Applications of Mathematics

Abdesslam Boulkhemair; Abdeljalil Nachaoui; Abdelkrim Chakib

A shape optimization approach for a class of free boundary problems of Bernoulli type

Applications of Mathematics, Vol. 58 (2013), No. 2, 205–221

Persistent URL: http://dml.cz/dmlcz/143163

A SHAPE OPTIMIZATION APPROACH FOR A CLASS OF FREE BOUNDARY PROBLEMS OF BERNOULLI TYPE

Abdesslam Boulkhemair, Abdeljalil Nachaoui, Nantes,
Abdelkrim Chakib, Beni-Mellal

(Received October 25, 2010)

Abstract. We are interested in an optimal shape design formulation for a class of free boundary problems of Bernoulli type. We show the existence of the optimal solution of this problem by proving continuity of the solution of the state problem with respect to the domain. The main tools in establishing such a continuity are a result concerning uniform continuity of the trace operator with respect to the domain and a recent result on the uniform Poincaré inequality for variable domains.

Keywords: shape optimization, Bernoulli, free boundary problem, exterior Bernoulli problem, optimal solution, state problem, continuity of the state problem, uniform tubular neighbourhood, diffeomorphism, uniform trace theorem, uniform Poincaré inequality

MSC 2010: 35J05, 35J20, 35J25, 35Qxx, 49-xx

1. Introduction

In this paper, we consider a shape optimization formulation to solve the so-called Bernoulli's problem. Many physical and industrial applications lead to such a problem which can be considered as a typical example of free boundary problems. This class of problems serves as mathematical models in fluid dynamics (see [13]), in insulation and electro-chemistry (see [1]), in electromagnetics (see [9], [10]) and in various other engineering fields (see [11]). The bidimensional free boundary value problem of Bernoulli type is stated as follows: Find a doubly connected domain Ω in $\mathbb{R}^2$ and a function $u\colon \Omega \to \mathbb{R}$ such that

$$(1)\qquad \begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on } \Gamma, \end{cases}$$

where f, g and h are given functions, ν is the exterior unit normal of the domain Ω, Γ_0 is the interior fixed part of the boundary $\partial\Omega$ and Γ the exterior free component of $\partial\Omega$ which is to be determined (see Fig. 1).

This problem has received a great deal of attention. The first theoretical results with elliptic solutions were carried out by Beurling [4]. Later, the problem has been extensively studied by several authors, see for example [2], [3], [7], [12], [18] and references therein. Among others, a way to solve this free boundary problem is to transform it into a shape optimization problem. Such an optimal shape design formulation of this problem was recently considered by Haslinger-Kozubek-Kunisch-Peichl in [14], where the Neumann boundary condition on Γ is included into a suitable least squares cost functional while the remaining Dirichlet boundary condition on Γ is considered as part of an appropriate state problem. The existence of the optimal solution of this formulation was established in [15], where the C^2-regularity of the free boundaries was used to construct a C^1-diffeomorphism of a uniform tubular neighborhood of the boundaries. However, this regularity of the free boundary occurs only when the given data of the boundary conditions are smooth enough (see [18], [12]). More recently, another optimal shape design formulation was used in the work of Ito-Kunisch-Peichl, [17], where the Dirichlet boundary condition on Γ was included into a suitable least squares cost functional while the remaining Neumann boundary condition on Γ was considered as part of an appropriate state problem. A numerical realization was investigated without theoretical justification of the existence of an optimal solution. This motivated us to carry out the existence analysis of an optimal solution for this formulation. This analysis uses a weaker regularity assumption on the free boundary than [15].

The main idea in our analysis is the construction of a C^1-diffeomorphism of a uniform tubular neighborhood of the boundary by using only the C^1-regularity of the boundary. By the way, this C^1-regularity is the basic assumption made by Kinderlehrer-Nirenberg, [18], to show that, under some additional regularity assumptions on the data, the free boundary is in fact more regular than C^1. The construction of such a diffeomorphism is the main ingredient in establishing the uniform continuity of the trace operator with respect to the domain, a result which is similar to that obtained in [6] for non closed boundaries. Then, we show the main result of this paper which is the continuity of the solution of the state problem with respect to the domain using an appropriate topology on an admissible family of domains. The proof relies on the uniform continuity of the trace operator with respect to the domain and on the uniform Poincaré inequality established in [5].

The outline of this paper is as follows. In Section 2, the shape optimization formulation of the free boundary Bernoulli problem is described using a suitable family of admissible domains. In Section 3, we show the existence of an optimal solution using the C^1-regularity for the boundaries and assuming the continuity with respect to the domain of the state problem. Section 4 is devoted to the proof of such a continuity.

2. Formulation of the problem

Let D be a fixed, connected and bounded open subset of $\mathbb{R}^2$. Let us consider the following 2-dimensional exterior Bernoulli problem:

$$\begin{cases} \text{Find } \Omega \subset D & \text{and } u \in H^1(\Omega) \text{ such that} \\ \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_0, \\ u = 0 & \text{on } \Gamma, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on } \Gamma, \end{cases} \tag{2}$$

where $f \in L^2(D)$, $g \in H^{1/2}(\Gamma_0)$ and $h \in H^1(D)$ are given functions and ν is the outward unit normal vector to Γ. An admissible domain Ω will be a doubly connected Lipschitz open subset of D. The boundary $\partial\Omega$ of Ω is the disjoint union of a fixed part Γ_0 and a free boundary unknown part Γ. We assume that Γ is exterior to Γ_0 and we denote by K the domain inside of Γ_0 (see Fig. 1).

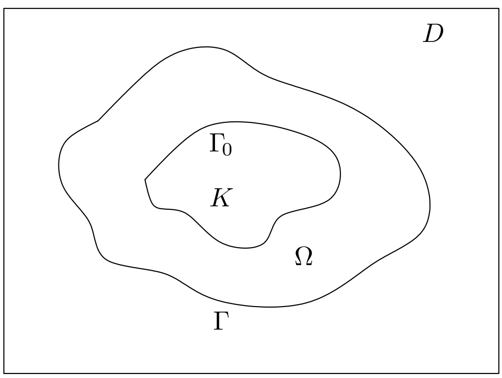

Figure 1. The considered domain $\Omega = \Omega(\varphi)$.

Since two boundary conditions must be satisfied on Γ in the boundary value problem (2), we can reformulate the free boundary problem as an optimal shape design one as follows: The Dirichlet boundary condition is included into a suitable least squares cost functional which can be minimized with respect to Γ, while the remaining condition is viewed as part of a state problem. The customary problem of shape optimization is then

$$\tag{3} \begin{cases} \text{Minimize } J(\Omega) = \displaystyle\int_\Gamma |u_\Omega|^2 \, \mathrm{d}\sigma \text{ for all } \Omega \in \mathcal{O}_{\mathrm{ad}}, \\ \text{where } u_\Omega \text{ is the solution of} \\ (\mathrm{PE}) \begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} = h & \text{on } \Gamma. \end{cases} \end{cases}$$

Here, $\mathcal{O}_{\mathrm{ad}}$ is the space of admissible domains. In order to define $\mathcal{O}_{\mathrm{ad}}$ and to give the description of an appropriate topology on it, we assume that the free boundary Γ is a parameterized curve defined by

$$\Gamma = \Gamma(\varphi) = \{\varphi(t) = (\varphi_1(t), \varphi_2(t));\ t \in \mathbb{R}\},$$

where $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ is a C^1, 1-periodic and injective function on $[0, 1[$. We shall also write $\Omega = \Omega(\varphi)$ to indicate the dependence on the parameterization φ. In fact, we shall also view φ as a mapping from the quotient space $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}^2$. It is well known that $\mathbb{R}/\mathbb{Z}$ is a compact metric space endowed with the distance $d(t + \mathbb{Z}, t' + \mathbb{Z}) = \inf\{|x - x'|/x \in t + \mathbb{Z}, x' \in t' + \mathbb{Z}\} = \inf\{|t - t' + k|/k \in \mathbb{Z}\}$, $t + \mathbb{Z}$ and $t' + \mathbb{Z}$ being two elements of $\mathbb{R}/\mathbb{Z}$.

We define V_{ad} to be the set of vector functions $\varphi \in C^1(\mathbb{R}, \mathbb{R}^2)$ such that

($\mathcal{H}_1$) φ is 1-periodic;

($\mathcal{H}_2$) there exist positive constants C_0, C_1 and C_2 such that

$$|\varphi(t)| \leqslant C_0, \quad \forall \bar{t} \in \mathbb{R}/\mathbb{Z},$$
$$C_1 d(\bar{t}, \bar{t'}) \leqslant |\varphi(t) - \varphi(t')| \leqslant C_2 d(\bar{t}, \bar{t'}) \quad \text{for all } \bar{t}, \bar{t'} \in \mathbb{R}/\mathbb{Z};$$

($\mathcal{H}_3$) $\overline{\Omega}(\varphi) \subset D$;

($\mathcal{H}_4$) there exists a positive constant γ such that

$$\operatorname{dist}(\Gamma_0, \Gamma(\varphi)) \geqslant \gamma.$$

Clearly, V_{ad} is a closed and bounded subset of $C^1(\mathbb{R},\mathbb{R}^2)$.

Now, the set U_{ad} of admissible functions will be any compact subset of V_{ad}. In other words, U_{ad} is a subset of V_{ad} whose elements and their derivatives are equicontinuous as it follows from the Ascoli-Arzelà theorem. An example of such a set U_{ad} is that of a closed subset of V_{ad} which is bounded in $C^{1,\delta}(\mathbb{R},\mathbb{R}^2)$ for some δ such that $0<\delta\leqslant 1$.

The set of admissible domains is then defined by

$$\mathcal{O}_{\mathrm{ad}}=\{\Omega=\Omega(\varphi)\subset D;\ \varphi\in U_{\mathrm{ad}}\}.$$

We shall also use the larger set

$$\widetilde{\mathcal{O}}_{\mathrm{ad}}=\{\Omega=\Omega(\varphi)\subset D;\ \varphi\in V_{\mathrm{ad}}\}$$

to state some intermediate results.

Remark 1. It follows from the assumptions on V_{ad} that the elements of $\widetilde{\mathcal{O}}_{\mathrm{ad}}$ (hence, those of $\mathcal{O}_{\mathrm{ad}}$ as well) are uniformly Lipschitz open sets in $\mathbb{R}^2$ and so they satisfy the uniform cone property; see [16], [20].

Remark 2. It is not clear in general whether the shape optimization formulation (3) is equivalent to the Bernoulli problem (2). However, we can say that it will be so if the set of admissible domains $\mathcal{O}_{\mathrm{ad}}$ is so large that it contains the domain which solves (2). This requires to know some regularity result on (2), that is, to know what is the regularity of the solution Γ of (2). In fact, in such generality, that is, for such general data f,g,h, we do not know whether one can solve (2). What is known is that, in the case $f=0$, $g=1$ and h being a Hölder function, Γ is of class $C^{1,\alpha}$ for some positive α (Alt-Caffarelli, [3]), and so, it is analytic (Lewy, [19], Kinderlehrer-Nirenberg, [18]). See also the discussion on this subject in [12]. Hence, we can state

Theorem 1. *In the case $f=0$, $g=1$ and $h\in C^{0,\alpha}(\Gamma_0)$ for some positive α, the Bernoulli problem* (2) *is equivalent to its shape optimization formulation* (3).

3. Existence of an optimal solution

We first give some notation and definitions which will be used in the sequel.

Let $H_{\Gamma_0}(\Omega)$ be the space defined by

$$H_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega); v|_{\Gamma_0} = 0\},$$

where $H^1(\Omega)$ is the usual Sobolev space equipped with the norm $\|\cdot\|_{1,\Omega}$ defined by

$$\|v\|_{1,\Omega} = (\|v\|_{0,\Omega}^2 + \|\nabla v\|_{0,\Omega}^2)^{1/2},$$
$$\|v\|_{0,\Omega} = \Big(\int_\Omega |v|^2 \,\mathrm{d}x\Big)^{1/2}.$$

The space $H_{\Gamma_0}(\Omega)$ is equipped with the norm

$$|v|_{1,\Omega} = \|\nabla v\|_{0,\Omega}.$$

We define the bilinear form a in $H_{\Gamma_0}(\Omega)$ by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x.$$

Let $u_0 \in H^1(D)$ be fixed and such that $u_0 = g$ on Γ_0. Then a variational formulation of the state problem (PE) is the following:

$$\text{(4)}\qquad \begin{cases} \text{find } w = u - u_0 \in H_{\Gamma_0}(\Omega) \text{ such that} \\ a(w,v) = -a(u_0,v) + \int_\Omega fv \,\mathrm{d}x + \int_\Gamma hv \,\mathrm{d}\sigma, \quad \forall v \in H_{\Gamma_0}(\Omega). \end{cases}$$

Since the given functions f, g and h are smooth enough, the existence and uniqueness of the solution of the variational problem (4) are ensured by Lax-Milgram's theorem. Thus, we can define the mapping $\Omega \mapsto w = w(\Omega)$ and denote its graph by

$$\mathcal{F} = \{(\Omega, w(\Omega)); \Omega \in \mathcal{O}_{\mathrm{ad}} \text{ and } w(\Omega) \text{ is the solution of (4) on } \Omega\}.$$

Now, the customary problem of shape optimization is

$$\text{(5)}\qquad \text{to minimize } J(\Omega) = J(\Omega, w(\Omega)) \text{ on } \mathcal{F}.$$

This minimization problem is usually solved by endowing the set $\mathcal{F}$ with a topology for which $\mathcal{F}$ is compact and J is lower semi-continuous. Let us therefore define the topology we shall work with. First, we define the convergence of a sequence $(\varphi_n)_n \subset V_{\text{ad}}$ by

$$\varphi_n \to \varphi \iff \begin{cases} \varphi_n \to \varphi \text{ uniformly on } [0,1], \\ \varphi_n' \to \varphi' \text{ uniformly on } [0,1] \end{cases} \tag{6}$$

that is, iff $\varphi_n \to \varphi$ in the C^1 topology. Then, the convergence of a sequence $(\Omega_n)_n = (\Omega(\varphi_n))_n \subset \widetilde{\mathcal{O}}_{\text{ad}}$ to $\Omega = \Omega(\varphi) \in \widetilde{\mathcal{O}}_{\text{ad}}$ is simply defined by

$$\Omega_n \to \Omega \iff \varphi_n \to \varphi. \tag{7}$$

If $w \in H^1(\Omega)$, we denote by $\tilde{w}$ a uniform extension of w from Ω to the fixed open bounded domain D. Note that the existence of such a uniform extension is ensured by the result of D. Chenais, [8], and Remark 1. We can then define the convergence of a sequence $(w_n)_n$ of solutions of (4) on $\Omega(\varphi_n)$ to the solution w of (4) on $\Omega(\varphi)$ by

$$w_n \to w \iff \tilde{w}_n \rightharpoonup \tilde{w} \text{ weakly in } H^1(D). \tag{8}$$

Finally, the topology we introduce on $\mathcal{F}$ is the one induced by the convergence defined by

$$(\Omega_n, w_n) \to (\Omega, w) \iff \begin{cases} \Omega_n \to \Omega, \\ w_n \to w. \end{cases} \tag{9}$$

We can now state

Theorem 2. *The minimization problem* (5) *admits a solution in* $\mathcal{F}$.

As already mentioned, the proof of this theorem is reduced to showing the compactness of $\mathcal{F}$ and the lower semi-continuity of J.

Concerning the compactness of $\mathcal{F}$ with respect to the convergence (9), note first that the compactness of $\mathcal{O}_{\text{ad}}$ with respect to the convergence (7) follows easily from the compactness of U_{ad} and the Ascoli-Arzelà theorem. Thus, the compactness of $\mathcal{F}$ will be a consequence of the continuity of the state problem (PE) with respect to the domain. The proof of this continuity turned out to be non trivial and will be done in the next section.

The proof of the lower semi-continuity (in fact, continuity!) of the functional J on $\mathcal{F}$ also uses arguments that will be developed in the next section. Therefore, we postpone it to the end of the paper.

The proof of continuity of the state problem with respect to the domain is based on essentially two ingredients that are the uniform Poincaré inequality and the uniform continuity of the trace operator with respect to the domain. The former ingredient is precisely

Theorem 3. *There exists a constant $M > 0$ such that*

$$\|u\|_{0,\Omega} \leqslant M\|\nabla u\|_{0,\Omega} \quad \forall u \in H_{\Gamma_0}(\Omega),\ \forall \Omega \in \widetilde{\mathcal{O}}_{\mathrm{ad}}. \tag{10}$$

The proof of this statement is non trivial. Anyhow, because of Remark 1, it follows from Corollary 3 (ii) of [5] and we refer to that paper.

As for the latter ingredient, it concerns the trace operator and reads as follows:

Theorem 4. *Let r be such that $\frac{1}{2} < r \leqslant 1$. Then there exists a constant K such that, for all $\varphi \in U_{\mathrm{ad}}$ and all u in $H^r(D)$,*

$$\|u\|_{0,\Gamma(\varphi)} \leqslant K\|u\|_{r,D},$$

where $\|\cdot\|_{0,\Gamma(\varphi)}$ is the $L^2(\Gamma(\varphi))$-norm and $\|\cdot\|_{r,D}$ is the $H^r(D)$-norm.

Clearly, this claims the uniform continuity of the trace operator with respect to all the boundaries $\Gamma(\varphi)$ or, equivalently, with respect to all the domains $\Omega(\varphi)$, $\varphi \in U_{\mathrm{ad}}$. The proof of this theorem is based on the construction of a C^1-diffeomorphism of a uniform tubular neighborhood of the free boundary onto a strip in the plane. We need some lemmas.

We start by showing that the derivatives of the elements of V_{ad} are uniformly bounded, that is

Lemma 1. *If $\varphi \in V_{\mathrm{ad}}$, then $C_1 \leqslant |\varphi'(t)| \leqslant C_2$ for all $t \in \mathbb{R}$.*

P r o o f. Let $t_0 \in [0,1[$, then we have $d(\overline{t_0}, \overline{t_0+h}) = \inf_{k\in\mathbb{Z}} |h+k| = |h|$ if $|h| < \frac{1}{2}$. Then

$$\Big|\frac{\varphi(t_0+h)-\varphi(t_0)}{h}\Big| = \Big|\frac{\varphi(t_0+h)-\varphi(t_0)}{d(\overline{t_0}, \overline{t_0+h})}\Big| \leqslant C_2 \quad \text{for all } h \in\ ]-1/2, 1/2[.$$

Taking the limit as $h \to 0$, we obtain that $|\varphi'(t_0)| \leqslant C_2$. By the same argument, we conclude that $|\varphi'(t_0)| \geqslant C_1$. □

We turn now to the construction of the announced diffeomorphism or, more precisely, its inverse. To this end, with a given $\varphi \in U_{\rm ad}$, one can associate the function $\Phi_j \in C^1((\mathbb{R}/\mathbb{Z}) \times \mathbb{R}, \mathbb{R}^2)$ defined by

$$\Phi_j(t,s) = \varphi(t) + s\psi_j(t), \tag{11}$$

where $\psi_j(t) = \int_{\mathbb{R}} \varphi'(t-\tau)^{\perp}\chi(j\tau)j\,\mathrm{d}\tau$, $j \geqslant 1$, $\varphi'^{\perp} = (-\varphi_2', \varphi_1')$ and $\chi \in C^{\infty}(\mathbb{R})$ is such that $\chi \geqslant 0$, $\int_{\mathbb{R}} \chi\,\mathrm{d}x = 1$ and $\chi(t) = 0$ if $|t| > 1$. Recall that $\varphi'(t)^{\perp}$ defines the normal direction to $\Gamma = \varphi(\mathbb{R})$ at $\varphi(t)$. Note also that ψ_j is just a regularized function of $\varphi'^{\perp}$ and as such it represents a good approximation for the latter as is well known. More precisely, we have

Lemma 2. *Given $\varepsilon > 0$, there exists j_ε such that, for all $j \geqslant j_\varepsilon$ and all $\varphi \in U_{\rm ad}$,*

$$\|\psi_j - \varphi'^{\perp}\|_{L^\infty} < \varepsilon.$$

Note that what will be important in the sequel is that j_ε is independent of $\varphi \in U_{\rm ad}$.

Proof. We have

$$\begin{aligned}|\psi_j(t) - \varphi'(t)^{\perp}| &= \left| \int_{\mathbb{R}} (\varphi'(t-\tau)^{\perp} - \varphi'(t)^{\perp})\chi(j\tau)j\,\mathrm{d}\tau \right| \\ &\leqslant \int_{\mathbb{R}} \left|\varphi'\Big(t - \frac{\tau}{j}\Big) - \varphi'(t)\right| \chi(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Since $U_{\rm ad}$ is compact, the functions φ' are equicontinuous and uniformly continuous when φ describes $U_{\rm ad}$ as follows from the Ascoli-Arzelà theorem. So, given $\varepsilon > 0$, there exists $\gamma_\varepsilon > 0$ independent of φ such that

$$\forall t, t' \in \mathbb{R},\ |t - t'| \leqslant \gamma_\varepsilon \quad \text{implies that} \quad |\varphi'(t) - \varphi'(t')| \leqslant \varepsilon,\ \forall \varphi \in U_{\rm ad}. \tag{12}$$

Hence, for j such that $|\tau|/j \leqslant 1/j \leqslant \gamma_\varepsilon$ and all $t \in \mathbb{R}$,

$$|\psi_j(t) - \varphi'(t)^{\perp}| \leqslant \int_{\mathbb{R}} \varepsilon\chi(\tau)\,\mathrm{d}\tau = \varepsilon.$$

So, we can take $j_\varepsilon \geqslant 1/\gamma_\varepsilon$. □

In the sequel and, in particular, in the next lemma, we shall take $j = j_\varepsilon$ and, for simplicity, we shall write ψ and Φ instead of ψ_{j_ε} and Φ_{j_ε} respectively. Of course, as one can guess, the number ε will be taken sufficiently small to be able to get the result.

Lemma 3. *There exists a small enough $s_0 > 0$ such that s_0 is independent of $\varphi \in U_{\rm ad}$ and the following three assertions hold.*

(i) *The Jacobian $J\Phi$ of Φ is such that*

$$|J\Phi| \geqslant \frac{1}{2}C_1^2 \text{ on } \mathbb{R} \times [-s_0, s_0]. \tag{13}$$

(ii) *There exists $C_3 > 0$ independent of φ such that*

$$\begin{gathered}|\Phi(t,s) - \Phi(t',s')| \leqslant C_3|(t-t', s-s')|,\\ \forall (t,s),(t',s') \in \mathbb{R} \times [-s_0, s_0].\end{gathered} \tag{14}$$

(iii) *Φ is injective in $(\mathbb{R}/\mathbb{Z}) \times [-s_0, s_0]$, and, more precisely,*

$$\begin{gathered}|\Phi(\bar{t},s) - \Phi(\bar{t'},s')| \geqslant \frac{C_1}{2\sqrt{2}}(d(\bar{t},\bar{t'})^2 + |s-s'|^2)^{1/2},\\ \forall (\bar{t},s),(\bar{t'},s') \in \mathbb{R}/\mathbb{Z} \times [-s_0, s_0],\end{gathered} \tag{15}$$

where C_1 is the same constant as that used in the definition of $V_{\rm ad}$.

Proof. The proof of assertions (i) and (ii) is not difficult and uses the same arguments as those used to prove Lemma 1 of [6]. So, we refer to that paper.

Let us show (iii). Let $(\bar{t},s)$, $(\bar{t'},s')$ be in $\mathbb{R}/\mathbb{Z} \times [-s_0, s_0]$. We have

$$\Phi(\bar{t},s) - \Phi(\bar{t'},s') = \varphi(t) - \varphi(t') + (s-s')\psi(t) - s'(\psi(t) - \psi(t')). \tag{16}$$

Now, let η be a small enough parameter to be determined later. We distinguish two cases:

First case, if $|s-s'| \leqslant \eta d(\bar{t},\bar{t'})$, we have

$$|\Phi(\bar{t},s) - \Phi(\bar{t'},s')| \geqslant C_1 d(\bar{t},\bar{t'}) - \eta d(\bar{t},\bar{t'})\|\psi\|_{L^\infty} - s_0\|\psi'\|_{L^\infty} d(\bar{t},\bar{t'}). \tag{17}$$

In fact, for the last term we have

$$|\psi(t) - \psi(t')| = |\psi(t+k) - \psi(t')| \leqslant \|\psi'\|_{L^\infty}|t-t'+k| \quad \text{for all } k \in \mathbb{Z};$$

hence

$$|\psi(t) - \psi(t')| \leqslant \|\psi'\|_{L^\infty} d(\bar{t},\bar{t'}).$$

Therefore,

$$\begin{aligned}|\Phi(\bar{t},s) - \Phi(\bar{t'},s')| &\geqslant C_1 d(\bar{t},\bar{t'}) - \eta d(\bar{t},\bar{t'})\|\psi\|_{L^\infty} - s_0\|\psi'\|_{L^\infty} d(\bar{t},\bar{t'})\\ &\geqslant (C_1 - \eta C_2 - s_0 C_2 k_\varepsilon\|\chi'\|_{L^1}) d(\bar{t},\bar{t'})\\ &\geqslant (C_1 - (\eta + s_0 k_\varepsilon\|\chi'\|_{L^1})C_2)\Big(\frac{1}{2}d(\bar{t},\bar{t'})^2 + \frac{1}{2}\frac{1}{\eta^2}|s-s'|^2\Big)^{1/2}.\end{aligned}$$

Assuming that $\eta \leqslant 1$, we obtain

$$|\Phi(\bar{t}, s) - \Phi(\bar{t'}, s')| \geqslant \frac{1}{\sqrt{2}}(C_1 - (\eta + s_0 k_\varepsilon \|\chi'\|_{L^1})C_2)(d(\bar{t}, \bar{t'})^2 + |s - s'|^2)^{1/2}.$$

Second case, if $|s - s'| \geqslant \eta d(\bar{t}, \bar{t'})$, we can write, for all $k \in \mathbb{Z}$,

$$\begin{aligned}\Phi(\bar{t}, s) - \Phi(\bar{t'}, s') = &-\Big((t' - t + k)\varphi'(t) + (t' - t + k) \\ &\times \int_0^1 (\varphi'(t + \tau(t' - t + k)) - \varphi'(t))\,\mathrm{d}\tau\Big) + (s - s')\varphi'(t)^\perp \\ &+ (s - s')(\psi(t) - \varphi'(t)^\perp) + s'(\psi(t) - \psi(t' + k)).\end{aligned}$$

We know that $\|\psi - \varphi'^\perp\|_{L^\infty} \leqslant \varepsilon$. On the other hand, it follows from the compactness of U_{ad} and the Ascoli-Arzelà theorem that there exists γ_ε independent of φ such that for all $t, t' \in \mathbb{R}$ such that $d(\bar{t}, \bar{t'}) \leqslant \gamma_\varepsilon$, we have $|\varphi'(t) - \varphi'(t')| \leqslant \varepsilon$. Hence, if $|s - s'| \leqslant \eta\gamma_\varepsilon$, we have

$$\left|\int_0^1 (\varphi'(t + \tau(t' - t + k)) - \varphi'(t))\,\mathrm{d}\tau\right| \leqslant \varepsilon.$$

Indeed, if k is such that $|t - t' + k| = d(\bar{t}, \bar{t'})$, we have

$$\inf_{l \in \mathbb{Z}} |t + \tau(t' - t + k) - t + l| \leqslant \inf_{l \in \mathbb{Z}} (|\tau|\,|(t' - t + k)| + |l|) \leqslant |\tau|\,|(t' - t + k)| \leqslant d(\bar{t}, \bar{t'}) \leqslant \gamma_\varepsilon.$$

Therefore, if $s_0 \leqslant \frac{1}{2}\eta\gamma_\varepsilon$, we have

$$\begin{aligned}|\Phi(\bar{t}, s) - \Phi(\bar{t'}, s')| &\geqslant |(t' - t + k)\varphi'(t) + (s - s')\varphi'(t)^\perp| \\ &\quad - \varepsilon|t' - t + k| - \varepsilon|s - s'| - s_0 C_2 k_\varepsilon \|\chi'\|_{L^1}|t' - t + k| \\ &\geqslant (|t' - t + k|^2|\varphi'(t)|^2 + |s - s'|^2|\varphi'(t)|^2)^{1/2} \\ &\quad - (\varepsilon + s_0 C_2 k_\varepsilon \|\chi'\|_{L^1})|t' - t + k| - \varepsilon|s - s'| \\ &\geqslant C_1(d(\bar{t}, \bar{t'})^2 + |s - s'|^2)^{1/2} \\ &\quad - (\varepsilon + s_0 C_2 k_\varepsilon \|\chi'\|_{L^1})|t' - t + k| - \varepsilon|s - s'|.\end{aligned}$$

Now, we take the inf with respect to k to obtain

$$\begin{aligned}&|\Phi(\bar{t}, s) - \Phi(\bar{t'}, s')| \\ &\qquad\geqslant C_1(d(\bar{t}, \bar{t'})^2 + |s - s'|^2)^{1/2} - (\varepsilon + s_0 C_2 k_\varepsilon \|\chi'\|_{L^1})d(\bar{t}, \bar{t'}) - \varepsilon|s - s'| \\ &\qquad\geqslant (C_1 - 2\varepsilon - s_0 C_2 k_\varepsilon \|\chi'\|_{L^1})(d(\bar{t}, \bar{t'})^2 + |s - s'|^2)^{1/2}.\end{aligned}$$

The constants ε, η and s_0 can be chosen, for example, such that

$$(\eta s_0 C_2 k_\varepsilon \|\chi'\|_{L^1}) C_2 \leqslant \frac{C_1}{2}, \quad 2\varepsilon + s_0 C_2 k_\varepsilon \|\chi'\|_{L^1} \leqslant \frac{C_1}{2} \quad \text{and} \quad s_0 \leqslant \frac{1}{2}\eta\gamma_\varepsilon.$$

Hence, it suffices to take

$$\eta = \frac{C_1}{4C_2}, \quad \varepsilon = \frac{C_1}{8} \quad \text{and} \quad s_0 = \min\Big\{\frac{C_1\gamma_\varepsilon}{8C_2}, \frac{C_1}{8C_2 k_\varepsilon \|\chi'\|_{L^1}}\Big\}.$$

This shows that there exists s_0 independent of φ such that Φ is injective in $\mathbb{R}/\mathbb{Z} \times [-s_0, s_0]$ and

$$|\Phi(\bar{t}, s) - \Phi(\bar{t'}, s')| \geqslant \frac{C_1}{2\sqrt{2}}(d(\bar{t}, \bar{t'})^2 + |s - s'|^2)^{1/2}, \quad \forall (\bar{t}, s),\ (\bar{t'}, s') \in \mathbb{R}/\mathbb{Z} \times [-s_0, s_0].$$

□

Corollary 1. *The function Φ defines a C^1 diffeomorphism from $(\mathbb{R}/\mathbb{Z}) \times]-s_0, s_0[$ onto an open neighbourhood of $\Gamma = \varphi(\mathbb{R}/\mathbb{Z})$. In particular, it is by restriction a diffeomorphism from $]0,1[\times]-s_0, s_0[$ onto a neighbourhood of $\Gamma \setminus \{\varphi(0)\}$.*

Proof of Theorem 4. Let us denote $\mathcal{I} =]0,1[$ and $\mathcal{J} =]-s_0, s_0[$ and let us consider $u \in H^r(D)$ where $\frac{1}{2} < r \leqslant 1$. We have

$$u(\varphi(t)) = u_{|\Gamma} \circ \varphi(t), \quad \forall t \in \mathbb{R},$$

and, on the other hand,

$$u(\varphi(t)) = u(\Phi(t, s))_{|s=0} \equiv v(t, s)_{|s=0}, \quad \forall t \in \mathbb{R},$$

where Φ is the function studied above and $v = u \circ \Phi$. From Corollary 1 we have that Φ is a C^1 diffeomorphism from $(\mathbb{R}/\mathbb{Z}) \times \mathcal{J}$ onto an open subset of $\mathbb{R}^2$ which is some tubular neighborhood of Γ, and thus $v \in H^r(\mathcal{I} \times \mathcal{J})$. Now,

$$\|u\|_{0,\Gamma(\varphi)} \leqslant C_2 \|u \circ \varphi\|_{0,\mathcal{I}} = C_2 \|v(t, s)_{|s=0}\|_{0,\mathcal{I}},$$

and, according to the standard result on the continuity of the trace operator from $H^r(\mathcal{I} \times \mathcal{J})$ to $L^2(\mathcal{I} \times \{0\})$, there exists a constant β, independent of v, such that $\|v_{|s=0}\|_{0,\mathcal{I}} \leqslant \beta \|v\|_{r,\mathcal{I}\times\mathcal{J}}$ for all $v \in H^r(\mathcal{I} \times \mathcal{J})$. Hence,

$$\|u\|_{0,\Gamma(\varphi)} \leqslant C_2\, \beta\, \|v\|_{r,\mathcal{I}\times\mathcal{J}}.$$

Using the same arguments as in [6], we can show that there exists a constant C_4 independent of φ, such that

$$\|v\|_{r,\mathcal{I}\times\mathcal{J}} \leqslant C_4 \|u\|_{r,D}.$$

This completes the proof of the theorem. □

Now, as a consequence of Theorem 4, we state and prove the following convergence result which will also be needed for the continuity of the state problem.

Corollary 2. *Let $(\varphi_n)_n \subset V_{\text{ad}}$ be a sequence such that $\varphi_n \to \varphi$ in the sense of* (6)*, that is in the C^1 topology, and let $u, h \in H^1(D)$. Then*

$$\lim_{n\to\infty} u \circ \varphi_n = u \circ \varphi \quad \textit{in } L^2([0,1]),$$

$$\textit{and} \quad \lim_{n\to\infty} \int_{\Gamma(\varphi_n)} uh \,\mathrm{d}\sigma = \int_{\Gamma(\varphi)} uh \,\mathrm{d}\sigma.$$

Proof. The first assertion is proved by using essentially a density result and the Lebesgue convergence theorem. Since it is proved in [6], Corollary 1, we refer to it.

As for the second assertion, we have

$$\begin{aligned}
&\left| \int_{\Gamma(\varphi_n)} uh \,\mathrm{d}\sigma - \int_{\Gamma(\varphi)} uh \,\mathrm{d}\sigma \right| \\
&\quad = \left| \int_0^1 (h\circ\varphi_n)(t)(u\circ\varphi_n)(t)|\varphi_n'(t)| - (h\circ\varphi)(t)(u\circ\varphi)(t)|\varphi'(t)| \,\mathrm{d}t \right| \\
&\quad \leqslant \left| \int_0^1 (h\circ\varphi_n - h\circ\varphi)(u\circ\varphi_n)|\varphi_n'| \,\mathrm{d}t \right| + \left| \int_0^1 (u\circ\varphi_n - u\circ\varphi)(h\circ\varphi_n)|\varphi_n'| \,\mathrm{d}t \right| \\
&\qquad + \left| \int_0^1 (h\circ\varphi)(u\circ\varphi)(|\varphi_n'| - |\varphi'|) \,\mathrm{d}t \right| \\
&\quad \leqslant \sqrt{C_2}\|u\|_{0,\Gamma(\varphi_n)}\|h\circ\varphi_n - h\circ\varphi\|_{0,[0,1]} + \sqrt{C_2}\|h\|_{0,\Gamma(\varphi_n)}\|u\circ\varphi_n - u\circ\varphi\|_{0,[0,1]} \\
&\qquad + \frac{1}{C_1}\sup_{[0,1]}|\varphi_n' - \varphi'|\, \|u\|_{0,\Gamma(\varphi)}\|h\|_{0,\Gamma(\varphi)}.
\end{aligned}$$

It follows from the first part that

$$\lim_{n\to\infty} \|u\circ\varphi_n - u\circ\varphi\|_{0,[0,1]} = \lim_{n\to\infty} \|h\circ\varphi_n - h\circ\varphi\|_{0,[0,1]} = 0.$$

Thus, Corollary 2 follows by applying Theorem 4 and using the C^1 convergence of (φ_n) to φ. □

In what follows, we shall prove the continuity of the state problem.

Let $(\Omega_n) = (\Omega(\varphi_n))$ be a sequence in $\mathcal{O}_{\text{ad}}$ such that Ω_n converges, in the sense of (7), to $\Omega = \Omega(\varphi) \in \mathcal{O}_{\text{ad}}$ and let $w_n = w(\Omega_n)$ be the solution of the problem (4) on Ω_n, that is,

$$(18)\quad \begin{cases} \text{find } w_n \in H_{\Gamma_0}(\Omega_n) \text{ such that} \\ \int_{\Omega_n} \nabla w_n \cdot \nabla v \,\mathrm{d}x = -\int_{\Omega_n} \nabla u_0 \cdot \nabla v \,\mathrm{d}x + \int_{\Omega_n} fv \,\mathrm{d}x + \int_{\Gamma(\varphi_n)} hv \,\mathrm{d}\sigma, \\ \hfill \forall v \in H_{\Gamma_0}(\Omega_n). \end{cases}$$

Now, we state the result on the continuity of the state problem.

Theorem 5. (i) *There exist an extension* $\tilde{w}_n$ *of* w_n *to* $H^1(D)$ *and a constant* C *independent of* n *such that*

$$\|\tilde{w}_n\|_{1,D} \leqslant C.$$

(ii) *There exists a subsequence of* $(\tilde{w}_n)_n$ *which is weakly convergent in* $H^1(D)$ *to a limit, denoted by* $\tilde{w}$, *which is the extension of the solution* w *of* (4) *to* Ω.

P r o o f. (i) Let w_n be the solution of (18). Using D. Chenais's result, [8], there exists $\tilde{w}_n$ an extension of w_n to $H^1(D)$ and a non negative constant $\widetilde{C}$ independent of n such that

$$\|\tilde{w}_n\|_{1,D} \leqslant \widetilde{C}\|w_n\|_{1,\Omega_n}. \tag{19}$$

Let us show that $\|w_n\|_{1,\Omega_n}$ is bounded with respect to n. Taking $v = w_n$ in (18), we can write

$$\int_{\Omega_n} \nabla w_n \cdot \nabla w_n \,\mathrm{d}x = -\int_{\Omega_n} \nabla u_0 \cdot \nabla w_n \,\mathrm{d}x + \int_{\Omega_n} f w_n + \int_{\Gamma(\varphi_n)} h w_n \,\mathrm{d}\sigma.$$

Hence,

$$|w_n|^2_{1,\Omega_n} \leqslant \|u_0\|_{1,D}|w_n|_{1,\Omega_n} + \|f\|_{0,D}\|w_n\|_{0,\Omega_n} + \|h\|_{0,\Gamma_n}\|w_n\|_{0,\Gamma_n}, \tag{20}$$

where Γ_n stands for $\Gamma(\varphi_n)$. Now, it follows from Theorem 3, inequality (19) and Theorem 2 that

$$\|w_n\|_{0,\Gamma_n} \leqslant K\|\tilde{w}_n\|_{1,D} \leqslant K\widetilde{C}\|w_n\|_{1,\Omega_n} \leqslant K\widetilde{C}\sqrt{1+M^2}|w_n|_{1,\Omega_n},$$

so that

$$|w_n|_{1,\Omega_n} \leqslant \|u_0\|_{1,D} + M\|f\|_{0,D} + C_2K^2\widetilde{C}\sqrt{1+M^2}\|h\|_{1,D}. \tag{21}$$

Hence,

$$\|w_n\|_{1,\Omega_n} \leqslant \sqrt{1+M^2}(\|u_0\|_{1,D} + M\|f\|_{0,D} + C_2K^2\widetilde{C}\sqrt{1+M^2}\|h\|_{1,D}), \tag{22}$$

which establishes the first part.

(ii) It follows from (i) that $(\tilde{w}_n)_n$ is a bounded sequence in $H^1(D)$. Therefore, we can extract a subsequence, denoted again by $(\tilde{w}_n)_n$, which weakly converges in $H^1(D)$ to a limit denoted by $\tilde{w}$. Note that $w = \tilde{w}|_\Omega$ is in $H_{\Gamma_0}(\Omega)$ as follows from the boundedness of the trace operator. To prove that w is the solution of (4) on Ω, it suffices to show that w satisfies the equation

$$(23)\quad \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x = -\int_\Omega \nabla u_0 \cdot \nabla v \,\mathrm{d}x + \int_\Omega f v \,\mathrm{d}x + \int_{\Gamma(\varphi)} h v \,\mathrm{d}\sigma, \quad \forall v \in H_{\Gamma_0}(\Omega).$$

In fact, it suffices to show that the variational equation (23) holds for all $v \in H_{\Gamma_0}(D) = \{\xi \in H^1(D); \xi = 0 \text{ on } \Gamma_0\}$. Note that, since for all v in $H_{\Gamma_0}(D)$, the restriction $v|_{\Omega_n}$ is in $H_{\Gamma_0}(\Omega_n)$ for all n, we have

$$(24)\quad \int_{\Omega_n} \nabla w_n \cdot \nabla v \,\mathrm{d}x = -\int_{\Omega_n} \nabla u_0 \cdot \nabla v \,\mathrm{d}x + \int_{\Omega_n} f v \,\mathrm{d}x + \int_{\Gamma(\varphi_n)} h v \,\mathrm{d}\sigma, \ \forall v \in H_{\Gamma_0}(D).$$

Now, we obtain (23) from (24) just by passing to the limit. Indeed, for $v \in H_{\Gamma_0}(D)$, let us define I_1, I_2, I_3 and I_4 by

$$\begin{aligned}
I_1 &= \int_{\Omega_n} \nabla \tilde{w}_n \cdot \nabla v \,\mathrm{d}x - \int_\Omega \nabla \tilde{w} \cdot \nabla v \,\mathrm{d}x \\
&= \int_D \chi_\Omega (\nabla \tilde{w}_n - \nabla \tilde{w}) \cdot \nabla v \,\mathrm{d}x + \int_D (\chi_{\Omega_n} - \chi_\Omega) \nabla \tilde{w}_n \cdot \nabla v \,\mathrm{d}x, \\
I_2 &= \int_\Omega \nabla u_0 \cdot \nabla v \,\mathrm{d}x - \int_{\Omega_n} \nabla u_0 \cdot \nabla v \,\mathrm{d}x \\
&= \int_D (\chi_{\Omega_n} - \chi_\Omega) \nabla u_0 \cdot \nabla v \,\mathrm{d}x, \\
I_3 &= \int_\Omega f v \,\mathrm{d}x - \int_{\Omega_n} f v \,\mathrm{d}x \\
&= \int_D (\chi_{\Omega_n} - \chi_\Omega) f v \,\mathrm{d}x, \\
I_4 &= \int_{\Gamma(\varphi)} h v \,\mathrm{d}\sigma - \int_{\Gamma(\varphi_n)} h v \,\mathrm{d}\sigma.
\end{aligned}$$

Clearly, Corollary 2 implies that $\lim_{n\to\infty} I_4 = 0$. As for the others, it follows from the convergence

$$\tilde{w}_n \rightharpoonup \tilde{w} \quad \text{in } H^1(D)\text{-weak},$$

the convergence of characteristic functions, due to Pironneau [16], [20],

$$\chi_{\Omega_n} \to \chi_\Omega \quad \text{in } L^\infty(D)\text{-weak*},$$

and the fact that $(\tilde{w}_n)$ is bounded in $H^1(D)$, that $\lim_{n\to\infty} I_1 = \lim_{n\to\infty} I_2 = \lim_{n\to\infty} I_3 = 0$.

This proves Theorem 5. □

To complete the proof of Theorem 2, let us establish

Theorem 6. *The functional* $J(\Omega, u) = \int_\Gamma |u|^2 \, \mathrm{d}\sigma$ *is continuous on* $\mathcal{F}$ *in the topology induced by the convergence* (9).

P r o o f. Let $((\Omega_n, u_n))_n$ be a sequence in $\mathcal{F}$, $\Omega_n = \Omega(\varphi_n)$, and assume that

$$(\Omega_n, u_n) \to (\Omega, u) \quad \text{as } n \to \infty,$$

where $\Omega = \Omega(\varphi)$ and $(\Omega, u) \in \mathcal{F}$. In what follows, the functions under consideration are of course the uniform extensions $\tilde{u}, \tilde{u}_n \in H^1(D)$, but for simplicity we shall drop the tilde. To show that $J(\Omega_n, u_n) \to J(\Omega, u)$, let us prove that $\sqrt{J(\Omega_n, u_n)} \to \sqrt{J(\Omega, u)}$. Letting $\|.\|$ stand for the L^2 norm on $[0,1]$, we can write

$$\begin{aligned}
(25) \quad & |\sqrt{J(\Omega_n, u_n)} - \sqrt{J(\Omega, u)}| \\
&= \big| \|u_n \circ \varphi_n \cdot |\varphi_n'|^{1/2}\| - \|u \circ \varphi \cdot |\varphi'|^{1/2}\| \big| \\
&\leqslant \|u_n \circ \varphi_n \cdot |\varphi_n'|^{1/2} - u \circ \varphi \cdot |\varphi'|^{1/2}\| \\
&\leqslant \|(u_n \circ \varphi_n - u \circ \varphi_n)|\varphi_n'|^{1/2}\| + \|(u \circ \varphi_n - u \circ \varphi)|\varphi_n'|^{1/2}\| \\
&\quad + \|u \circ \varphi(|\varphi_n'|^{1/2} - |\varphi'|^{1/2})\| \\
&\leqslant \|u_n - u\|_{0,\Gamma_n} + \sqrt{C_2}\|u \circ \varphi_n - u \circ \varphi\| \\
&\quad + \frac{1}{\sqrt{C_1}} \|u\|_{0,\Gamma} \sup_{[0,1]} \big| |\varphi_n'|^{1/2} - |\varphi'|^{1/2} \big| \\
&\leqslant K\|u_n - u\|_{r,D} + \sqrt{C_2}\|(u \circ \varphi_n - u \circ \varphi)\| + \frac{1}{2C_1}\|u\|_{0,\Gamma} \sup_{[0,1]} |\varphi_n' - \varphi'|,
\end{aligned}$$

where we have used Theorem 4. Then, Theorem 6 follows from Corollary 2 and the compactness of the injection of $H^1(D)$ into $H^r(D)$, of course by taking $\frac{1}{2} < r < 1$. □

Acknowledgement. We are grateful to the anonymous referee for his remarks and suggestions that let us improve the paper.

References

[1] *A. Acker*: An extremal problem involving current flow through distributed resistance. SIAM J. Math. Anal. *12* (1981), 169–172.
[2] *A. Acker, R. Meyer*: A free boundary problem for the p-Laplacian: uniqueness, convexity, and successive approximation of solutions. Electron. J. Differ. Equ. *8* (1995), 1–20.
[3] *H. W. Alt, L. A. Caffarelli*: Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math. *325* (1981), 105–144.
[4] *A. Beurling*: On free-boundary problems for the Laplace equations. Sem. analytic functions *1* (1958), 248–263.

[5] *A. Boulkhemair, A. Chakib*: On the uniform Poincaré inequality. Commun. Partial Differ. Equations *32* (2007), 1439–1447.

[6] *A. Boulkhemair, A. Chakib, A. Nachaoui*: Uniform trace theorem and application to shape optimization. Appl. Comput. Math. *7* (2008), 192–205.

[7] *P. Cardaliaguet, R. Tahraoui*: Some uniqueness results for Bernoulli interior free-boundary problems in convex domains. Electron. J. Differ. Equ. *102* (2002), 1–16.

[8] *D. Chenais*: On the existence of a solution in a domain identification problem. J. Math. Anal. Appl. *1975*, 189–219.

[9] *M. Crouzeix*: Variational approach of a magnetic shaping problem. Eur. J. Mech, B *10* (1991), 527–536.

[10] *J. Descloux*: Stability of the solutions of the bidimensional magnetic shaping problem in absence of surface tension. Eur. J. Mech, B *10* (1991), 513–526.

[11] *A. Fasano*: Some free boundary problems with industrial applications. Shape optimization and free boundaries, Proc. NATO ASI, Sémin. Math. Supér., Montréal/Can. 1990 (1992), 113–142.

[12] *M. Flucher, M. Rumpf*: Bernoulli's free-boundary problem, qualitative theory and numerical approximation. J. Reine Angew. Math. *486* (1997), 165–205.

[13] *A. Friedman*: Free boundary problem in fluid dynamics, variational methods for equilibrium problems of fluids. Astérisque *118* (1984), 55–67.

[14] *J. Haslinger, T. Kozubek, K. Kunisch, G. Peichl*: Shape optimization and fictitious domain approach for solving free boundary problems of Bernoulli type. Comput. Optim. Appl. *26* (2003), 231–251.

[15] *J. Haslinger, T. Kozubek, K. Kunisch, G. Peichl*: An embedding domain approach for a class of 2-d shape optimization problems: Mathematical analysis. J. Math. Anal. Appl. *290* (2004), 665–685.

[16] *A. Henrot, M. Pierre*: Variation and Optimization of Formes. A Geometric Analysis. (Variation et optimisation de formes. Une analyse géométrique.) Mathématiques & Applications 48. Springer, Berlin, 2005. (In French.)

[17] *K. Ito, K. Kunisch, G. Peichl*: Variational approach to shape derivatives for a class of Bernoulli problems. J. Math. Anal. Appl. *314* (2006), 126–149.

[18] *D. Kinderlehrer, L. Nirenberg*: Regularity in free boundary problems. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. *4* (1977), 373–391.

[19] *H. Lewy*: A note on harmonic functions and a hydrodynamical application. Proc. Am. Math. Soc. *3* (1952), 111–113.

[20] *O. Pironneau*: Optimal Shape Design for Elliptic Systems. Springer Series in Computational Physics. Springer, New York, 1984.

Authors' addresses: *Abdesslam Boulkhemair*, *Abdeljalil Nachaoui*, Laboratoire de Mathématiques Jean Leray, CNRS UMR6629/Université de Nantes, 2, rue de la Houssinière, BP 92208, 44322 Nantes, France, e-mail: `boulkhemair-a@univ-nantes.fr`, `nachaoui-a@univ-nantes.fr`; *Abdelkrim Chakib*, Département de Mathématiques Appliquées et Informatique, FST de Beni-Mellal, Université Sultan Moulay Slimane, B.P. 523, Beni-Mellal, Morocco, e-mail: `chakib@fstbm.ac.ma`.