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SEEMINGLY UNRELATED REGRESSION MODELS

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Abstract. The cross-covariance matrix of observation vectors in two linear statistical models need not be zero matrix. In such a case the problem is to find explicit expressions for the best linear unbiased estimators of both model parameters and estimators of variance components in the simplest structure of the covariance matrix. Univariate and multivariate forms of linear models are dealt with.

Keywords: seemingly unrelated regression, linear statistical model, variance components, BLUE, MINQUE

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Introduction

Suppose a medicament must be tested on a group of animals. The aim of the experiment is to recognize whether the medicament influences the relationship between a value of a physiological parameter and characteristics (weight, age, etc.) of animals.

The measured values of physiological parameter on n animals are given by an n-dimensional vector $\boldsymbol{\mu}$, the characteristics of animals are given by an $n \times k$ matrix $\mathbf{X}$. The dependence of $\boldsymbol{\mu}$ on $\mathbf{X}$ is assumed to be of the form $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is a k-dimensional unknown vector parameter. The observation vector $\mathbf{Y}_1$ before the experiment is $\mathbf{Y}_1 \sim (\mathbf{X}\boldsymbol{\beta}_1, \boldsymbol{\Sigma}_{1,1})$, where the mean value $E(\mathbf{Y}_1)$ of the observation vector $\mathbf{Y}_1$ is $E(\mathbf{Y}_1) = \mathbf{X}\boldsymbol{\beta}_1$ and the covariance matrix $\mathrm{Var}(\mathbf{Y}_1)$ is $\mathrm{Var}(\mathbf{Y}_1) = \boldsymbol{\Sigma}_{1,1}$. The observation vector $\mathbf{Y}_2$ after using the medicament is $\mathbf{Y}_2 \sim (\mathbf{X}\boldsymbol{\beta}_2, \boldsymbol{\Sigma}_{2,2})$.

Since the same animals are investigated before and after the experiment, it must be assumed that $\mathrm{cov}(\mathbf{Y}_1, \mathbf{Y}_2) = \boldsymbol{\Sigma}_{1,2} \neq \mathbf{0}$

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Thus the whole experiment can be described as

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim \left[\begin{pmatrix} \mathbf{X}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X} \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{1,2} \\ \boldsymbol{\Sigma}_{2,1} & \boldsymbol{\Sigma}_{2,2} \end{pmatrix} \right].$$

To be a little more general, let the model be

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{1,2} \\ \boldsymbol{\Sigma}_{2,1}, & \boldsymbol{\Sigma}_{2,2} \end{pmatrix} \right]. \tag{1}$$

The problem is to estimate the vector parameters $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$ and, in the case $\mathbf{X}_1 = \mathbf{X}_2$, to test the hypothesis $H_0\colon\ \boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ against $H_1\colon\ \boldsymbol{\beta}_1 \neq \boldsymbol{\beta}_2$.

Some partial solutions are given in the following text. It is assumed that the rank of the matrix $\mathbf{X}_1$ is $r(\mathbf{X}_1) = k < n$ and the same is valid for the matrix $\mathbf{X}_2$. The matrix $\begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, \boldsymbol{\Sigma}_{1,2} \\ \boldsymbol{\Sigma}_{2,1}, \boldsymbol{\Sigma}_{2,2} \end{pmatrix}$ is assumed to be positive definite, i.e. the model (1) is regular.

The seemingly unrelated regression model was introduced by A. Zellner in the year 1962. There are relatively many papers and also chapters in monographs on econometrics treating the model, e.g. among others see [1], [2], [3], [7], [10].

1. Univariate models

The best linear unbiased estimator (BLUE) of the vector $\begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}$ in the model (1) is

$$\begin{aligned} \begin{pmatrix} \hat{\boldsymbol{\beta}}_1 \\ \hat{\boldsymbol{\beta}}_2 \end{pmatrix} = & \left[\begin{pmatrix} \mathbf{X}_1', & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}^{1,1}, & \boldsymbol{\Sigma}^{1,2} \\ \boldsymbol{\Sigma}^{2,1}, & \boldsymbol{\Sigma}^{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2 \end{pmatrix} \right]^{-1} \\ & \times \begin{pmatrix} \mathbf{X}_1', & \mathbf{0} \\ \mathbf{0}, & \mathbf{X}_2' \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}^{1,1}, & \boldsymbol{\Sigma}^{1,2} \\ \boldsymbol{\Sigma}^{2,1}, & \boldsymbol{\Sigma}^{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}, \end{aligned}$$

where

$$\begin{pmatrix} \boldsymbol{\Sigma}^{1,1}, & \boldsymbol{\Sigma}^{1,2} \\ \boldsymbol{\Sigma}^{2,1}, & \boldsymbol{\Sigma}^{2,2} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{1,2} \\ \boldsymbol{\Sigma}_{2,1}, & \boldsymbol{\Sigma}_{2,2} \end{pmatrix}^{-1}.$$

In order to express the estimators $\hat{\boldsymbol{\beta}}_1$ and $\hat{\boldsymbol{\beta}}_2$ separately, the following lemma will be used.

Lemma 1.1. *In the regular model*

$$\begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} \sim \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{pmatrix} \mathbf{T}_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{T}_{2,2} \end{pmatrix} \right]$$

the BLUE of $\boldsymbol{\beta}_2$ *is*

$$\begin{aligned} \hat{\boldsymbol{\beta}}_2 = \{&\mathbf{X}_2'[\mathbf{T}_{2,2} + \mathbf{D}(\mathbf{X}_1'\mathbf{T}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}']^{-1}\mathbf{X}_2\}^{-1}\mathbf{X}_2'[\mathbf{T}_{2,2} + \mathbf{D}(\mathbf{X}_1'\mathbf{T}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{D}']^{-1} \\ &\times [\boldsymbol{\eta}_2 - \mathbf{D}(\mathbf{X}_1'\mathbf{T}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{T}_{1,1}^{-1}\boldsymbol{\eta}_1]. \end{aligned}$$

Proof. Proof is given in [4], p. 326. □

Theorem 1.2. *The BLUEs of* $\boldsymbol{\beta}_1$ *and* $\boldsymbol{\beta}_2$ *in the model* (1) *are*

$$\begin{aligned} \hat{\boldsymbol{\beta}}_1 &= \left\{\mathbf{X}_1'[\boldsymbol{\Sigma}_{11.2} + \boldsymbol{\Sigma}_{1,2}\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_{2,2}^{-1}\boldsymbol{\Sigma}_{2,1}]^{-1}\mathbf{X}_1\right\}^{-1} \\ &\quad \times \mathbf{X}_1'[\boldsymbol{\Sigma}_{11.2} + \boldsymbol{\Sigma}_{1,2}\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_{2,2}^{-1}\boldsymbol{\Sigma}_{2,1}]^{-1} \\ &\quad \times \left\{\mathbf{Y}_1 - \boldsymbol{\Sigma}_{1,2}\boldsymbol{\Sigma}_{2,2}^{-1}[\mathbf{I} - \mathbf{X}_2(\mathbf{X}_2'\boldsymbol{\Sigma}_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_{2,2}^{-1}]\mathbf{Y}_2\right\}, \\ \hat{\boldsymbol{\beta}}_2 &= \left\{\mathbf{X}_2'[\boldsymbol{\Sigma}_{22.1} + \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\boldsymbol{\Sigma}_{1,2}]^{-1}\mathbf{X}_2\right\}^{-1} \\ &\quad \times \mathbf{X}_2'[\boldsymbol{\Sigma}_{22.1} + \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\boldsymbol{\Sigma}_{1,2}]^{-1} \\ &\quad \times \left\{\mathbf{Y}_2 - \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}[\mathbf{I} - \mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}]\mathbf{Y}_1\right\}, \end{aligned}$$

where

$$\boldsymbol{\Sigma}_{11.2} = \boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,2}\boldsymbol{\Sigma}_{2,2}^{-1}\boldsymbol{\Sigma}_{2,1}, \quad \boldsymbol{\Sigma}_{22.1} = \boldsymbol{\Sigma}_{2,2} - \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}\boldsymbol{\Sigma}_{1,2}.$$

Proof. Let the model (1) be transformed in the following way

$$\begin{aligned} \begin{pmatrix} \mathbf{I}, & \mathbf{0} \\ -\boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}, & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} &= \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 - \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{Y}_1 \end{pmatrix} \\ &\sim \left[\begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ -\boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_{1,1}, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_{22.1} \end{pmatrix} \right]. \end{aligned}$$

Let in Lemma 1.1

$$\boldsymbol{\eta}_1 = \mathbf{Y}_1, \quad \boldsymbol{\eta}_2 = \mathbf{Y}_2 - \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{Y}_1, \quad \mathbf{T}_{1,1} = \boldsymbol{\Sigma}_{1,1}, \quad \mathbf{T}_{2,2} = \boldsymbol{\Sigma}_{22.1}.$$

Then

$$\begin{aligned} \hat{\boldsymbol{\beta}}_2 &= \left\{\mathbf{X}_2'[\boldsymbol{\Sigma}_{22.1} + \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\boldsymbol{\Sigma}_{1,2}]^{-1}\mathbf{X}_2\right\}^{-1} \\ &\quad \times \mathbf{X}_2'[\boldsymbol{\Sigma}_{22.1} + \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\boldsymbol{\Sigma}_{1,2}]^{-1} \\ &\quad \times \left\{\mathbf{Y}_2 - \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}_{1,1}^{-1}[\mathbf{I} - \mathbf{X}_1(\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol{\Sigma}_{1,1}^{-1}]\mathbf{Y}_1\right\}. \end{aligned}$$

Because of the structure of the model (1), the expression for the BLUE $\hat{\boldsymbol\beta}_1$ can be written as

$$\begin{aligned}\hat{\boldsymbol\beta}_1 = &\left\{\mathbf{X}_1'[\boldsymbol\Sigma_{11.2} + \boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_{2,2}^{-1}\boldsymbol\Sigma_{2,1}]^{-1}\mathbf{X}_1\right\}^{-1}\\ &\times \mathbf{X}_1'[\boldsymbol\Sigma_{11.2} + \boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_{2,2}^{-1}\boldsymbol\Sigma_{2,1}]^{-1}\\ &\times \left\{\mathbf{Y}_1 - \boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}[\mathbf{I} - \mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_{2,2}^{-1}]\mathbf{Y}_2\right\}.\end{aligned}$$

□

Lemma 1.3. *The covariance matrices of $\hat{\boldsymbol\beta}_1$ and $\hat{\boldsymbol\beta}_2$ from Theorem* 1.2 *are*

$$\begin{aligned}\mathrm{Var}(\hat{\boldsymbol\beta}_1) &= \left\{\mathbf{X}_1'[\boldsymbol\Sigma_{11.2} + \boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_{2,2}^{-1}\boldsymbol\Sigma_{2,1}]^{-1}\mathbf{X}_1\right\}^{-1},\\ \mathrm{Var}(\hat{\boldsymbol\beta}_2) &= \left\{\mathbf{X}_2'[\boldsymbol\Sigma_{22.1} + \boldsymbol\Sigma_{2,1}\boldsymbol\Sigma_{1,1}^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol\Sigma_{1,1}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol\Sigma_{1,1}^{-1}\boldsymbol\Sigma_{1,2}]^{-1}\mathbf{X}_2\right\}^{-1},\\ \mathrm{cov}(\hat{\boldsymbol\beta}_1, \hat{\boldsymbol\beta}_2) &= \mathrm{Var}(\hat{\boldsymbol\beta}_1)\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\mathbf{X}_2)^{-1}\\ &= \left\{\mathrm{Var}(\hat{\boldsymbol\beta}_2)\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\boldsymbol\Sigma_{2,1}\boldsymbol\Sigma_{1,1}^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\mathbf{X}_1)^{-1}\right\}'\\ &= \mathrm{Var}(\hat{\boldsymbol\beta}_1)[\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2 - \mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2\\ &\quad\times(\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\boldsymbol\Sigma_{2,1}\boldsymbol\Sigma_{1,1}^{-1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\mathbf{X}_1)^{-1}\\ &\quad\times\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2]\,\mathrm{Var}(\hat{\boldsymbol\beta}_2).\end{aligned}$$

Proof. The expressions for $\mathrm{Var}(\hat{\boldsymbol\beta}_1)$ and $\mathrm{Var}(\hat{\boldsymbol\beta}_2)$ can be obtained directly from the expressions for $\hat{\boldsymbol\beta}_1$ and $\hat{\boldsymbol\beta}_2$. The expression for $\mathrm{cov}(\hat{\boldsymbol\beta}_1, \hat{\boldsymbol\beta}_2)$ can be obtained as

$$(\mathbf{I}, \mathbf{0})\begin{pmatrix}\mathbf{X}_1'\boldsymbol\Sigma^{1,1}\mathbf{X}_1, & \mathbf{X}_1'\boldsymbol\Sigma^{1,2}\mathbf{X}_2\\ \mathbf{X}_2'\boldsymbol\Sigma^{2,1}\mathbf{X}_1, & \mathbf{X}_2'\boldsymbol\Sigma^{2,2}\mathbf{X}_2\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{0}\\ \mathbf{I}\end{pmatrix}.$$

The identity

$$\begin{pmatrix}\mathbf{A}, & \mathbf{B}\\ \mathbf{B}', & \mathbf{C}\end{pmatrix}^{-1} = \begin{pmatrix}\alpha\alpha, & \alpha\beta\\ \beta\alpha, & \beta\beta\end{pmatrix}$$

with

$$\begin{aligned}\alpha\alpha &= (\mathbf{A} - \mathbf{B}'\mathbf{C}^{-1}\mathbf{B})^{-1},\\ \alpha\beta &= -(\mathbf{A} - \mathbf{B}'\mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{B}\mathbf{C}^{-1} = (\beta\alpha)',\\ \beta\beta &= (\mathbf{C} - \mathbf{B}'\mathbf{A}^{-1}\mathbf{B})^{-1},\end{aligned}$$

which is valid for any positive definite matrix $\begin{pmatrix}\mathbf{A}, & \mathbf{B}\\ \mathbf{B}', & \mathbf{C}\end{pmatrix}$, is used in what follows. Thus

$$\begin{aligned}\mathrm{cov}(\hat{\boldsymbol\beta}_1, \hat{\boldsymbol\beta}_2) &= -[\mathbf{X}_1'\boldsymbol\Sigma^{1,1}\mathbf{X}_1 - \mathbf{X}_1'\boldsymbol\Sigma^{1,2}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma^{2,2}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma^{2,1}]^{-1}\\ &\quad\times\mathbf{X}_1'\boldsymbol\Sigma^{1,2}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma^{2,2}\mathbf{X}_2)^{-1}\\ &= \mathrm{Var}(\hat{\boldsymbol\beta}_1)\mathbf{X}_1'\boldsymbol\Sigma^{1,2}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma^{2,2}\mathbf{X}_2)^{-1}\end{aligned}$$

$$= \operatorname{Var}(\hat{\boldsymbol\beta}_1)\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2\left[\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\mathbf{X}_2\right]^{-1}$$
$$= \operatorname{Var}(\hat{\boldsymbol\beta}_1)\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\mathbf{X}_2)^{-1}(\operatorname{Var}(\hat{\boldsymbol\beta}_2))^{-1}\operatorname{Var}(\hat{\boldsymbol\beta}_2).$$

Since

$$(\operatorname{Var}(\hat{\boldsymbol\beta}_2))^{-1} = \mathbf{X}_2'\boldsymbol\Sigma^{2,2}\mathbf{X}_2 - \mathbf{X}_2'\boldsymbol\Sigma^{2,1}\mathbf{X}_1(\mathbf{X}_1'\boldsymbol\Sigma^{1,1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol\Sigma^{1,2}\mathbf{X}_2,$$

we have

$$\begin{aligned}
&\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\mathbf{X}_2)^{-1}(\operatorname{Var}(\hat{\boldsymbol\beta}_2))^{-1}\\
&\quad= \mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2\\
&\qquad - \mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2(\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\mathbf{X}_2)^{-1}\mathbf{X}_2'\boldsymbol\Sigma_{22.1}^{-1}\boldsymbol\Sigma_{2,1}\boldsymbol\Sigma_{1,1}^{-1}\mathbf{X}_1\\
&\qquad \times (\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\mathbf{X}_1)^{-1}\mathbf{X}_1'\boldsymbol\Sigma_{11.2}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}_2.
\end{aligned}$$

□

To test the hypothesis (in the case of normality) $H_0\colon \boldsymbol\beta_1 = \boldsymbol\beta_2$ on the base of the BLUEs of $\boldsymbol\beta_1$ and $\boldsymbol\beta_2$ is simple if the covariance matrix $\begin{pmatrix}\boldsymbol\Sigma_{1,1}, & \boldsymbol\Sigma_{1,2}\\ \boldsymbol\Sigma_{2,1}, & \boldsymbol\Sigma_{2,2}\end{pmatrix}$ is given.

It is possible to base the test on the estimators

$$\tilde{\boldsymbol\beta}_i = (\mathbf{X}\boldsymbol\Sigma_{i,i}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Sigma_{i,i}^{-1}\mathbf{Y}_1, \quad i = 1,2,$$

as well. Here

$$\begin{aligned}
\begin{pmatrix}\tilde{\boldsymbol\beta}_1\\ \tilde{\boldsymbol\beta}_2\end{pmatrix} &\sim N_{2k}\left[\begin{pmatrix}\boldsymbol\beta_1\\ \boldsymbol\beta_2\end{pmatrix}, \begin{pmatrix}\operatorname{Var}(\tilde{\boldsymbol\beta}_1), & \operatorname{cov}(\tilde{\boldsymbol\beta}_1,\tilde{\boldsymbol\beta}_2)\\ \operatorname{cov}(\tilde{\boldsymbol\beta}_2,\tilde{\boldsymbol\beta}_1), & \operatorname{Var}(\tilde{\boldsymbol\beta}_2)\end{pmatrix}\right],\\
\operatorname{Var}(\tilde{\boldsymbol\beta}_1) &= (\mathbf{X}'\boldsymbol\Sigma_{1,1}^{-1}\mathbf{X})^{-1},\\
\operatorname{Var}(\tilde{\boldsymbol\beta}_2) &= (\mathbf{X}'\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X})^{-1},\\
\operatorname{cov}(\tilde{\boldsymbol\beta}_1,\tilde{\boldsymbol\beta}_2) &= (\mathbf{X}'\boldsymbol\Sigma_{1,1}^{-1}\mathbf{X})^{-1}\mathbf{X}'\boldsymbol\Sigma_{1,1}^{-1}\boldsymbol\Sigma_{1,2}\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X}(\mathbf{X}'\boldsymbol\Sigma_{2,2}^{-1}\mathbf{X})^{-1}\\
&= [\operatorname{cov}(\tilde{\boldsymbol\beta}_2,\tilde{\boldsymbol\beta}_1)]'.
\end{aligned}$$

Since the matrices $\boldsymbol\Sigma_{1,1}$, $\boldsymbol\Sigma_{1,2}$, $\boldsymbol\Sigma_{2,2}$ are usually unknown, a way out is to assume a simple structure with a few unknown estimable parameters. The simplest structure seems to be

$$\boldsymbol\Sigma_{1,1} = \sigma_1^2\mathbf{I}, \quad \boldsymbol\Sigma_{1,2} = c\mathbf{I}, \quad \boldsymbol\Sigma_{2,2} = \sigma_2^2\mathbf{I}.$$

Thus the model (1) in the case $\mathbf{X}_1 = \mathbf{X}_2$ can be written as

$$\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} \sim N_{2n}\left[(\mathbf{I}_2\otimes\mathbf{X})\begin{pmatrix}\boldsymbol\beta_1\\ \boldsymbol\beta_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2, & c\\ c, & \sigma_2^2\end{pmatrix}\otimes\mathbf{I}_n\right].$$

Lemma 1.4. *Let* $\boldsymbol{\vartheta} = (\sigma_1^2, \sigma_2^2, c)' = (\vartheta_1, \vartheta_2, \vartheta_3)'$. *Then the* $\boldsymbol{\vartheta}_0$*-MINQUE (minimum norm quadratic unbiased estimator) of* $\boldsymbol{\vartheta}$ *is* $\hat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}\hat{\gamma}$, *where* $\boldsymbol{\vartheta}_0 = (\sigma_{1,0}^2, \sigma_{2,0}^2, c_0)'$ *is an approximate value of the vector* $\boldsymbol{\vartheta}$,

$$\mathbf{S} = \frac{n-k}{(\sigma_{1,0}^2\sigma_{2,0}^2 - c_0^2)^2}\begin{pmatrix} \sigma_{2,0}^4, & c_0^2, & -2c_0\sigma_{2,0}^2 \\ c_0^2, & \sigma_{1,0}^4, & -2c_0\sigma_{1,0}^2 \\ -2c_0\sigma_{2,0}^2, & -2c_0\sigma_{1,0}^2, & 2(\sigma_{1,0}^2\sigma_{2,0}^2 + c_0^2) \end{pmatrix}$$

(it is assumed to be regular), $\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)'$,

$$\hat{\gamma}_1 = \frac{1}{(\sigma_{1,0}^2\sigma_{2,0}^2 - c_0^2)^2}\,(\sigma_{2,0}^4\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_1 - 2c_0\sigma_{2,0}^2\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_2 + c_0^2\mathbf{Y}_2'\mathbf{M}_X\mathbf{Y}_2),$$

$$\hat{\gamma}_2 = \frac{1}{(\sigma_{1,0}^2\sigma_{2,0}^2 - c_0^2)^2}\,(c_0^2\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_1 - 2c_0\sigma_{1,0}^2\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_2 + \sigma_{1,0}^4\mathbf{Y}_2'\mathbf{M}_X\mathbf{Y}_2),$$

$$\hat{\gamma}_3 = \frac{1}{(\sigma_{1,0}^2\sigma_{2,0}^2 - c_0^2)^2}(-2c_0\sigma_{2,0}^2\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_1 + 2(\sigma_{1,0}^2\sigma_{2,0}^2 + c_0^2)\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_2 - 2c_0\sigma_{1,0}^2\mathbf{Y}_2'\mathbf{M}_X\mathbf{Y}_2).$$

In the case of normality of $\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix}$ *it is valid that*

$$\operatorname*{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}}) = 2\mathbf{S}^{-1}.$$

Proof. The covariance matrix $\begin{pmatrix}\sigma_1^2, & c\\ c, & \sigma_2^2\end{pmatrix}\mathbf{I}_n$ can be rewritten as

$$\begin{pmatrix}\sigma_1^2, & c\\ c, & \sigma_2^2\end{pmatrix} \otimes \mathbf{I}_n = \sigma_1^2\mathbf{V}_1 + \sigma_2^2\mathbf{V}_2 + c\mathbf{V}_3,$$

where

$$\mathbf{V}_1 = \begin{pmatrix}1, & 0\\ 0, & 0\end{pmatrix}\otimes\mathbf{I}_n, \quad \mathbf{V}_2 = \begin{pmatrix}0, & 0\\ 0, & 1\end{pmatrix}\otimes\mathbf{I}_n, \quad \mathbf{V}_3 = \begin{pmatrix}0, & 1\\ 1, & 0\end{pmatrix}\otimes\mathbf{I}_n.$$

Now the procedure for estimation of $\boldsymbol{\vartheta}$ in the model

$$\boldsymbol{\eta} \sim N_n\left(\mathbf{A\Theta}, \sum_{i=1}^{p}\vartheta_i\mathbf{V}_i\right)$$

can be used (in more detail see in [9]). Here

$$\hat{\boldsymbol{\vartheta}} = \mathbf{S}^{-1}_{(M_A\Sigma_0M_A)^+}\begin{pmatrix}\boldsymbol{\eta}'(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\mathbf{V}_1(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\boldsymbol{\eta}\\ \vdots \\ \boldsymbol{\eta}'(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\mathbf{V}_p(\mathbf{M}_A\boldsymbol{\Sigma}_0\mathbf{M}_A)^+\boldsymbol{\eta}\end{pmatrix},$$

$$\boldsymbol{\Sigma}_0 = \sum_{i=1}^{p}\vartheta_{i,0}\mathbf{V}_i \quad \text{and} \quad \operatorname{Var}_{\vartheta_0}(\hat{\boldsymbol{\vartheta}}) = 2\mathbf{S}^{-1}_{(M_A\Sigma_0M_A)^+}.$$

If the relationships $(\mathbf{X}_1 = \mathbf{X}_2)$

$$\boldsymbol{\Sigma}_0 = \begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix} \otimes \mathbf{I}_n, \quad \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} = \mathbf{I}_2 \otimes \mathbf{M}_X,$$

$$\left\{ \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \left[\begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix} \otimes \mathbf{I}_n \right] \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \right\}^+ = \begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix}^{-1} \otimes \mathbf{M}_X,$$

$$\left\{ \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \left[\begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix} \otimes \mathbf{I}_n \right] \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \right\}^+ \left[\begin{pmatrix} 1, & 0 \\ 0, & 0 \end{pmatrix} \otimes \mathbf{I}_n \right]$$
$$\times \left\{ \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \left[\begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix} \otimes \mathbf{I}_n \right] \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \right\}^+$$
$$= \frac{1}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} \begin{pmatrix} \sigma_{2,0}^4, & -c_0 \sigma_{2,0}^2 \\ -c_0 \sigma_{2,0}^2, & c_0^2 \end{pmatrix} \otimes \mathbf{M}_X = \mathbf{A}_1 \otimes \mathbf{M}_X,$$

$$\left\{ \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \left[\begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix} \otimes \mathbf{I}_n \right] \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \right\}^+ \left[\begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \otimes \mathbf{I}_n \right]$$
$$\times \left\{ \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \left[\begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix} \otimes \mathbf{I}_n \right] \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \right\}^+$$
$$= \frac{1}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} \begin{pmatrix} c_0^2, & -c_0 \sigma_{1,0}^2 \\ -c_0 \sigma_{1,0}^2, & \sigma_{1,0}^4 \end{pmatrix} \otimes \mathbf{M}_X = \mathbf{A}_2 \otimes \mathbf{M}_X,$$

$$\left\{ \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \left[\begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix} \otimes \mathbf{I}_n \right] \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \right\}^+ \left[\begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \otimes \mathbf{I}_n \right]$$
$$\times \left\{ \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \left[\begin{pmatrix} \sigma_{1,0}^2, & c_0 \\ c_0, & \sigma_{2,0}^2 \end{pmatrix} \otimes \mathbf{I}_n \right] \mathbf{M}_{\begin{pmatrix} X, & 0 \\ 0, & X \end{pmatrix}} \right\}^+$$
$$= \frac{1}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} \begin{pmatrix} -2c_0 \sigma_{2,0}^2, & \sigma_{1,0}^2 \sigma_{2,0}^2 + c_0^2 \\ \sigma_{1,0}^2 \sigma_{2,0}^2 + c_0^2, & -2c_0 \sigma_{1,0}^2 \end{pmatrix} \otimes \mathbf{M}_X = \mathbf{A}_3 \otimes \mathbf{M}_X,$$

$$\mathrm{Tr}\left\{ (\mathbf{A}_1 \otimes \mathbf{M}_X) \left[\begin{pmatrix} 1, & 0 \\ 0, & 0 \end{pmatrix} \otimes \mathbf{I}_n \right] \right\} = \frac{\sigma_{2,0}^4}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} (n-k),$$

$$\mathrm{Tr}\left\{ (\mathbf{A}_1 \otimes \mathbf{M}_X) \left[\begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \otimes \mathbf{I}_n \right] \right\} = \frac{c_0^2}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} (n-k),$$

$$\mathrm{Tr}\left\{ (\mathbf{A}_1 \otimes \mathbf{M}_X) \left[\begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \otimes \mathbf{I}_n \right] \right\} = \frac{-2c_0 \sigma_{2,0}^2}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} (n-k),$$

$$\mathrm{Tr}\left\{ (\mathbf{A}_2 \otimes \mathbf{M}_X) \left[\begin{pmatrix} 0, & 0 \\ 0, & 1 \end{pmatrix} \otimes \mathbf{I}_n \right] \right\} = \frac{\sigma_{1,0}^4}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} (n-k),$$

$$\mathrm{Tr}\left\{ (\mathbf{A}_2 \otimes \mathbf{M}_X) \left[\begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \otimes \mathbf{I}_n \right] \right\} = \frac{-2c_0 \sigma_{1,0}^2}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} (n-k),$$

$$\mathrm{Tr}\left\{ (\mathbf{A}_3 \otimes \mathbf{M}_X) \left[\begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \otimes \mathbf{I}_n \right] \right\} = \frac{2(\sigma_{1,0}^2 \sigma_{2,0}^2 + c_0^2)}{(\sigma_{1,0}^2 \sigma_{2,0}^2 - c_0^2)^2} (n-k),$$

are used, then the proof can be easily finished. □

In practice the iteration procedure for the determination of the estimators $\hat{\sigma}_1^2$, $\hat{\sigma}_2^2$, $\hat{c}$ is used.

If the separate estimators of σ_1^2, σ_2^2, c are not interesting for a statistician and the test of the hypothesis $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ is important only, then the following procedure (see the following lemma) can be used.

Lemma 1.5. *If* $\operatorname{Var}\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix} = \begin{pmatrix}\sigma_1^2, & c\\ c, & \sigma_2^2\end{pmatrix} \otimes \mathbf{I}_n$ *and* $\begin{pmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{pmatrix}$ *is normally distributed, then*

$$\begin{aligned}\widehat{(\sigma_1^2 + \sigma_2^2 - 2c)} &= \frac{1}{n-k}(\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_1 - 2\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_2 + \mathbf{Y}_2'\mathbf{M}_X\mathbf{Y}_2)\\ &\sim (\sigma_1^2 + \sigma_2^2 - 2c)\frac{\chi^2_{n-k}(0)}{n-k}\end{aligned}$$

and the test statistic for the hypothesis $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$ *is*

$$\frac{n-k}{k}\,\frac{\mathbf{Y}_1'\mathbf{P}_X\mathbf{Y}_1 - 2\mathbf{Y}_1'\mathbf{P}_X\mathbf{Y}_2 + \mathbf{Y}_2'\mathbf{P}_X\mathbf{Y}_2}{\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_1 - 2\mathbf{Y}_1'\mathbf{M}_X\mathbf{Y}_2 + \mathbf{Y}_2'\mathbf{M}_X\mathbf{Y}_2} \sim F_{k,n-k}.$$

P r o o f. The statement is a direct consequence of the assumption

$$\mathbf{Y}_1 - \mathbf{Y}_2 \sim N_k[\mathbf{X}(\boldsymbol{\beta}_1 - \boldsymbol{\beta}_2), (\sigma_1^2 + \sigma_2^2 - 2c)\mathbf{I}_n].$$

□

The treated problem has quite different features in the multivariate case; see the following section.

2. Multivariate case

If an experimental animal is characterized by the measurement of m physiological parameters, then the model of the experiment is ($\mathbf{X}_1 = \mathbf{X}_2$)

$$\text{(2)}\quad \operatorname{vec}(\underline{\mathbf{Y}}_1, \underline{\mathbf{Y}}_2) \sim \left[\begin{pmatrix}\mathbf{I}_m \otimes \mathbf{X}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{I}_m \otimes \mathbf{X}\end{pmatrix}\begin{pmatrix}\operatorname{vec}(\mathbf{B}_1)\\ \operatorname{vec}(\mathbf{B}_2)\end{pmatrix}, \begin{pmatrix}\boldsymbol{\Sigma}_{1,1} \otimes \mathbf{I}_n, & \boldsymbol{\Sigma}_{1,2} \otimes \mathbf{I}_n\\ \boldsymbol{\Sigma}_{2,1} \otimes \mathbf{I}_n, & \boldsymbol{\Sigma}_{2,2} \otimes \mathbf{I}_n\end{pmatrix}\right],$$

where $\underline{\mathbf{Y}}_1$, $\underline{\mathbf{Y}}_2$ are $n \times m$ observation matrices, $\mathbf{B}_1$, $\mathbf{B}_2$ are $k \times m$ parameter matrices and

$$\begin{aligned}&\operatorname{Var}[\operatorname{vec}(\underline{\mathbf{Y}}_1)] = \boldsymbol{\Sigma}_{1,1} \otimes \mathbf{I}_n, \quad \operatorname{cov}[\operatorname{vec}(\mathbf{Y}_1), \operatorname{vec}(\mathbf{Y}_2)] = \boldsymbol{\Sigma}_{1,2} \otimes \mathbf{I}_n,\\ &\operatorname{cov}[\operatorname{vec}(\mathbf{Y}_2), \operatorname{vec}(\mathbf{Y}_1)] = \boldsymbol{\Sigma}_{2,1} \otimes \mathbf{I}_n, \quad \operatorname{Var}[\operatorname{vec}(\underline{\mathbf{Y}}_2)] = \boldsymbol{\Sigma}_{2,2} \otimes \mathbf{I}_n\end{aligned}$$

(more details on structures of multivariate models see in [5]).

Lemma 2.1. *The BLUEs of* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *in the model* (2) *are*

$$\widehat{\mathbf{B}}_i = (\mathbf{X}_i'\mathbf{X}_i)^{-1}\mathbf{X}_i'\underline{\mathbf{Y}}_i, \quad i = 1, 2,$$

and

$$\mathrm{Var}\begin{pmatrix}\mathrm{vec}(\widehat{\mathbf{B}}_1)\\ \mathrm{vec}(\widehat{\mathbf{B}}_2)\end{pmatrix} = \begin{pmatrix}\mathbf{\Sigma}_{1,1}, & \mathbf{\Sigma}_{1,2}\\ \mathbf{\Sigma}_{2,1}, & \mathbf{\Sigma}_{2,2}\end{pmatrix} \otimes (\mathbf{X}'\mathbf{X})^{-1}.$$

P r o o f. Since

$$\mathrm{Var}\begin{pmatrix}\mathrm{vec}(\widehat{\underline{\mathbf{Y}}}_1)\\ \mathrm{vec}(\widehat{\underline{\mathbf{Y}}}_2)\end{pmatrix} = \begin{pmatrix}\mathbf{\Sigma}_{1,1}, & \mathbf{\Sigma}_{1,2}\\ \mathbf{\Sigma}_{2,1}, & \mathbf{\Sigma}_{2,2}\end{pmatrix} \otimes \mathbf{I}_n,$$

the estimator is obviously independent of $\begin{pmatrix}\mathbf{\Sigma}_{1,1}, \, \mathbf{\Sigma}_{1,2}\\ \mathbf{\Sigma}_{2,1}, \, \mathbf{\Sigma}_{2,2}\end{pmatrix}$. However the BLUEs can be obtained directly from Theorem 1.2 as well. Thus, e.g. the estimator of $\mathbf{B}_1$ is

$$\begin{aligned}
\mathrm{vec}(\widehat{\mathbf{B}}_1) = \Big(&(\mathbf{I}_m \otimes \mathbf{X}')\big\{(\mathbf{\Sigma}_{11.2} \otimes \mathbf{I}_n) + [(\mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}) \otimes \mathbf{I}_n](\mathbf{I}_m \otimes \mathbf{X})\\
&\times [(\mathbf{I}_m \otimes \mathbf{X}')(\mathbf{\Sigma}_{2,2}^{-1} \otimes \mathbf{I}_n)(\mathbf{I}_m \otimes \mathbf{X})]^{-1}(\mathbf{I}_m \otimes \mathbf{X}')[(\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1}) \otimes \mathbf{I}_n]\big\}^{-1}\\
&\times (\mathbf{I}_m \otimes \mathbf{X})\Big)^{-1}(\mathbf{I}_m \otimes \mathbf{X}')\big\{(\mathbf{\Sigma}_{11.2} \otimes \mathbf{I}_n) + [(\mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}) \otimes \mathbf{I}_n](\mathbf{I}_m \otimes \mathbf{X})\\
&\times [(\mathbf{I}_m \otimes \mathbf{X}')(\mathbf{\Sigma}_{2,2}^{-1} \otimes \mathbf{I}_n)(\mathbf{I}_m \otimes \mathbf{X})]^{-1}(\mathbf{I}_m \otimes \mathbf{X}')[(\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1}) \otimes \mathbf{I}_n]\big\}^{-1}\\
&\times \big\{\mathrm{vec}(\underline{\mathbf{Y}}_1) - [(\mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}) \otimes \mathbf{I}_n]\,\mathrm{vec}(\underline{\mathbf{Y}}_2) + [(\mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}) \otimes \mathbf{I}_n](\mathbf{I}_m \otimes \mathbf{X})\\
&\times \big[(\mathbf{I}_m \otimes \mathbf{X}')(\mathbf{\Sigma}_{2,2}^{-1} \otimes \mathbf{I}_n)(\mathbf{I}_m \otimes \mathbf{X})\big]^{-1}(\mathbf{I}_m \otimes \mathbf{X}')(\mathbf{\Sigma}_{2,2}^{-1} \otimes \mathbf{I}_n)\mathrm{vec}(\underline{\mathbf{Y}}_2)\big\}\\
= \Big(&(\mathbf{I}_m \otimes \mathbf{X}')\big\{\mathbf{\Sigma}_{11.2} \otimes \mathbf{I}_n + (\mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1}) \otimes \mathbf{P}_X\big\}^{-1}(\mathbf{I}_m \otimes \mathbf{X})\Big)^{-1}(\mathbf{I}_m \otimes \mathbf{X}')\\
&\times \big\{\mathbf{\Sigma}_{11.2} \otimes \mathbf{I}_n + (\mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1}) \otimes \mathbf{P}_X\big\}^{-1}\\
&\times \mathrm{vec}(\underline{\mathbf{Y}}_1 - \underline{\mathbf{Y}}_2\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1} + \mathbf{P}_X\underline{\mathbf{Y}}_2\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1})\\
= \big\{&(\mathbf{I}_m \otimes \mathbf{X}')[\mathbf{\Sigma}_{11.2}^{-1} \otimes \mathbf{M}_X + (\mathbf{\Sigma}_{11.2} + \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1})^{-1} \otimes \mathbf{P}_X](\mathbf{I}_m \otimes \mathbf{X})\big\}^{-1}\\
&\times (\mathbf{I}_m \otimes \mathbf{X}')[\mathbf{\Sigma}_{11.2}^{-1} \otimes \mathbf{M}_X + (\mathbf{\Sigma}_{11.2} + \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1})^{-1} \otimes \mathbf{P}_X]\\
&\times \mathrm{vec}(\underline{\mathbf{Y}}_1 - \mathbf{M}_X\mathbf{Y}_2\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1})\\
= [&\mathbf{\Sigma}_{1,1}^{-1} \otimes (\mathbf{X}'\mathbf{X})]^{-1}(\mathbf{\Sigma}_{1,1}^{-1} \otimes \mathbf{X}')\,\mathrm{vec}(\underline{\mathbf{Y}}_1 - \mathbf{M}_X\mathbf{Y}_2\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1})\\
= \;&\mathrm{vec}[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}_1].
\end{aligned}$$

The expressions for $\widehat{\mathbf{B}}_2$ and $\mathrm{Var}\begin{pmatrix}\mathrm{vec}(\widehat{\mathbf{B}}_1)\\ \mathrm{vec}(\widehat{\mathbf{B}}_2)\end{pmatrix}$ are obvious. □

Lemma 2.2. *Let* $\mathbf{T}$ *be an* $n \times n$ *p.s.d. matrix,* $\mathbf{\Sigma}$ *be an* $m \times m$ *p.d. matrix,* $r(\mathbf{T}) > m$ *and*

$$\mathrm{vec}(\underline{\boldsymbol{\xi}}) \sim N_{nm}(\mathbf{0}, \mathbf{\Sigma} \otimes \mathbf{T}).$$

Then

$$\underline{\boldsymbol{\xi}}'\mathbf{T}^-\underline{\boldsymbol{\xi}} \sim W_m[r(\mathbf{T}), \mathbf{\Sigma}]$$

(the Wishart distribution with $r(\mathbf{T})$ *degrees of freedom and with the covariance matrix* $\mathbf{\Sigma}$*).*

Proof. Let $\mathbf{T} = \sum_{i=1}^{r(T)} \lambda_i \mathbf{f}_i \mathbf{f}_i'$ be the spectral decomposition of the matrix $\mathbf{T}$. Let

$$\mathbf{J} = (\mathbf{f}_1, \dots, \mathbf{f}_{r(T)}) \begin{pmatrix} \sqrt{\lambda_1}, & \dots, & 0 \\ \vdots & \ddots & \vdots \\ 0, & \dots, & \sqrt{\lambda_{r(T)}} \end{pmatrix}$$

and

$$\mathbf{K} = (\mathbf{f}_1, \dots, \mathbf{f}_{r(T)}) \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}}, & \dots, & 0 \\ \vdots & \ddots & \vdots \\ 0, & \dots, & \frac{1}{\sqrt{\lambda_{r(T)}}} \end{pmatrix}.$$

Then $\mathbf{J}$ is an $n \times r(\mathbf{T})$ matrix with the property $\mathbf{T} = \mathbf{J}\mathbf{J}'$ and $\mathbf{K}$ is an $n \times r(\mathbf{T})$ matrix with the property $\mathbf{K}\mathbf{K}' = \mathbf{T}^+$, i.e. $\mathbf{J}'\mathbf{K} = \mathbf{I}_{r(T)}$. Then

$$(\mathbf{I}_m \otimes \mathbf{K}')\,\mathrm{vec}(\underline{\boldsymbol{\xi}}) \sim N_{mr(T)}(\mathbf{0}, \mathbf{\Sigma} \otimes \mathbf{I}_{r(T)}) \;\Rightarrow\; \underline{\boldsymbol{\xi}}'\mathbf{K}\mathbf{K}'\underline{\boldsymbol{\xi}} = \underline{\boldsymbol{\xi}}'\mathbf{T}^+\underline{\boldsymbol{\xi}} \sim W_m[r(\mathbf{T}), \mathbf{\Sigma}].$$

Since

$$\forall\{i = 1, \dots, m\}\; P\big\{\{\underline{\boldsymbol{\xi}}\}_{\cdot,i} \in \mathcal{M}(\mathbf{T})\big\} = 1$$

it is valid that $\underline{\boldsymbol{\xi}}'\mathbf{T}^+\underline{\boldsymbol{\xi}} = \underline{\boldsymbol{\xi}}'\mathbf{T}^-\underline{\boldsymbol{\xi}}$ for any g-inverse $\mathbf{T}^-$ of $\mathbf{T}$ (more details on g-inverses see in [8]). □

Lemma 2.3. *Let* $\mathbf{X}_1 = \mathbf{X}_2$ *and* $(\underline{\mathbf{Y}}_1, \underline{\mathbf{Y}}_2)$ *be normally distributed. Then*

$$\begin{pmatrix} \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1, & \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 \\ \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1, & \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2 \end{pmatrix} \sim W_{2m}\left[n - k, \begin{pmatrix} \mathbf{\Sigma}_{1,1}, & \mathbf{\Sigma}_{1,2} \\ \mathbf{\Sigma}_{2,1}, & \mathbf{\Sigma}_{2,2} \end{pmatrix}\right].$$

Proof. Since

$$\mathrm{vec}(\underline{\mathbf{Y}}_1, \underline{\mathbf{Y}}_2) \sim N_{2nm}\left[(\mathbf{I}_{2m} \otimes \mathbf{X})\,\mathrm{vec}(\mathbf{B}_1, \mathbf{B}_2), \begin{pmatrix} \mathbf{\Sigma}_{1,1}, & \mathbf{\Sigma}_{1,2} \\ \mathbf{\Sigma}_{2,1}, & \mathbf{\Sigma}_{2,2} \end{pmatrix} \otimes \mathbf{I}_n\right],$$

it is valid that

$$(\mathbf{I}_{2m} \otimes \mathbf{M}_X)\,\mathrm{vec}(\underline{\mathbf{Y}}_1, \underline{\mathbf{Y}}_2) \sim N_{2nm}\left[\mathbf{0}, \begin{pmatrix} \mathbf{\Sigma}_{1,1}, & \mathbf{\Sigma}_{1,2} \\ \mathbf{\Sigma}_{2,1}, & \mathbf{\Sigma}_{2,2} \end{pmatrix} \otimes \mathbf{M}_X\right].$$

With respect to Lemma 2.2

$$\begin{pmatrix} \underline{\mathbf{Y}}_1' \\ \underline{\mathbf{Y}}_2' \end{pmatrix} \mathbf{M}_X (\underline{\mathbf{Y}}_1, \underline{\mathbf{Y}}_2) \sim W_{2m}\left[n-k, \begin{pmatrix} \mathbf{\Sigma}_{1,1}, & \mathbf{\Sigma}_{1,2} \\ \mathbf{\Sigma}_{2,1}, & \mathbf{\Sigma}_{2,2} \end{pmatrix}\right].$$

□

In the following lemma the test statistics for the hypotheses

(i) $\mathbf{h}'(\mathbf{B}_1 - \mathbf{B}_2) = \mathbf{0}$,

(ii) $(\mathbf{B}_1 - \mathbf{B}_2)\mathbf{l} = \mathbf{0}$, and

(iii) $\mathbf{B}_1 - \mathbf{B}_2 = \mathbf{0}$

are given in the case of normality of the matrix $(\underline{\mathbf{Y}}_1, \underline{\mathbf{Y}}_2)$.

Lemma 2.4.

(i) *Let* $\mathbf{h} \in \mathbb{R}^k$ *and the hypothesis be* $\mathbf{h}'(\mathbf{B}_1 - \mathbf{B}_2) = \mathbf{0}$. *Then*

$$\frac{\mathbf{h}'(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)(\underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1 - \mathbf{Y}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 - \mathbf{Y}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1 + \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2)^{-1}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)'\mathbf{h}}{\mathbf{h}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{h}} \times \frac{n-k-m+1}{m} \sim F_{m,n-k-m+1}.$$

(ii) *Let* $\mathbf{l} \in \mathbb{R}^m$ *and the hypothesis be* $(\mathbf{B}_1 - \mathbf{B}_2)\mathbf{l} = \mathbf{0}$. *Then*

$$\frac{n-k}{k}\,\frac{\mathbf{l}'(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)'\mathbf{X}'\mathbf{X}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)\mathbf{l}}{\mathbf{l}'(\underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1 - \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 - \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1 + \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2)\mathbf{l}} \sim F_{k,n-k}.$$

(iii) *Let the hypothesis be* $\mathbf{B}_1 - \mathbf{B}_2 = \mathbf{0}$. *Let* $\mathbf{U} = \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1 - \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 - \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1 + \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2$. *Then*

$$-\left[n - \frac{1}{2}(k+m+1)\right]\log\frac{\det(\mathbf{U})}{\det[\mathbf{U} + (\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)'\mathbf{X}'\mathbf{X}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)]} \to \chi^2_{km}.$$

P r o o f. The matrix $\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2$ and the matrix $\mathbf{W} = \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1 - \mathbf{Y}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 - \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1 + \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2$ are independent.

(i) Under the null hypothesis

$$\frac{\mathrm{vec}[\mathbf{h}'(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)]}{\sqrt{\mathbf{h}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{h}}} \sim N_m(\mathbf{0}, \mathbf{\Sigma}_{1,1} - \mathbf{\Sigma}_{1,2} - \mathbf{\Sigma}_{2,1} + \mathbf{\Sigma}_{2,2})$$

and

$$\begin{aligned}\mathbf{W} &= \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1 - \mathbf{Y}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 - \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1 + \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2\\ &\sim W_m(n-k, \boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,2} - \boldsymbol{\Sigma}_{2,1} + \boldsymbol{\Sigma}_{2,2}).\end{aligned}$$

With respect to the Hotelling theorem [6] it is valid that

$$\frac{\mathbf{h}'(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)\mathbf{W}^{-1}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)'\mathbf{h}}{\mathbf{h}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{h}}\,\frac{(n-k-m+1)}{m} \sim F_{m,n-k-m+1}.$$

(ii) It is valid that

$$\begin{aligned}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)\mathbf{l} &\sim N_k[(\mathbf{B}_1 - \mathbf{B}_2)\mathbf{l}, \mathbf{l}'(\boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,2} - \boldsymbol{\Sigma}_{2,1} + \boldsymbol{\Sigma}_{2,2})\mathbf{l}(\mathbf{X}'\mathbf{X})^{-1}],\\ \mathbf{l}'\mathbf{W}\mathbf{l} &\sim \chi^2_{n-k}\mathbf{l}'(\boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,2} - \boldsymbol{\Sigma}_{2,1} + \boldsymbol{\Sigma}_{2,2})\mathbf{l}.\end{aligned}$$

Thus

$$\frac{\mathbf{l}'(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)'\mathbf{X}'\mathbf{X}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)\mathbf{l}/k}{\mathbf{l}'\mathbf{W}\mathbf{l}/(n-k)} \sim F_{k,n-k}.$$

(iii) The relationships

$$\begin{aligned}\operatorname{vec}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2) &\sim N_{km}[\mathbf{0}, (\boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,2} - \boldsymbol{\Sigma}_{2,1} + \boldsymbol{\Sigma}_{2,2}) \otimes (\mathbf{X}'\mathbf{X})^{-1}]\\ &\Rightarrow (\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)'\mathbf{X}'\mathbf{X}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2) \sim W_m(k, \boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,2} - \boldsymbol{\Sigma}_{2,1} + \boldsymbol{\Sigma}_{2,2})\end{aligned}$$

and

$$\begin{aligned}&\underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1 - \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 - \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1 + \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2\\ &\quad\sim W_m(n-k, \boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,2} - \boldsymbol{\Sigma}_{2,1} + \boldsymbol{\Sigma}_{2,2})\end{aligned}$$

imply, with respect to the Wilks-Bartlett theorem [6], the statement. □

If $\begin{pmatrix}\boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{1,2}\\ \boldsymbol{\Sigma}_{2,1}, & \boldsymbol{\Sigma}_{2,2}\end{pmatrix} = \begin{pmatrix}\sigma_1^2, & c\\ c, & \sigma_2^2\end{pmatrix} \otimes \mathbf{I}_n$, then the hypothesis $\mathbf{B}_1 = \mathbf{B}_2$ can be tested with the help of the following lemma.

Lemma 2.5. *If* $\begin{pmatrix}\boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{1,2}\\ \boldsymbol{\Sigma}_{2,1}, & \boldsymbol{\Sigma}_{2,2}\end{pmatrix} = \begin{pmatrix}\sigma_1^2, & c\\ c, & \sigma_2^2\end{pmatrix} \otimes \mathbf{I}_n$ *then the test statistic is*

$$\frac{\operatorname{Tr}[(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)'\mathbf{X}'\mathbf{X}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)]}{\operatorname{Tr}(\underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1 - \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 - \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1 + \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2)}\,\frac{m(n-k)}{km} \sim F_{km,m(n-k)}.$$

P r o o f. The expression

$$[\operatorname{vec}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)]'[(\sigma_1^2 + \sigma_2^2 - 2c)\mathbf{I}_m \otimes (\mathbf{X}'\mathbf{X})^{-1}]^{-1} \operatorname{vec}(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2) \sim \chi^2_{km}$$

can be transwritten as

$$\frac{\mathrm{Tr}[(\widehat{\mathbf{B}}_1 - \widehat{\mathbf{B}}_2)'\mathbf{X}'\mathbf{X}(\widehat{\mathbf{B}}1 - \widehat{\mathbf{B}}_2)]}{\sigma_1^2 + \sigma_2^2 - 2c}.$$

Since

$$\mathrm{Tr}(\underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_1 - \underline{\mathbf{Y}}_1'\mathbf{M}_X\underline{\mathbf{Y}}_2 - \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_1 + \underline{\mathbf{Y}}_2'\mathbf{M}_X\underline{\mathbf{Y}}_2) \sim (\sigma_1^2 + \sigma_2^2 - 2c)\chi^2_{m(n-k)},$$

the statement is obvious. □

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