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TWO IDENTITIES RELATED TO DIRICHLET  
CHARACTER OF POLYNOMIALS

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*Abstract.* Let  $q$  be a positive integer,  $\chi$  denote any Dirichlet character mod  $q$ . For any integer  $m$  with  $(m, q) = 1$ , we define a sum  $C(\chi, k, m; q)$  analogous to high-dimensional Kloosterman sums as follows:

$$C(\chi, k, m; q) = \sum_{a_1=1}^{q'} \sum_{a_2=1}^{q'} \dots \sum_{a_k=1}^{q'} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}),$$

where  $a \cdot \bar{a} \equiv 1 \pmod{q}$ . The main purpose of this paper is to use elementary methods and properties of Gauss sums to study the computational problem of the absolute value  $|C(\chi, k, m; q)|$ , and give two interesting identities for it.

*Keywords:* Dirichlet character of polynomials, sum analogous to Kloosterman sum, identity, Gauss sum

*MSC 2010:* 11L05

## 1. INTRODUCTION

Let  $q$  be a positive integer,  $\chi$  denote any Dirichlet character mod  $q$ . For any integer  $m$  with  $(m, q) = 1$ , we define a sum analogous to Kloosterman sums  $C(\chi, k, m; q)$  as follows:

$$C(\chi, k, m; q) = \sum_{a_1=1}^{q'} \sum_{a_2=1}^{q'} \dots \sum_{a_k=1}^{q'} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}),$$

where  $a \cdot \bar{a} \equiv 1 \pmod{q}$ .

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It is clear that this sum very similar to a hyper-Kloosterman sum, it is also a special case of the general character sums of the polynomials

$$(1) \quad \sum_{x_1=N_1+1}^{N_1+M_1} \sum_{x_2=N_2+1}^{N_2+M_2} \cdots \sum_{x_s=N_s+1}^{N_s+M_s} \chi(f(x_1, x_2, \dots, x_s)),$$

where  $M_i$  and  $N_i$  are any positive integers, and  $f(x_1, x_2, \dots, x_s)$  is a polynomial.

It is a very important and difficult problem in analytic number theory to give a sharper upper bound estimate for (1). If  $q = p$  is an odd prime and  $s = 1$ , then Weil (see [1]) obtained the following important conclusion:

Let  $\chi$  be a  $q$ th-order character mod  $p$ . If  $f(x)$  is not a perfect  $q$ th power mod  $p$ , then we have the estimate

$$(2) \quad \sum_{x=N+1}^{N+M} \chi(f(x)) \ll p^{\frac{1}{2}} \ln p,$$

where  $A \ll B$  denotes  $|A| < cB$  for some constant  $c$ , which in this case depends only on the degree of  $f(x)$ . Some related results can also be found in [2], [5] and [6]. The main term  $p^{\frac{1}{2}}$  in (2) is the best possible. In fact, Zhang Wenpeng and Yao Weili [6] found some polynomials  $f(x) = (x - r)^m(x - s)^n$  such that

$$\sum_{a=1}^q \chi((a - r)^m(a - s)^n) = \sqrt{q} \chi((sm - rm)^m(rn - sn)^n) \bar{\chi}((m + n)^{m+n}),$$

where  $(r - s, q) = 1$ ,  $m, n$  and  $\chi$  also satisfy some special conditions.

In this paper, we shall use elementary methods and properties of Gauss sums to study the computational problem of  $C(\chi, k, m; q)$ , and give two interesting identities. That is, we shall prove the following conclusions:

**Theorem 1.** *Let  $q$  be an odd number,  $k$  and  $m$  be positive integers such that  $(km, q) = 1$  and  $(k + 1, \varphi(q)) = 1$ . Then for any primitive character  $\chi$  mod  $q$ , we have the identity*

$$|C(\chi, k, m; q)| = \left| \sum_{a_1=1}^q \sum_{a_2=1}^q \cdots \sum_{a_k=1}^q \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \right| = \sqrt{q^k},$$

where  $\bar{a}$  denotes the solution of the congruent equation  $ax \equiv 1 \pmod{q}$ .

**Theorem 2.** Let  $q$  be an odd perfect square,  $k$  and  $m$  be positive integers such that  $(km, q) = 1$  and  $(k + 1, \varphi(q)) = 1$ . Then for any primitive character  $\chi \pmod{q}$ , we have the identity

$$\left| \sum'_{a=1}^q \chi(a + m\bar{a}^k) \right| = \sqrt{q}.$$

For the general integer  $q \geq 3$  (or the general positive integer  $k$ ), whether there exists an identity for

$$\left| \sum'_{a=1}^q \chi(a + m\bar{a}^k) \right|,$$

is an open problem.

## 2. SEVERAL LEMMAS

To complete the proof of our Theorems, we need the following several lemmas.

**Lemma 1.** Let  $p$  be an odd prime,  $\alpha$  be a positive integer,  $\chi$  be any primitive character  $\pmod{p^\alpha}$ . Then for any integer  $m$  and positive integer  $k$  with  $(m(k + 1), p(p - 1)) = 1$ , we have the identity

$$\left| \sum'_{a_1=1}^{p^\alpha} \sum'_{a_2=1}^{p^\alpha} \dots \sum'_{a_k=1}^{p^\alpha} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \right| = p^{k\alpha/2},$$

where  $\sum'_{a=1}^{p^\alpha}$  denotes the summation over all  $1 \leq a \leq p^\alpha$  such that  $(a, p) = 1$ .

**P r o o f.** From the properties of the classical Gauss sums and the reduced residue system  $\pmod{p^\alpha}$  we have (in what follows we use  $e(t) = e^{2\pi it}$ )

$$\begin{aligned} (3) \quad & \sum'_{a_1=1}^{p^\alpha} \sum'_{a_2=1}^{p^\alpha} \dots \sum'_{a_k=1}^{p^\alpha} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \\ &= \frac{1}{\tau(\bar{\chi})} \sum'_{a_1=1}^{p^\alpha} \dots \sum'_{a_k=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \bar{\chi}(b) e\left(\frac{b(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k})}{p^\alpha}\right) \\ &= \frac{1}{\tau(\bar{\chi})} \sum'_{a_1=1}^{p^\alpha} \dots \sum'_{a_k=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \bar{\chi}(b) e\left(\frac{a_1 + a_2 + \dots + a_k + mb^{k+1}\overline{a_1 a_2 \dots a_k}}{p^\alpha}\right) \\ &= \frac{1}{\tau(\bar{\chi})} \sum'_{a_1=1}^{p^\alpha} \dots \sum'_{a_k=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \bar{\chi}(b) e\left(\frac{a_1 b^{k+1} + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}}{p^\alpha}\right). \end{aligned}$$

Since  $(k+1, p(p-1)) = 1$ , there exist two integers  $r$  and  $s$  such that  $r\varphi(p^\alpha) + s(k+1) = 1$ . If  $b$  runs through a reduced residue system mod  $p^\alpha$ , then  $b^{k+1}$  also runs through a reduced residue system mod  $p^\alpha$ . So from (3) we have

$$\begin{aligned}
 (4) \quad & \sum_{a_1=1}^{p^\alpha} \sum_{a_2=1}^{p^\alpha} \dots \sum_{a_k=1}^{p^\alpha} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \\
 &= \frac{1}{\tau(\overline{\chi})} \sum_{a_1=1}^{p^\alpha} \dots \sum_{a_k=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \overline{\chi}(b^{s(k+1)}) e\left(\frac{a_1 b^{k+1} + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}}{p^\alpha}\right) \\
 &= \frac{1}{\tau(\overline{\chi})} \sum_{a_1=1}^{p^\alpha} \dots \sum_{a_k=1}^{p^\alpha} \sum_{b=1}^{p^\alpha} \overline{\chi}^s(b) e\left(\frac{a_1 b + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}}{p^\alpha}\right) \\
 &= \frac{\tau(\overline{\chi}^s)}{\tau(\overline{\chi})} \sum_{a_1=1}^{p^\alpha} \dots \sum_{a_k=1}^{p^\alpha} \chi^s(a_1) e\left(\frac{a_2 + a_3 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}}{p^\alpha}\right) \\
 &= \frac{\tau^2(\overline{\chi}^s)}{\tau(\overline{\chi})} \sum_{a_2=1}^{p^\alpha} \dots \sum_{a_k=1}^{p^\alpha} \chi^s(m\overline{a_2 a_3 \dots a_k}) e\left(\frac{a_2 + a_3 \dots + a_k}{p^\alpha}\right) \\
 &= \frac{\tau^{k+1}(\overline{\chi}^s)}{\tau(\overline{\chi})} \chi^s(m).
 \end{aligned}$$

Note that  $|\tau(\overline{\chi})| = |\tau(\overline{\chi}^s)| = p^{\alpha/2}$ ,  $|\chi^s(m)| = 1$ , from (4) we may immediately deduce

$$\left| \sum_{a_1=1}^{p^\alpha} \sum_{a_2=1}^{p^\alpha} \dots \sum_{a_k=1}^{p^\alpha} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \right| = p^{k\alpha/2}.$$

This proves Lemma 1. □

Now we introduce a hyper-Kloosterman sum by

$$K(q, m+1, z) = \sum_{\substack{x_1, x_2, \dots, x_m \pmod q \\ (x_1, p) = \dots = (x_m, p) = 1}} e\left(\frac{x_1 + \dots + x_m + z\overline{x_1 \dots x_m}}{q}\right)$$

for  $q = p^\alpha$ ,  $m \geq 1$ , and  $z$  not divisible by  $p$ . Define an exponential sum by

$$I(q, m, z) = \sum_{\substack{x \pmod q \\ (x, p) = 1}} e\left(\frac{mx + z\overline{x^m}}{q}\right).$$

Under the above notations, we have the following:

**Lemma 2.** Let  $p$  be an odd prime,  $q = p^{2\alpha}$ . Then for any integers  $m \geq 1$  and  $z$  with  $(z, p) = 1$ , we have the identity

$$K(q, m + 1, z) = q^{(m-1)/2} I(q, m, z).$$

*Proof.* This identity follows from R. A. Smith [3] or Yangbo Ye [4]. □

**Lemma 3.** Let  $p$  be an odd prime,  $\alpha$  be any positive integer,  $\chi$  be any primitive character mod  $p^{2\alpha}$ . Then for any integer  $m$  and positive integer  $k$  with  $(k + 1, p(p - 1)) = (km, p) = 1$ , we have the identity

$$\begin{aligned} & \left| \sum_{a_1=1}^{p^{2\alpha}'} \sum_{a_2=1}^{p^{2\alpha}'} \dots \sum_{a_k=1}^{p^{2\alpha}'} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \right| \\ &= p^{\alpha(k-1)} \left| \sum_{a=1}^{p^{2\alpha}'} \chi(ka + m\overline{a^k}) \right|. \end{aligned}$$

*Proof.* From Lemma 2 and the method of proving (4) we have

$$\begin{aligned} (5) \quad & \sum_{a_1=1}^{p^{2\alpha}'} \sum_{a_2=1}^{p^{2\alpha}'} \dots \sum_{a_k=1}^{p^{2\alpha}'} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \\ &= \frac{1}{\tau(\overline{\chi})} \sum_{b=1}^{p^{2\alpha}} \overline{\chi}(b) \sum_{a_1=1}^{p^{2\alpha}'} \dots \sum_{a_k=1}^{p^{2\alpha}'} e\left(\frac{a_1 + a_2 + \dots + a_k + mb^{k+1}\overline{a_1 a_2 \dots a_k}}{p^{2\alpha}}\right) \\ &= \frac{p^{\alpha(k-1)}}{\tau(\overline{\chi})} \sum_{b=1}^{p^{2\alpha}} \overline{\chi}(b) \sum_{a=1}^{p^{2\alpha}'} e\left(\frac{ka + mb^{k+1}\overline{a^k}}{p^{2\alpha}}\right) \\ &= \frac{p^{\alpha(k-1)}}{\tau(\overline{\chi})} \sum_{b=1}^{p^{2\alpha}} \overline{\chi}(b) \sum_{a=1}^{p^{2\alpha}'} e\left(\frac{kab + m\overline{a^k}}{p^{2\alpha}}\right) \\ &= \frac{p^{\alpha(k-1)}}{\tau(\overline{\chi})} \sum_{a=1}^{p^{2\alpha}} \sum_{b=1}^{p^{2\alpha}} \overline{\chi}(b) e\left(\frac{b(ka + m\overline{a^k})}{p^{2\alpha}}\right) \\ &= p^{\alpha(k-1)} \cdot \sum_{a=1}^{p^{2\alpha}'} \chi(ka + m\overline{a^k}), \end{aligned}$$

from which Lemma 3 follows. □

**Lemma 4.** Let  $q_1$  and  $q_2$  be two positive integers with  $(q_1, q_2) = 1$ ,  $\chi_1 \bmod q_1$  and  $\chi_2 \bmod q_2$ . Then for any integers  $m \geq 1$  and  $n$  with  $(n, q_1 q_2) = 1$ , we have

$$\begin{aligned} & \sum_{a_1=1}^{q_1 q_2'} \sum_{a_2=1}^{q_1 q_2'} \cdots \sum_{a_m=1}^{q_1 q_2'} \chi_1 \chi_2 (a_1 + a_2 + \cdots + a_m + n \bar{a}_1 \bar{a}_2 \cdots \bar{a}_m) \\ &= \sum_{a_1=1}^{q_1'} \sum_{a_2=1}^{q_1'} \cdots \sum_{a_m=1}^{q_1'} \chi_1 (a_1 + a_2 + \cdots + a_m + n \bar{a}_1 \bar{a}_2 \cdots \bar{a}_m) \\ & \quad \times \sum_{b_1=1}^{q_2'} \sum_{b_2=1}^{q_2'} \cdots \sum_{b_m=1}^{q_2'} \chi_2 (b_1 + b_2 + \cdots + b_m + n \bar{b}_1 \bar{b}_2 \cdots \bar{b}_m) \end{aligned}$$

*Proof.* Since  $(q_1, q_2) = 1$ , from the properties of the reduced residue system mod  $q_1 q_2$  we have

$$\begin{aligned} & \sum_{a_1=1}^{q_1 q_2'} \sum_{a_2=1}^{q_1 q_2'} \cdots \sum_{a_m=1}^{q_1 q_2'} \chi_1 \chi_2 (a_1 + a_2 + \cdots + a_m + n \bar{a}_1 \bar{a}_2 \cdots \bar{a}_m) \\ &= \sum_{a_1=1}^{q_1'} \sum_{b_1=1}^{q_2'} \sum_{a_2=1}^{q_1'} \sum_{b_2=1}^{q_2'} \cdots \sum_{a_m=1}^{q_1'} \sum_{b_m=1}^{q_2'} \chi_1 \chi_2 \left( \sum_{i=1}^m (a_i q_2 + b_i q_1) + n \prod_{i=1}^m \overline{a_i q_2 + b_i q_1} \right) \\ &= \sum_{a_1=1}^{q_1'} \sum_{a_2=1}^{q_1'} \cdots \sum_{a_m=1}^{q_1'} \chi_1 \left( \sum_{i=1}^m a_i q_2 + n \prod_{i=1}^m \overline{a_i q_2} \right) \\ & \quad \times \sum_{b_1=1}^{q_2'} \sum_{b_2=1}^{q_2'} \cdots \sum_{b_m=1}^{q_2'} \chi_2 \left( \sum_{i=1}^m b_i q_1 + n \prod_{i=1}^m \overline{b_i q_1} \right) \\ &= \sum_{a_1=1}^{q_1'} \sum_{a_2=1}^{q_1'} \cdots \sum_{a_m=1}^{q_1'} \chi_1 (a_1 + a_2 + \cdots + a_m + n \bar{a}_1 \bar{a}_2 \cdots \bar{a}_m) \\ & \quad \times \sum_{b_1=1}^{q_2'} \sum_{b_2=1}^{q_2'} \cdots \sum_{b_m=1}^{q_2'} \chi_2 (b_1 + b_2 + \cdots + b_m + n \bar{b}_1 \bar{b}_2 \cdots \bar{b}_m). \end{aligned}$$

This proves Lemma 4. □

### 3. PROOF OF THE THEOREMS

In this section, we shall complete the proof of our Theorems. First we prove Theorem 1. Let  $q = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$  denote the factorization of  $q$  into prime powers. Then for any primitive character  $\chi \bmod q$ , we have  $\chi = \chi_1 \chi_2 \cdots \chi_r$ , where  $\chi_i$  is a primitive character mod  $p_i^{\alpha_i}$ ,  $i = 1, 2, \dots, r$ . From Lemma 1 and Lemma 4 with

$(km, q) = 1, (k + 1, \varphi(q)) = 1$  we have

$$\begin{aligned} & \left| \sum_{a_1=1}^{q'} \sum_{a_2=1}^{q'} \dots \sum_{a_k=1}^{q'} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \right| \\ &= \prod_{i=1}^r \left| \sum_{a_1=1}^{p_i^{\alpha_i}} \sum_{a_2=1}^{p_i^{\alpha_i}} \dots \sum_{a_k=1}^{p_i^{\alpha_i}} \chi(a_1 + a_2 + \dots + a_k + m\overline{a_1 a_2 \dots a_k}) \right| \\ &= \prod_{i=1}^r \sqrt{p_i^{\alpha_i}} = \sqrt{q}. \end{aligned}$$

This proves Theorem 1. □

Now we prove Theorem 2. Applying Lemma 1 and Lemma 3 we have the identity

$$\begin{aligned} p^{\alpha k} &= \left| \sum_{a_1=1}^{p^{2\alpha}} \sum_{a_2=1}^{p^{2\alpha}} \dots \sum_{a_k=1}^{p^{2\alpha}} \chi(a_1 + a_2 + \dots + a_k + m\overline{ka_1 a_2 \dots a_k}) \right| \\ &= p^{\alpha(k-1)} \cdot \left| \sum_{a=1}^{p^{2\alpha}} \chi(ka + m\overline{a^k}) \right| \end{aligned}$$

or

$$(6) \quad \left| \sum_{a=1}^{p^{2\alpha}} \chi(a + m\overline{a^k}) \right| = p^{\alpha}.$$

From (6), Lemma 4 and the properties of the primitive character  $\chi \pmod q$  we may immediately deduce the identity

$$\left| \sum_{a=1}^q \chi(a + m\overline{a^k}) \right| = \sqrt{q}.$$

This completes the proof of Theorem 2. □

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