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Communications in Mathematics, Vol. 20 (2012), No. 2, 81--88

Persistent URL: `http://dml.cz/dmlcz/143141`

VOLUME
20/2012
No. 2
ISSN 1804-1388

COMMUNICATIONS IN MATHEMATICS

Available online at
http://cm.osu.cz

UNIVERSITAS OSTRAVIENSIS

Communications in Mathematics 20 (2012) 81–88

On the Diophantine equation $x^2 + 2^\alpha 5^\beta 17^\gamma = y^n$

Hemar Godinho, Diego Marques, Alain Togbé

Abstract. In this paper, we find all solutions of the Diophantine equation $x^2 + 2^\alpha 5^\beta 17^\gamma = y^n$ in positive integers $x, y \geq 1$, $\alpha, \beta, \gamma, n \geq 3$ with $\gcd(x, y) = 1$.

1 Introduction

There are many results concerning the generalized Ramanujan-Nagell equation

$$x^2 + C = y^n, \tag{1}$$

where $C > 0$ is a given integer and x, y, n are positive integer unknowns with $n \geq 3$. Results obtained for general superelliptic equations clearly provide effective finiteness results for this equation, too (see for example [8], [31], [32] and the references given there). The first result concerning the above equation was due to V. A. Lebesgue [23] and it goes back to 1850, where he proved that the above equation has no solutions for $C = 1$. More recently, other values of C were considered. Tengely [33] solved the equation with $C = b^2$, b odd and $3 \leq b \leq 501$. The case where $C = p^k$, a power of a prime number, was studied in [5], [21], [20] for $p = 2$, in [6], [4], [24] for $p = 3$, in [1], [2] for $p = 5$, and in [27] for $p = 7$. The case $C = p^{2k}$ with $2 \leq p < 100$ prime and $\gcd(x, y) = 1$ was solved by Bérczes and Pink [9]. For arbitrary primes, some advances can be found in [7]. In [13], the cases with $1 \leq C \leq 100$ were completely solved. The solutions for the cases $C = 2^a \cdot 3^b$, $C = 2^a \cdot 5^b$ and $C = 5^a \cdot 13^b$, when x and y are coprime, can be found in [3], [25], [26], respectively. Recent progress on the subject were made in the cases $C = 5^a \cdot 11^b$, $C = 2^a \cdot 11^b$, $C = 2^a \cdot 3^b \cdot 11^c$, $C = 2^a \cdot 5^b \cdot 13^c$ and can be found in [16], [15], [14], [18]. For related results concerning equation (1) see [10], [22], [29], [30] and the references given there. For a survey concerning equation (1) see [12].

In this paper, we are interested in solving the Diophantine equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = y^n, \quad \gcd(x, y) = 1\,,\ x, y \geq 1\,,\ \alpha, \beta, \gamma \geq 0\,,\ n \geq 3\,. \tag{2}$$

2010 MSC: Primary 11D61, Secondary 11Y50
Key words: Diophantine equation, exponential equation, primitive divisor theorem

Our result is the following.

Theorem 1. *The equation (2) has no solution except for:*

$n = 3$	*the solutions given in Table 1;*
$n = 4$	*the solutions given in Table 2;*
$n = 5$	$(x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0)$;
$n = 6$	$(x, y, \alpha, \beta, \gamma) = (7, 3, 3, 1, 1), (23, 3, 3, 2, 0)$;
$n = 8$	$(x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1), (79, 3, 6, 1, 0)$.

One can deduce from the above result the following corollary.

Corollary 1. *The equation*

$$x^2 + 5^k 17^l = y^n, \quad x \geq 1\,,\ y \geq 1\,,\ \gcd(x, y) = 1\,,\ n \geq 3\,,\ k \geq 0\,,\ l \geq 0 \qquad (3)$$

has only the solutions

$$(x, y, k, l, n) = (94, 21, 2, 1, 3)\,,\ (2034, 161, 3, 2, 3)\,,\ (8, 3, 0, 1, 4)\,.$$

Therefore, our work extends that of Pink and Rábai [28]. We will follow the standard approach to work on equation (2) but with another version of MAGMA (V2.18-6) that gives better results when we deal with the corresponding elliptic curves.

2 The case $n = 3$

Lemma 1. *When $n = 3$, all the solutions to equation (2) are given in Table 1.*
For $n = 6$, we have $(x, y, \alpha, \beta, \gamma) = (7, 3, 3, 1, 1)\,,\ (23, 3, 3, 2, 0)$.

Proof. Equation (2) can be rewritten as

$$\left(\frac{x}{z^3}\right)^2 + A = \left(\frac{y}{z^2}\right)^3, \qquad (4)$$

where A is sixth-power free and defined implicitly by $2^\alpha 5^\beta 17^\gamma = Az^6$. One can see that $A = 2^{\alpha_1} 5^{\beta_1} 17^{\gamma_1}$ with $\alpha_1,\ \beta_1,\ \gamma_1,\ \in \{0, 1, 2, 3, 4, 5\}$. We thus get

$$V^2 = U^3 - 2^{\alpha_1} 5^{\beta_1} 17^{\beta_1}, \qquad (5)$$

with $U = y/z^2,\ V = x/z^3$ and $\alpha_1,\ \beta_1,\ \gamma_1\ \in \{0, 1, 2, 3, 4, 5\}$. We need to determine all the $\{2, 5, 17\}$-integral points on the above 216 elliptic curves. Recall that if $\mathcal{S}$ is a finite set of prime numbers, then an $\mathcal{S}$-integer is rational number a/b with coprime integers a and b, where the prime factors of b are in $\mathcal{S}$. We use the command *SIntegralPoints* of `MAGMA` [17] to determine all the $\{2, 5, 17\}$-integer points on the above elliptic curves. Here are a few remarks about the computations:

1. We eliminate the solutions with $UV = 0$ because they yield to $xy = 0$.

Table 1: Solutions for $n = 3$.

α_1	β_1	γ_1	z	α	β	γ	x	y
1	0	0	1	1	0	0	5	3
1	0	0	$2 \cdot 5$	7	6	0	383	129
2	0	0	1	2	0	0	11	5
4	0	1	5	4	6	1	5369	321
3	0	2	5	3	6	2	167589	3041
1	1	1	2^2	13	1	1	93	89
1	1	1	5	1	7	1	1531	171
1	1	1	1	1	1	1	453	59
3	1	1	1	3	1	1	7	9
1	1	2	1	1	1	2	63	19
2	1	2	1	2	1	2	59	21
1	1	3	2	7	1	3	5471	321
1	1	3	5	1	7	3	17052501	66251
3	2	0	1	3	2	0	23	9
3	2	0	2	9	2	0	17771	681
5	2	0	1	5	2	0	261	41
0	2	1	1	0	2	1	94	21
0	2	1	2	6	2	1	55157	1449
3	3	1	2	9	3	1	10763	489
3	3	1	2^2	15	3	1	4617433	27729
0	3	2	1	0	3	2	2034	161
3	3	5	2^5	33	3	5	2037783243169	160733121
1	4	0	1	1	4	0	9	11
4	4	1	$2 \cdot 5$	10	10	1	3274947	22169
5	4	2	$2 \cdot 5$	11	10	2	699659581	788121
1	5	0	17	1	5	6	916769	9971
1	5	1	17	1	5	7	846227	14859
1	5	1	2	7	5	1	17579	681

2. We consider only solutions such that the numerators of U and V are coprime.

3. If U and V are integers then $z = 1$. So $\alpha_1 = \alpha$, $\beta_1 = \beta$, and $\gamma_1 = \gamma$.

4. If U and V are rational numbers which are not integers, then z is determined by the denominators of U and V. The numerators of these rational numbers give x and y. Then α, β, γ are computed knowing that $2^\alpha 5^\beta 17^\gamma = Az^6$.

Therefore, we first determine $(U, V, \alpha_1, \beta_1, \gamma_1)$ and then we use the relations

$$U = \frac{y}{z^2}, \quad V = \frac{x}{z^3}, \quad 2^\alpha 5^\beta 17^\gamma = Az^6,$$

to find the solutions $(x, y, \alpha, \beta, \gamma)$ listed in Table 1.

For $n = 6$, equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = y^6 \tag{6}$$

becomes equation

$$x^2 + 2^\alpha 5^\beta 17^\gamma = \left(y^2\right)^3 . \tag{7}$$

We look in the list of solutions of Table 1 and observe that y is a perfect square only when $y = 9$ corresponding to two solutions. Therefore, the only solutions to equation (2) for $n = 6$ are the two solutions listed in Theorem 1. This completes the proof of Lemma 1. □

Remark 1. Notice that with the old version of `MAGMA`, it was difficult to determine the rational points of certain elliptic curves when $2^\alpha 5^\beta 17^\gamma$ is very high. That is the case of the following elliptic curves:

$$V^2 = U^3 - 2^3 \cdot 5^5 \cdot 17^5, \quad V^2 = U^3 - 2^5 \cdot 5^1 \cdot 17^4.$$

We thank the team `MAGMA`, particularly Steve Donnelly for the new version (Magma V2.18-6) and their help.

3 The case $n = 4$

Here, we have the following result.

Lemma 2. *If $n = 4$, then the only solutions to equation (2) are given in Table 2.*

If $n = 8$, then the only solution to equation (2) is $(x, y, \alpha, \beta, \gamma) = (47, 3, 8, 0, 1)$, $(79, 3, 6, 1, 0)$.

Table 2: Solutions for $n = 4$.

α_1	β_1	γ_1	z	α	β	γ	x	y
1	0	0	2	5	0	0	7	3
0	1	0	2	4	1	0	1	3
0	0	1	2^2	8	0	1	1087	33
0	0	1	1	0	0	1	8	3
0	0	1	2^2	8	0	1	47	9
1	0	1	2	5	0	1	9	5
3	0	1	2	7	0	1	15	7
3	0	1	2^2	11	0	1	495	23
2	1	0	2	6	1	0	79	9
2	2	1	2	6	2	1	409	21
3	2	2	2	7	2	2	511	33
1	0	3	2^2	9	0	3	4785	71

Proof. Equation (2) can be written as

$$\left(\frac{x}{z^2}\right)^2 + A = \left(\frac{y}{z}\right)^4 , \tag{8}$$

where A is fourth-power free and defined implicitly by $2^\alpha\, 5^\beta\, 17^\gamma = Az^4$. One can see that $A = 2^{\alpha_1}\, 5^{\beta_1}\, 17^{\gamma_1}$ with $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0, 1, 2, 3\}$. Hence, the problem consists of determining the $\{2, 5, 17\}$-integer points on the totality of the 64 elliptic curves

$$V^2 = U^4 - 2^{\alpha_1}\, 5^{b_1}\, 17^{\gamma_1}, \tag{9}$$

with $U = y/z,\ V = x/z^2$ and $\alpha_1,\ \beta_1,\ \gamma_1 \in \{0, 1, 2, 3\}$. Here, we use the command *SIntegralQuarticPoints* of `MAGMA` [17] to determine the $\{2, 5, 17\}$-integer points on the above elliptic curves. As in Section 2, we first find $(U, V, \alpha_1, \beta_1, \gamma_1)$, and then using the coprimality conditions on x and y and the definition of U and V, we determine all the corresponding solutions $(x, y, \alpha, \beta, \gamma)$ listed in Table 2.

Looking in the list of solutions of equation Table 2, we observe the 2 solutions in Table 2 whose values for y are perfect squares. Thus, the only solutions to equation (2) with $n = 8$ are those listed in Theorem 1. This concludes the proof of Lemma 2. $\square$

4 The case $n \geq 5$

The aim of this section is to determine all solutions of equation (2), for $n \geq 5$ and to prove its unsolubility for $n = 7$ and $n \geq 9$. The cases when n is of the form $2^a 3^b$ were treated in previous sections. So, apart from these cases, in order to prove that (2) has no solution for $n \geq 7$, it suffices to consider n prime. In fact, if $(x, y, \alpha, \beta, \gamma, n)$ is a solution for (2) and $n = pk$, where $p \geq 7$ is prime and $k > 1$, then $(x, y^k, \alpha, \beta, \gamma, p)$ is also a solution. So, from now on, n will denote a prime number.

Lemma 3. *The Diophantine equation* (2) *has no solution with $n \geq 5$ prime except for*

$$n = 5 \qquad (x, y, \alpha, \beta, \gamma) = (401, 11, 1, 3, 0)\,.$$

Proof. Let $(x, y, \alpha, \beta, \gamma, n)$ be a solution for (2). We claim that y is odd. In fact, if y is even and since $\gcd(x, y) = 1$, one has that x is odd, and then $-2^\alpha 5^\beta 17^\gamma \equiv x^2 - y^n \equiv 1 \pmod 4$, but this contradicts the fact that $-2^\alpha 5^\beta 17^\gamma \equiv 0, 2$ or $3 \pmod 4$ (according to $\alpha \geq 2, \alpha = 1$ or $\alpha = 0$, respectively). Now, write equation (2) as $x^2 + dz^2 = y^n$, where

$$d = 2^{\alpha - 2\lfloor \alpha/2 \rfloor} 5^{\beta - 2\lfloor \beta/2 \rfloor} 17^{\gamma - 2\lfloor \gamma/2 \rfloor}\,,$$

and $z = 2^{\lfloor \alpha/2 \rfloor} 5^{\lfloor \beta/2 \rfloor} 17^{\lfloor \gamma/2 \rfloor}$. Since $x - 2\lfloor x/2 \rfloor \in \{0, 1\}$, we have

$$d \in \{1, 2, 5, 10, 17, 34, 85, 170\}\,.$$

We then factor the previous equation over $\mathbb{K} = \mathbb{Q}[\mathrm{i}\sqrt{d}] = \mathbb{Q}[\sqrt{-d}]$ as

$$(x + \mathrm{i}\sqrt{d}z)(x - \mathrm{i}\sqrt{d}z) = y^n.$$

Now, we claim that the ideals $(x + \mathrm{i}\sqrt{d}z)\mathcal{O}_{\mathbb{K}}$ and $(x - \mathrm{i}\sqrt{d}z)\mathcal{O}_{\mathbb{K}}$ are coprime. If this is not the case, there must exist a prime ideal $\mathfrak{p}$ containing these ideals. Therefore, $x \pm \mathrm{i}\sqrt{d}z$ and y^n (and so y) belong to $\mathfrak{p}$. Thus $2x \in \mathfrak{p}$ and hence either 2 or x belongs to $\mathfrak{p}$. Since $\gcd(2,y)=\gcd(x,y)=1$, then 1 belongs to the ideals $\langle 2,y\rangle$ and $\langle x,y\rangle$, then $1\in\mathfrak{p}$ leading to an absurdity of $\mathfrak{p}=\mathcal{O}_{\mathbb{K}}$. By the unique factorization of ideals, it follows that $(x+\mathrm{i}\sqrt{d}z)\mathcal{O}_{\mathbb{K}}=\mathfrak{j}^n$, for some ideal $\mathfrak{j}$ of $\mathcal{O}_{\mathbb{K}}$. Using Mathematica's command *NumberFieldClassNumber[Sqrt[-d]]*, we obtain that the class number of $\mathbb{K}$ is either $1,2,4$ or 12 and so coprime to n, then $\mathfrak{j}$ is a principal ideal yielding

$$x+\mathrm{i}\sqrt{d}z=\varepsilon\eta^n, \tag{10}$$

for some $\eta\in\mathcal{O}_{\mathbb{K}}$ and ε a unit of $\mathbb{K}$. Since the group of units of $\mathbb{K}$ is a subset of $\{\pm1,\pm\mathrm{i}\}$ and n is odd, then ε is a n-th power. Thus, (10) can be reduced to $x+\mathrm{i}\sqrt{d}z=\eta^n$. Since $\mathbb{K}$ is an imaginary quadratic field and $-d\not\equiv 1 \pmod 4$, then $\{1,\mathrm{i}\sqrt{d}\}$ is an integral basis and we can write $\eta=u+\mathrm{i}\sqrt{d}v$, for some integers u and v. We then get

$$\frac{\eta^n-\overline{\eta}^n}{\eta-\overline{\eta}}=\frac{2^{\lfloor\alpha/2\rfloor}5^{\lfloor\beta/2\rfloor}17^{\lfloor\gamma/2\rfloor}}{v}, \tag{11}$$

where, as usual, $\overline{w}$ denotes the complex conjugate of w.

Let $(L_m)_{m\ge0}$ be the Lucas sequence given by

$$L_m=\frac{\eta^m-\overline{\eta}^m}{\eta-\overline{\eta}},\quad \text{for}\quad m\ge 0\,.$$

We recall that the Primitive Divisor Theorem for Lucas sequences ensures for primes $n\ge5$, that there exists a *primitive divisor* for L_n, except for the finitely many (*defective*) pairs $(\eta,\overline{\eta})$ given in Table 1 of [11] (a primitive divisor of L_n is a prime that divides L_n but does not divide $(\eta-\overline{\eta})^2\prod_{j=1}^{n-1}L_j$). And a helpful property of a primitive divisor p is that $p\equiv\pm1 \pmod n$.

For $n=5$, we find in Table 1 in [11] that L_5 has a primitive divisor except for $(u,d,v)=(1,10,1)$ which leads to a number $\eta=1+\mathrm{i}\sqrt{10}\in\mathbb{Q}[\mathrm{i}\sqrt{10}]$ ($d=10$ is one of the possible values of d described in the beginning of this proof), which gives the solution with $n=5$.

Apart from this case, let p be a primitive divisor of L_n, $n\ge7$. The identity (11) implies that $p\in\{2,5,17\}$ and so $p=17$, since $p\not\equiv\pm1 \pmod n$, for $p=2{,}5$. Hence, n is a prime dividing 17 ± 1 and so $n=2$ or 3 which contradicts the fact that $n\ge7$. This completes the proof of Theorem 1. □

Acknowledgement

The authors are grateful to the anonymous referees for insightful and valuable comments that helped to improve the manuscript. The first author was partially supported by a grant from CNPq-Brazil. The second author is grateful to FAP-DF, FEMAT-Brazil and CNPq-Brazil for the financial support. The third author thanks Professor I. Pink for sending a copy of his manuscript [28]. He was also partially supported by Purdue University North Central.

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Received: 23 July, 2012
Accepted for publication: 27 November, 2012
Communicated by: Attila Bérczes