by $\tilde{g}O(X,T)$.
(2) The family of r-fuzzy F_σ sets in (X,T) is denoted by $F_\sigma(X,T)$.

2. m^*-FUZZY BASICALLY DISCONNECTED SPACES

In this section, the concepts of m^*r-fuzzy $\tilde{g}$-open F_σ sets and m^*-fuzzy basically disconnected spaces are introduced. Some interesting properties and characterizations are studied.

Definition 2.1. Let X be a nonempty set and I^X a collection of all fuzzy sets in X. A subfamily m_X of I^X is called a minimal structure (briefly, m-structure) on X if $\bar{0} \in m_X$ and $\bar{1} \in m_X$.

Definition 2.2. Let (X, T) be a smooth fuzzy topological space, $r \in I_0$. Then the collection of the families $\tilde{g}O(X, T)$ and $F_\sigma(X, T)$ which is finer than the smooth fuzzy topology T on X is a minimal* structure (briefly, m^*-structure) on X, denoted by m^*_X. A nonempty set X with an m^*-structure m^*_X on X is denoted by (X, m^*_X) (or briefly, (X, m^*)) and it is called an m^*-smooth fuzzy space. Each member of m^*_X is said to be m^*r-fuzzy $\tilde{g}$-open F_σ and the complement of an m^*r-fuzzy $\tilde{g}$-open F_σ set is said to be m^*r-fuzzy $\tilde{g}$-closed G_δ.

Definition 2.3. A minimal structure m^*_X on a nonempty set X is said to have property $\mathcal{B}$ if the union of any family of m^*r-fuzzy $\tilde{g}$-open F_σ sets belonging to m^*_X belongs to m^*_X, $r \in I_0$.

Definition 2.4. Let (X, T) be a smooth fuzzy topological space with an m^*-structure m^*_X determined by T and let m^*_X have property $\mathcal{B}$. For any $\lambda \in I^X$ and $r \in I_0$, the $m^*_X r$-fuzzy $\tilde{g}G_\delta$-closure of λ and the $m^*_X r$-fuzzy $\tilde{g}F_\sigma$ interior of λ are defined as follows:

(1) $C_{m^*}(\lambda, r) = \bigwedge\{\mu \colon \lambda \leqslant \mu, \mu \text{ is } m^*r\text{-fuzzy } \tilde{g}\text{-closed } G_\delta\}$;
(2) $I_{m^*}(\lambda, r) = \bigvee\{\mu \colon \lambda \geqslant \mu, \mu \text{ is } m^*r\text{-fuzzy } \tilde{g}\text{-open } F_\sigma\}$.

R e m a r k 2.1. Let (X, T) be a smooth fuzzy topological space, $r \in I_0$. For any $\lambda \in I^X$, if $m^*_X = T$, then

(1) $C_{m^*}(\lambda, r) = C_T(\lambda, r)$;
(2) $I_{m^*}(\lambda, r) = I_T(\lambda, r)$.

Notation 2.1. Let (X, T) be a smooth fuzzy topological space with an m^*-structure determined by T. For $r \in I_0$, any $\lambda \in I^X$ which is both m^*r-fuzzy $\tilde{g}$-open F_σ and m^*r-fuzzy $\tilde{g}$-closed G_δ is denoted by m^*r-fuzzy $\tilde{g}$-COGF.

Definition 2.5. Let (X, T) be a smooth fuzzy topological space with an m^*-structure m^*_X determined by T and let m^*_X have property $\mathcal{B}$. The m^*-smooth fuzzy space (X, m^*) is said to be m^*-fuzzy basically disconnected if the m^*r-fuzzy $\tilde{g}G_\delta$-closure of every m^*r-fuzzy $\tilde{g}$-open F_σ set is m^*r-fuzzy $\tilde{g}$-open F_σ, $r \in I_0$.

Proposition 2.1. *For a smooth fuzzy topological space with an m^*-structure on X determined by T where m^*_X has property $\mathcal{B}$, the following conditions are equivalent:*

(a) *(X, m^*) is an m^*-fuzzy basically disconnected space.*
(b) *For each m^*r-fuzzy $\tilde{g}$-closed G_δ set λ, $I_{m^*}(\lambda, r)$ is m^*r-fuzzy $\tilde{g}$-closed G_δ, $r \in I_0$.*
(c) *For each m^*r-fuzzy $\tilde{g}$-open F_σ set λ,*
$C_{m^*}(\lambda, r) + C_{m^*}((\bar{1} - C_{m^*}(\lambda, r)), r) = \bar{1}$, $r \in I_0$.
(d) *For every pair of m^*r-fuzzy $\tilde{g}$-open F_σ sets λ and μ with $C_{m^*}(\lambda, r) + \mu = \bar{1}$, we have $C_{m^*}(\lambda, r) + C_{m^*}(\mu, r) = \bar{1}$, $r \in I_0$.*

Proof. (a) $\Rightarrow$ (b). Let $\lambda \in I^X$ be any m^*r-fuzzy $\tilde{g}$-closed G_δ set. Then $\bar{1} - \lambda$ is m^*r-fuzzy $\tilde{g}$-open F_σ. Now, $C_{m^*}(\bar{1} - \lambda, r) = \bar{1} - I_{m^*}(\lambda, r)$. By (a), $C_{m^*}(\bar{1} - \lambda, r)$ is m^*r-fuzzy $\tilde{g}$-open F_σ, which implies that $I_{m^*}(\lambda, r)$ is m^*r-fuzzy $\tilde{g}$-closed G_δ.

(b) $\Rightarrow$ (c). Let λ be any m^*r-fuzzy $\tilde{g}$-open F_σ set. Then

$$C_{m^*}(\lambda, r) + C_{m^*}((\bar{1} - C_{m^*}(\lambda, r)), r) = C_{m^*}(\lambda, r) + C_{m^*}((I_{m^*}(\bar{1} - \lambda, r)), r). \tag{2.1}$$

Since λ is m^*r-fuzzy $\tilde{g}$-open F_σ, $\bar{1} - \lambda$ is m^*r-fuzzy $\tilde{g}$-closed G_δ. Hence by (b), $I_{m^*}(\bar{1} - \lambda, r)$ is m^*r-fuzzy $\tilde{g}$-closed G_δ. Therefore, by (2.1)

$$\begin{aligned} C_{m^*}(\lambda, r) + C_{m^*}((\bar{1} - C_{m^*}(\lambda, r)), r) &= C_{m^*}(\lambda, r) + (I_{m^*}(\bar{1} - \lambda, r)) \\ &= C_{m^*}(\lambda, r) + \bar{1} - C_{m^*}(\lambda, r) \\ &= \bar{1}. \end{aligned}$$

Therefore, $C_{m^*}(\lambda, r) + C_{m^*}((\bar{1} - C_{m^*}(\lambda, r)), r) = \bar{1}$.

(c) $\Rightarrow$ (d). Let λ and μ be m^*r-fuzzy $\tilde{g}$-open F_σ sets with

$$C_{m^*}(\lambda, r) + \mu = \bar{1}. \tag{2.2}$$

Then by (c) we have $\bar{1} = C_{m^*}(\lambda, r) + C_{m^*}((\bar{1} - C_{m^*}(\lambda, r)), r) = C_{m^*}(\lambda, r) + C_{m^*}(\mu, r)$.

Therefore, $C_{m^*}(\lambda, r) + C_{m^*}(\mu, r) = \bar{1}$.

(d) $\Rightarrow$ (a). Let λ be an m^*r-fuzzy $\tilde{g}$-open F_σ set. Put $\mu = \bar{1} - C_{m^*}(\lambda, r)$. Then $C_{m^*}(\lambda, r) + \mu = \bar{1}$. Therefore by (d), $C_{m^*}(\lambda, r) + C_{m^*}(\mu, r) = \bar{1}$. This implies that $C_{m^*}(\lambda, r)$ is m^*r-fuzzy $\tilde{g}$-open F_σ and so (X, T) is m^*-fuzzy basically disconnected. □

Proposition 2.2. *Let (X,T) be a smooth fuzzy topological space with an m^*-structure m_X^* determined by T and let m_X^* possess property $\mathcal{B}$. Then (X,m^*) is m^*-fuzzy basically disconnected if and only if for all m^*r-fuzzy $\tilde{g}$-open F_σ sets λ and m^*r-fuzzy $\tilde{g}$-closed sets μ such that $\lambda \leqslant \mu$, $C_{m^*}(\lambda,r) \leqslant I_{m^*}(\mu,r)$, $r \in I_0$.*

P r o o f. Let λ be m^*r-fuzzy $\tilde{g}$-open F_σ and let μ be m^*r-fuzzy $\tilde{g}$-closed G_δ with $\lambda \leqslant \mu$. Then by (b) of Proposition 2.1, $I_{m^*}(\mu,r)$ is m^*r-fuzzy $\tilde{g}$-closed G_δ. Also, since λ is m^*r-fuzzy $\tilde{g}$-open F_σ, $C_{m^*}(\lambda,r) \leqslant I_{m^*}(\mu,r)$.

Conversely, let μ be any m^*r-fuzzy $\tilde{g}$-closed G_δ set. Then $I_{m^*}(\mu,r) \in I^X$ is m^*r-fuzzy $\tilde{g}$-open F_σ and $I_{m^*}(\mu,r) \leqslant \mu$. Therefore by assumption, $C_{m^*}((I_{m^*}(\mu,r),r)) \leqslant I_{m^*}(\mu,r)$. This implies that $I_{m^*}(\mu,r)$ is m^*r-fuzzy $\tilde{g}$-closed G_δ. Hence by (b) of Proposition 2.1, it follows that (X,m^*) is m^*-fuzzy basically disconnected. □

R e m a r k 2.2. Let (X,m^*) be any m^*-fuzzy basically disconnected space. Let $\{\lambda_i, \bar{1}-\mu_i/i \in \mathbb{N}\}$ be collection such that $\lambda_i's$ are m^*r-fuzzy $\tilde{g}$-open F_σ and $\mu_i's$ are m^*r-fuzzy $\tilde{g}$-closed G_δ, and let λ and μ be m^*r-fuzzy $\tilde{g}$-COGF sets. If $\lambda_i \leqslant \lambda \leqslant \mu_j$ and $\lambda_i \leqslant \mu \leqslant \mu_j$ for all $i,j \in \mathbb{N}$, then there exists an m^*r-fuzzy $\tilde{g}$-COGF set γ such that $C_{m^*}(\lambda_i,r) \leqslant \gamma \leqslant I_{m^*}(\mu_j,r)$ for all $i,j \in \mathbb{N}$, $r \in I_0$.

P r o o f. By Proposition 2.2, $C_{m^*}(\lambda_i,r) \leqslant C_{m^*}(\lambda,r) \wedge I_{m^*}(\mu,r) \leqslant I_{m^*}(\mu_j,r)$ for all $i,j \in \mathbb{N}$. Therefore, $\gamma = C_{m^*}(\lambda,r) \wedge I_{m^*}(\mu,r)$ is an m^*r-fuzzy $\tilde{g}$-COGF set satisfying the required conditions. □

Proposition 2.3. *Let (X,m^*) be any m^*-fuzzy basically disconnected space. Let $\{\lambda_l\}_{l\in Q}$ and $\{\mu_l\}_{l\in Q}$ be monotone increasing collections of m^*r-fuzzy $\tilde{g}$-open F_σ sets and m^*r-fuzzy $\tilde{g}$-closed G_δ sets of (X,m^*) and suppose that $\lambda_{q_1} \leqslant \mu_{q_2}$ whenever $q_1 < q_2$ (Q is the set of all rational numbers). Then there exists a monotone increasing collection $\{\gamma_l\}_{l\in Q}$ of m^*r-fuzzy $\tilde{g}$-COGF sets of (X,m^*) such that $C_{m^*}(\lambda_{q_1},r) \leqslant \gamma_{q_2}$ and $\gamma_{q_1} \leqslant I_{m^*}(\mu_{q_2},r)$ whenever $q_1 < q_2$, $r \in I_0$.*

P r o o f. Let us arrange all rational numbers into a sequence $\{q_n\}$ (without repetitions). For every $n \geqslant 2$, we shall define inductively a collection $\{\gamma_{q_i}/1 \leqslant i \leqslant n\} \subset I^X$ such that

$$(\mathbf{S_n}) \quad C_{m^*}(\lambda_q,r) \leqslant \gamma_{q_i} \quad \text{if } q < q_i, \qquad \gamma_{q_i} \leqslant I_{m^*}(\mu_q,r) \quad \text{if } q_i < q, \quad \text{for all } i < n$$

By Proposition 2.2, the countable collections $\{C_{m^*}(\lambda_q,r)$ and $\{I_{m^*}(\mu_q,r)\}$ satisfy $C_{m^*}(\lambda_{q_1},r) \leqslant I_{m^*}(\mu_{q_2},r)$ if $q_1 < q_2$. By Remark 2.2, there exists an m^*r-fuzzy $\tilde{g}$-COGF set δ_1 such that $C_{m^*}(\lambda_{q_1},r) \leqslant \delta_1 \leqslant I_{m^*}(\mu_{q_2},r)$. Setting $\gamma_{q_1} = \delta_1$, we get $(\mathbf{S_2})$.

Define $\psi = \bigvee\{\gamma_{q_i} \colon i < n, q_i < q_n\} \vee \lambda_{q_n}$ and $\varphi = \bigwedge\{\gamma_{q_j} \colon j < n, q_j > q_n\} \wedge \mu_{q_n}$. Then we have $C_{m^*}(\gamma_{q_i},r) \leqslant C_{m^*}(\psi,r) \leqslant I_{m^*}(\gamma_{q_j},r)$ and $C_{m^*}(\gamma_{q_i},r) \leqslant I_{m^*}(\varphi,r) \leqslant$

$I_{m^*}(\gamma_{q_j}, r)$ whenever $q_i < q_n < q_j$ $(i, j < n)$, as well as $\lambda_q \leqslant C_{m^*}(\psi, r) \leqslant \mu_q$ and $\lambda_q \leqslant I_{m^*}(\varphi, r) \leqslant \mu_{q'}$ whenever $q < q_n < q'$. This shows that the countable collections $\{\gamma_{q_i} \colon i < n, q_i < q_n\} \cup \{\lambda_q \colon q < q_n\}$ and $\{\gamma_{q_j} \colon j < n, q_j > q_n\} \cup \{\mu_q \colon q > q_n\}$ together with ψ and φ fulfil all the conditions of Remark 2.2. Hence, there exists an m^*r-fuzzy $\tilde{g}$-COGF set δ_n such that $C_{m^*}(\delta_n, r) \leqslant \mu_q$ if $q_n < q$, $\lambda_q \leqslant I_{m^*}(\delta_n, r)$ if $q < q_n$, $C_{m^*}(\gamma_{q_i}, r) \leqslant I_{m^*}(\delta_n, r)$ if $q_i < q_n C_{m^*}(\delta_n, r) \leqslant I_{m^*}(\gamma_{q_i}, r)$ if $q_n < q_j$ where $1 \leqslant i, j \leqslant n-1$. Now, setting $\gamma_{q_n} = \delta_n$ we obtain the fuzzy sets $\gamma_{q_1}, \gamma_{q_2}, \gamma_{q_3}, \ldots, \gamma_{q_n}$ that satisfy $(\mathbf{S_{n+1}})$. Therefore, the collection $\{\gamma_{q_i} \colon i = 1, 2, \ldots\}$ has the required property. □

3. Properties and characterizations of m^*-fuzzy basically disconnected spaces

In this section, the concept of m^*-fuzzy continuous functions is introduced. In this regard, various properties and characterizations are discussed.

Definition 3.1. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces with the m^*-structures m_1^* and m_2^* determined by T and S respectively, and let both m_1^* and m_2^* have property $\mathcal{B}$. A function $f \colon (X, m_1^*) \to (Y, m_2^*)$ is called m^*-fuzzy irresolute if $f^{-1}(\lambda) \in I^X$ is m^*r-fuzzy $\tilde{g}$-closed G_δ, for every m^*r-fuzzy closed $\tilde{g}$-G_δ set $\lambda \in I^Y$, $r \in I_0$.

Definition 3.2. Let Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces with the m^*-structures m_1^* and m_2^* determined by T and S respectively, and let both m_1^* and m_2^* have property $\mathcal{B}$. A function $f \colon (X, m_1^*) \to (Y, m_2^*)$ is called m^*-fuzzy open if $f(\lambda) \in I^Y$ is m^*r-fuzzy $\tilde{g}$-open F_σ, for every m^*r-fuzzy $\tilde{g}$-open F_σ set $\lambda \in I^X$, $r \in I_0$.

Proposition 3.1. *Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces with the m^*-structures m_1^* and m_2^* determined by T and S respectively, and let both m_1^* and m_2^* have property $\mathcal{B}$. A function $f \colon (X, m_1^*) \to (Y, m_2^*)$ is m^*-fuzzy irresolute iff $f(C_{m^*}(\lambda, r)) \leqslant C_{m^*}(f(\lambda), r)$, for every $\lambda \in I^X$, $r \in I_0$.*

P r o o f. Suppose that f is m^*-fuzzy irresolute and $\lambda \in I^X$. Then $C_{m^*}(f(\lambda), r) \in I^Y$ is m^*r-fuzzy $\tilde{g}$-closed G_δ. By hypothesis, $f^{-1}(C_{m^*}(f(\lambda), r)) \in I^X$ is m^*r-fuzzy $\tilde{g}$-closed G_δ. Also, $\lambda \leqslant f^{-1}(f(\lambda)) \leqslant f^{-1}(C_{m^*}(f(\lambda), r))$. Hence by the definition of the m^*r-fuzzy $\tilde{g}G_\delta$ closure, $C_{m^*}(\lambda, r) \leqslant f^{-1}(C_{m^*}(f(\lambda), r))$. That is, $f(C_{m^*}(\lambda, r)) \leqslant C_{m^*}(f(\lambda), r)$.

Conversely, suppose that $\lambda \in I^Y$ is m^*r-fuzzy $\tilde{g}$-closed G_δ. Now by hypothesis, $f(C_{m^*}(f^{-1}(\lambda), r)) \leqslant C_{m^*}(f(f^{-1}(\lambda)), r)$. This implies $C_{m^*}(f^{-1}(\lambda), r) \leqslant f^{-1}(\lambda)$. So

$f^{-1}(\lambda) = C_{m^*}(f^{-1}(\lambda), r)$. That is, $f^{-1}(\lambda) \in I^X$ is m^*r-fuzzy $\tilde{g}$-closed G_δ and so f is m^*-fuzzy irresolute. □

Proposition 3.2. *Let Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces with the m^*-structures m_1^* and m_2^* determined by T and S respectively, and let both m_1^* and m_2^* have property $\mathcal{B}$. Let $f\colon (X, m_1^*) \to (Y, m_2^*)$ be an m^*-fuzzy open surjective function. Then $f^{-1}(C_{m^*}(\lambda, r)) \leqslant C_{m^*}(f^{-1}(\lambda), r)$ for every $\lambda \in I^Y$, $r \in I_0$.*

Proof. Let $\lambda \in I^Y$ and let $\mu = f^{-1}(\bar{1} - \lambda)$. Then $I_{m^*}(f^{-1}(\bar{1} - \lambda), r) = I_{m^*}(\mu, r) \in I^X$ is m^*r-fuzzy $\overline{g}$-open F_σ. Now, $I_{m^*}(\mu, r) \leqslant \mu$. Hence, $f(I_{m^*}(\mu, r)) \leqslant f(\mu)$. That is, $I_{m^*}(f(I_{m^*}(\mu, r)), r) \leqslant I_{m^*}(f(\mu), r)$. Since f is m^*-fuzzy open, $f(I_{m^*}(\mu, r)) \in I^Y$ is m^*r-fuzzy $\tilde{g}$-open F_σ. Therefore, $f(I_{m^*}(\mu, r)) \leqslant I_{m^*}(f(\mu), r))$ $= I_{m^*}(\bar{1} - \lambda, r)$. Hence, $I_{m^*}(f^{-1}(\bar{1} - \lambda), r) = I_{m^*}(\mu, r) \leqslant f^{-1}(I_{m^*}(\bar{1} - \lambda), r)$. Therefore, $\bar{1} - I_{m^*}(f^{-1}(\bar{1} - \lambda), r) = \bar{1} - I_{m^*}(\mu, r) \geqslant \bar{1} - f^{-1}(I_{m^*}(\bar{1} - \lambda), r)$. Hence, $f^{-1}(\bar{1} - I_{m^*}(\bar{1} - \lambda), r) \leqslant C_{m^*}((\bar{1} - f^{-1}(\bar{1} - \lambda)), r)$. Therefore, $f^{-1}(C_{m^*}(\lambda, r)) \leqslant C_{m^*}(f^{-1}(\lambda), r)$. □

Proposition 3.3. *Let (X, m_1^*) be any m^*-fuzzy basically disconnected space and let (Y, S) be any smooth fuzzy topological space with an m^*-structure m_2^* determined by S where m_2^* has property $\mathcal{B}$. Let $f\colon (X, m_1^*) \to (Y, m_2^*)$ be an m^*-fuzzy irresolute, m^*-fuzzy open and surjective function. Then (Y, m_2^*) is m^*-fuzzy basically disconnected.*

Proof. Let $\lambda \in I^Y$ be any m^*-fuzzy $\tilde{g}$-open F_σ set. Since f is m^*-fuzzy irresolute, $f^{-1}(\lambda) \in I^X$ is m^*r-fuzzy $\tilde{g}$-open F_σ. Therefore by an assumption on (X, m_1^*), it follows that $C_{m^*}(f^{-1}(\lambda), r) \in I^X$ is m^*r-fuzzy $\tilde{g}$-open F_σ. As f is m^*-fuzzy open, $f(C_{m^*}(f^{-1}(\lambda), r)) \in I^Y$ is m^*r-fuzzy $\tilde{g}$-open F_σ. By Proposition 3.2, $f^{-1}(C_{m^*}(\lambda, r)) \leqslant C_{m^*}(f^{-1}(\lambda), r)$ and hence, $f(f^{-1}(C_{m^*}(\lambda, r))) = C_{m^*}(\lambda, r) \leqslant f(C_{m^*}(f^{-1}(\lambda), r)) \leqslant C_{m^*}(f(f^{-1}(\lambda), r)) = C_{m^*}(\lambda, r)$ by Proposition 3.1. Thus $C_{m^*}(\lambda, r) = f(C_{m^*}(f^{-1}(\lambda), r))$ and therefore, $C_{m^*}(\lambda, r) \in I^Y$ is m^*r-fuzzy $\tilde{g}$-open F_σ, proving that (Y, m_2^*) is m^*-fuzzy basically disconnected. □

Definition 3.3. Let (X, T) be a smooth fuzzy topological space with an m^*-structure m_X^* determined by T and let m_X^* possess property $\mathcal{B}$. A function $f\colon X \to R(I)$ is called lower (upper) m^*-fuzzy continuous if $f^{-1}(R_t)(f^{-1}(L_t))$ is m^*r-fuzzy $\tilde{g}$-open F_σ (m^*r-fuzzy $\tilde{g}$-open F_σ/m^*r-fuzzy $\tilde{g}$-closed G_δ), for each $t \in \mathbb{R}$, $r \in I_0$.

Proposition 3.4. *Let (X, T) be a smooth fuzzy topological space with an m^*-structure m^*_X determined by T and let m^*_X have property $\mathcal{B}$. For $\lambda \in I^X$ and $r \in I_0$, let $f\colon X \to R(I)$ be such that*

$$f(x)(t) = \begin{cases} 1 & \text{if } t < 0, \\ \lambda(x) & \text{if } 0 \leqslant t \leqslant 1, \\ 0 & \text{if } t > 0 \end{cases}$$

for all $x \in X$. Then f is lower (upper) m^*-fuzzy continuous iff λ is m^*r-fuzzy $\tilde{g}$-open F_σ (m^*r-fuzzy $\tilde{g}$-open F_σ/m^*r-fuzzy $\tilde{g}$-closed G_δ), $r \in I_0$.

P r o o f.

$$f^{-1}(R_t) = \begin{cases} 1 & \text{if } t < 0, \\ \lambda & \text{if } 0 \leqslant t \leqslant 1, \\ 0 & \text{if } t > 1 \end{cases}$$

implies that f is lower m^*-fuzzy continuous iff λ is m^*r-fuzzy $\tilde{g}$-open F_σ.

$$f^{-1}(L_t) = \begin{cases} 1 & \text{if } t < 0, \\ \lambda & \text{if } 0 < t \leqslant 1, \\ 0 & \text{if } t > 1 \end{cases}$$

implies that f is upper m^*-fuzzy continuous iff λ is m^*r-fuzzy $\tilde{g}$-closed G_δ. □

Proposition 3.5. *Let (X, T) be a smooth fuzzy topological space with an m^*-structure m^*_X determined by T and let m^*_X have property $\mathcal{B}$; let $\lambda \in I^X$. Then 1_λ is lower (upper) m^*-fuzzy continuous iff λ is m^*r-fuzzy $\tilde{g}$-open F_σ (m^*r-fuzzy $\tilde{g}$-open F_σ/m^*r-fuzzy $\tilde{g}$-closed G_δ), $r \in I_0$.*

P r o o f. The proof follows from Proposition 3.4. □

Definition 3.4. Let (X, T) and (Y, S) be any two smooth fuzzy topological spaces with the m^*-structures m^*_1 and m^*_2 determined by T and S respectively and let both m^*_1 and m^*_2 have property $\mathcal{B}$. A function $f\colon (X, m^*_1) \to (Y, m^*_2)$ is called strongly m^*-fuzzy continuous if $f^{-1}(\lambda) \in I^X$ is m^*r-fuzzy $\tilde{g}$-open F_σ/m^*r-fuzzy $\tilde{g}$-closed G_δ, for every m^*r-fuzzy $\tilde{g}$-open F_σ set $\lambda \in I^Y$, $r \in I_0$.

Proposition 3.6. *Let (X, T) be a smooth fuzzy topological space with an m^*-structure m^*_X determined by T and let m^*_X possess property $\mathcal{B}$. Then for $r \in I_0$, the following conditions are equivalent:*

(a) *(X, m^*) is an m^*-fuzzy basically disconnected space.*

(b) *If $g, h\colon X \to R(I)$ where g is lower m^*-fuzzy continuous, h is upper m^*-fuzzy continuous, then there exists $f \in C_{Sm^*}(X, m^*)$ such that $g \leqslant f \leqslant h$.*
$[C_{Sm^*}(X, m^*)$ = collection of all strongly m^*-fuzzy continuous functions on X with values in $R(I)]$.

(c) *If $\bar{1} - \lambda$, μ are m^*r-fuzzy $\tilde{g}$-open F_σ sets such that $\mu \leqslant \lambda$, then there exists a strongly m^*-fuzzy continuous $f\colon X \to I^X$ such that $\mu \leqslant (\bar{1} - L_1)f \leqslant R_0 f \leqslant \lambda$.*

P r o o f. (a) $\Rightarrow$ (b). Define $H_k = L_k h$ and $G_k = (\bar{1} - R_k)g$, $k \in Q$. Thus we have two monotone increasing families of m^*r-fuzzy $\tilde{g}$-open F_σ sets and m^*r-fuzzy $\tilde{g}$-closed sets of (X, m^*). Moreover $H_k \leqslant G_s$ if $k < s$. By Proposition 2.3, there exists a monotone increasing family $\{F_k\}_{k \in Q}$ of m^*r-fuzzy $\tilde{g}$-COGF sets of (X, m^*) sets such that $C_{m^*}(H_k, r) \leqslant F_s$ and $F_k \leqslant I_{m^*}(G_s, r)$ whenever $k < s$. Letting $V_t = \bigwedge\limits_{k<t} (\bar{1} - F_k)$ for all $t \leqslant R$, we define a monotone decreasing family $\{V_t\colon t \in \mathbb{R}\} \subset I^X$. Moreover, we have $C_{m^*}(V_t, r) \leqslant I_{m^*}(V_s, r)$ whenever $s < t$. We have

$$\begin{aligned}\bigvee_{t\in\mathbb{R}} V_t &= \bigvee_{t\in\mathbb{R}} \bigwedge_{k<t} (\bar{1} - F_k) \geqslant \bigvee_{t\in\mathbb{R}} \bigwedge_{k<t} (\bar{1} - G_k) \\ &= \bigvee_{t\in\mathbb{R}} \bigwedge_{k<t} g^{-1}(R_k) = g^{-1}\Big(\bigvee_{t\in\mathbb{R}} R_t\Big) = \bar{1}.\end{aligned}$$

Similarly, $\bigwedge\limits_{t\in\mathbb{R}} V_t = 0$. We now define a function $f\colon X \to R(I)$ possessing the required properties. Let $f(x)(t) = V_t(x)$ for all $x \in X$ and $t \in \mathbb{R}$. By the above discussion it follow that f is well defined. To prove f is strongly m^*-fuzzy continuous, we observe that

$$\bigvee_{s>t} V_s = \bigvee_{s>t} I_{m^*}(V_s, r) \quad \text{and}$$
$$\bigwedge_{s<t} V_s = \bigwedge_{s<t} C_{m^*}(V_s, r).$$

Then $f^{-1}(R_t) = \bigvee\limits_{s>t} V_s = \bigvee\limits_{s>t} I_{m^*}(V_s, r)$ is m^*r-fuzzy $\tilde{g}$-COGF. And $f^{-1}(L_t') = \bigwedge\limits_{s<t} V_s = \bigwedge\limits_{s<t} C_{m^*}(V_s, r)$ is m^*r-fuzzy $\tilde{g}$-COGF. Therefore, f is strongly m^*-fuzzy continuous. To conclude the proof it remains to show that $g \leqslant f \leqslant h$. That is, $g^{-1}(\bar{1} - L_t) \leqslant f^{-1}(\bar{1} - L_t) \leqslant h^{-1}(\bar{1} - L_t)$ and $g^{-1}(R_t) \leqslant f^{-1}(R_t) \leqslant h^{-1}(R_t)$ for each $t \in \mathbb{R}$. We have

$$\begin{aligned}g^{-1}(\bar{1} - L_t) &= \bigwedge_{s<t} g^{-1}(\bar{1} - L_s) = \bigwedge_{s<t} \bigwedge_{k<s} g^{-1}(R_k) \\ &= \bigwedge_{s<t} \bigwedge_{k<s} (\bar{1} - G_k) \leqslant \bigwedge_{s<t} \bigwedge_{k<s} (\bar{1} - F_k) \\ &= \bigwedge_{s<t} V_s = f^{-1}(\bar{1} - L_t) \quad \text{and}\end{aligned}$$

$$
\begin{aligned}
f^{-1}(\bar{1}-L_t) &= \bigwedge_{s<t} V_s = \bigwedge_{s<t}\bigwedge_{k<s}(\bar{1}-F_k)\\
&\leqslant \bigwedge_{s<t}\bigwedge_{k<s}(\bar{1}-H_k) = \bigwedge_{s<t}\bigwedge_{k<s} h^{-1}(\bar{1}-L_k)\\
&= \bigwedge_{s<t} h^{-1}(\bar{1}-Ls) = h^{-1}(\bar{1}-L_t).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
g^{-1}(R_t) &= \bigvee_{s>t} g^{-1}(R_s) = \bigvee_{s>t}\bigvee_{k>s} g^{-1}(R_k)\\
&= \bigvee_{s>t}\bigvee_{k>s}(\bar{1}-G_k)) \leqslant \bigvee_{s>t}\bigwedge_{k<s}(\bar{1}-F_k)\\
&= \bigvee_{s>t} V_s = f^{-1}(R_t) \quad \text{and}\\
f^{-1}(R_t) &= \bigvee_{s>t} V_s = \bigvee_{s>t}\bigwedge_{k<s}(\bar{1}-F_k)\\
&\leqslant \bigvee_{s>t}\bigvee_{k>s}(\bar{1}-H_k) = \bigvee_{s>t}\bigvee_{k>s} h^{-1}(\bar{1}-L_k)\\
&= \bigvee_{s>t} h^{-1}(R_s) = h^{-1}(R_t).
\end{aligned}
$$

Thus, (b) is proved.

(b) $\Rightarrow$ (c). Suppose that λ is m^*r-fuzzy $\tilde{g}$-closed G_δ and μ is m^*r-fuzzy $\tilde{g}$-open F_σ such that $\mu \leqslant \lambda$. Then $1_\mu \leqslant 1_\lambda$ where 1_μ, 1_λ are lower and upper m^*-fuzzy continuous functions, respectively. Hence by (b), there exists a strong m^*-fuzzy continuous function $f\colon X \to R(I)$ such that $1_\mu \leqslant f \leqslant 1_\lambda$. Clearly, $f(x) \in I^X$ for all $x \in X$ and $\mu = (\bar{1}-L_1)1_\mu \leqslant (\bar{1}-L_1)f \leqslant R_0 f \leqslant R_0 1_\lambda = \lambda$. Therefore, $\mu \leqslant (\bar{1}-L_1)f \leqslant R_0 f \leqslant \lambda$.

(c) $\Rightarrow$ (a). $(\bar{1}-L_1)f$ and $R_0 f$ are m^*r-fuzzy $\tilde{g}$-COGF sets. By Proposition 2.2, (X, m^*) is an m^*-fuzzy basically disconnected space. □

4. Tietze extension theorem

In this section, Tietze Extension Theorem for m^*-fuzzy basically disconnected spaces is studied.

Proposition 4.1. *Let (X, m^*) be an m^*-fuzzy basically disconnected space and let $A \subset X$ be such that 1_A is m^*r-fuzzy $\tilde{g}$-open F_σ. Let $f\colon (A, m^*/A) \to I^X$ be strong m^*-fuzzy continuous. Then f has a strong m^*-fuzzy continuous extension over (X, m^*), $r \in I_0$.*

P r o o f. Let $g, h\colon X \to I^X$ be such that $g = f = h$ on A and $g(x) = 0$, $h(x) = 1$ if $x \notin A$. We now have

$$R_t g = \begin{cases} \mu_t \wedge 1_A & \text{if } t \leqslant 0, \\ 1 & \text{if } t < 0 \end{cases}$$

where μ_t is m^*r-fuzzy $\tilde{g}$-open F_σ and is such that $\mu_t/A = R_t f$ and

$$L_t h = \begin{cases} \lambda_t \wedge 1_A & \text{if } t \leqslant 1, \\ 1 & \text{if } t > 1 \end{cases}$$

where λ_t is m^*r-fuzzy $\tilde{g}$-COGF and is such that $\lambda_t/A = L_t f$. Thus, g is lower m^*-fuzzy continuous and h is upper m^*-fuzzy continuous with $g \leqslant h$. By Proposition 3.6, there is a strong m^*-fuzzy continuous function $F\colon X \to I^X$ such that $g \leqslant F \leqslant h$. Hence $F \equiv f$ on A. □

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