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Abstract. The inverse problem of the calculus of variations in a nonholonomic setting is studied. The concept of constraint variationality is introduced on the basis of a recently discovered nonholonomic variational principle. Variational properties of first order mechanical systems with general nonholonomic constraints are studied. It is shown that constraint variationality is equivalent with the existence of a closed representative in the class of 2-forms determining the nonholonomic system. Together with the recently found constraint Helmholtz conditions this result completes basic geometric properties of constraint variational systems. A few examples of constraint variational systems are discussed.

1 Introduction

The covariant local inverse problem of the calculus of variations for second order ordinary differential equations means to find necessary and sufficient conditions under which a system of equations

$$A_\sigma(t,q^\nu,\dot q^\nu) + B_{\sigma\rho}(t,q^\nu,\dot q^\nu)\ddot q^\rho = 0\,, \quad 1 \le \sigma \le m \tag{1}$$

for curves $\mathbb{R} \ni t \to (q^\nu(t)) \in \mathbb{R}^m$, is variational "as it stands", i.e. to determine if there exists a Lagrangian $L(t,q^\nu,\dot q^\nu)$ such that the functions on the left-hand-sides are Euler-Lagrange expressions of L:

$$A_\sigma + B_{\sigma\rho}\ddot q^\rho = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\sigma}\,, \tag{2}$$

and, moreover, in the affirmative case to find a formula for computing a Lagrangian.

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Solution to this problem is very well known: conditions for variationality are the celebrated *Helmholtz conditions* [3], and a corresponding Lagrangian is then given by the famous *Vainberg-Tonti integral formula* [15], [16].

In this paper we are interested in a generalization of the inverse problem to nonholonomic mechanics. Some aspects based on an analogy with certain properties of unconstrained variational equations have already been studied (see e.g. [1], [9], [12]). However, only recently a variational principle for nonholonomic systems has been found [6], which opened a new way to formulate the problem and search for a solution in a parallel to the unconstrained case. Here we follow this way and introduce the concept of *constraint variationality* on the basis of the constraint variational principle, in a spirit as it is understood for unconstrained equations.

Namely, given a constraint Q by k first order ordinary differential equations

$$\dot{q}^{m-k+a} = g^a(t, q^\sigma, \dot{q}^l)\,, \quad 1 \le a \le k, \tag{3}$$

where $1 \le \sigma \le m$ and $1 \le l \le m-k$, the generalized ("nonholonomic") Euler--Lagrange equations represent a system of $m-k$ second order ordinary differential equations on the constraint submanifold $Q \subset J^1(\mathbb{R} \times \mathbb{R}^m)$. It is interesting that in this case one has $k+1$ "Lagrange functions" where k is the number of constraint equations. This rather mysterious property of nonholonomic systems is related to the fact that the corresponding Lagrangian 1-form has $k+1$ generic components, and is not reduced to a horizontal form (which is determined by a single function) as happens by circumstance in the unconstrained case.

The nonholonomic inverse problem concerns a system of *mixed first order and second order* ordinary differential equations

$$\dot{q}^{m-k+a} - g^a(t, q^\sigma, \dot{q}^l) = 0\,, \quad 1 \le a \le k\,, \tag{4}$$

$$\bar{A}_s(t, q^\sigma, \dot{q}^l) + \bar{B}_{sr}(t, q^\sigma, \dot{q}^l)\ddot{q}^r = 0\,, \quad 1 \le s \le m-k\,. \tag{5}$$

The first order equations give rise to a nonholonomic constraint submanifold $Q \subset J^1(\mathbb{R} \times \mathbb{R}^m)$ of corank k, while the second order equations then represent the dynamics on the constraint submanifold Q. The problem now is to find necessary and sufficient conditions under which equations (5) "as they stand" become the constrained Euler-Lagrange equations, and in the affirmative case, to find a corresponding constraint Lagrange 1-form.

It is known that in the unconstrained case variationality is equivalent with the possibility to extend the Euler-Lagrange form to a closed 2-form. Helmholtz conditions then become nothing but the closedness conditions, and the Vainberg--Tonti formula appears by application of the Poincaré Lemma. The main result we achieve in this paper means that the solution of the inverse problem in the nonholonomic setting has the same geometric properties: namely, that the constraint variationality is equivalent with the property that the corresponding equations can be represented by a closed form defined on the constraint Q. The closedness conditions are the constraint Helmholtz conditions obtained in our older paper [9].

It is worth mention that given an unconstrained Lagrangian system, the corresponding constrained system is constraint variational for *any* nonholonomic constraint. On the other hand, however, a nonholonomic system which is constraint variational may arise from a *non-variational unconstrained system*. Moreover, such an unconstrained system *need not be unique* in the sense that the corresponding unconstrained systems are *generically different*.

Due to the above properties, the range of applications of constraint variationality conditions is broader than that of Helmholtz conditions for unconstrained systems. At the end of the paper we illustrate on a few examples some possible applications of constraint Helmholtz conditions with the stress on rather unexpected properties of constraint variationality.

2 Unconstrained mechanical systems and variationality

Throughout the paper we consider a fibred manifold $\pi\colon Y \to \mathbb{R}$, $\dim Y = m+1$, and the corresponding jet bundles $\pi_r\colon J^rY \to \mathbb{R}$ where $r = 1, 2$. We denote by $\pi_{1,0}\colon J^1Y \to Y$, $\pi_{2,0}\colon J^2Y \to Y$ and $\pi_{2,1}\colon J^2Y \to J^1Y$ the canonical projections. Recall that a section δ of π_r is called *holonomic*, if it is of the form $\delta = J^r\gamma$ for a section γ of π.

A form η on J^rY is called *horizontal*, if $i_\xi\eta = 0$ for every π_r-vertical vector field ξ, and is called *contact*, if $J^r\gamma^*\eta = 0$ for every section γ of π. We shall use the following basis of 1-forms on J^1Y and J^2Y respectively, adapted to the contact structure:

$$(dt, \omega^\sigma, d\dot{q}^\sigma), \quad (dt, \omega^\sigma, \dot{\omega}^\sigma, d\ddot{q}^\sigma)\,,$$

where

$$\omega^\sigma = dq^\sigma - \dot{q}^\sigma dt, \quad \dot{\omega}^\sigma = d\dot{q}^\sigma - \ddot{q}^\sigma dt.$$

For every k-form η on J^1Y there exists a unique decomposition

$$\pi_{2,1}^*\eta = p_{k-1}\eta + p_k\eta,$$

where $p_{k-1}\eta$ and $p_k\eta$ is the $(k-1)$-contact component and the k-contact component of η, respectively, containing in every its term exactly $(k-1)$, respectively k, factors ω^σ and $\dot{\omega}^\sigma$. If $p_k\eta = 0$ we say that η is $(k-1)$contact. Similarly, if $p_{k-1}\eta = 0$ we speak about a k-contact form.

A first order mechanical system is described by a dynamical form E on J^2Y with components affine in the second derivatives; in fibered coordinates,

$$E = E_\sigma(t, q^\nu, \dot{q}^\nu, \ddot{q}^\nu)dq^\sigma \wedge dt\,, \tag{6}$$

where

$$E_\sigma = A_\sigma(t, q^\lambda, \dot{q}^\lambda) + B_{\sigma\nu}(t, q^\lambda, \dot{q}^\lambda)\ddot{q}^\nu. \tag{7}$$

A section γ of π is called a *path of E* if $E_\sigma \circ J^2\gamma = 0$. This condition gives a system of m second order ordinary differential equations

$$A_\sigma\Big(t, q^\lambda, \frac{dq^\lambda}{dt}\Big) + B_{\sigma\nu}\Big(t, q^\lambda, \frac{dq^\lambda}{dt}\Big)\frac{d^2q^\nu}{dt^2} = 0\,, \tag{8}$$

which have the meaning of the equations of motion.

If E is a dynamical form with components affine in the second derivatives then in a neighborhood of every point in J^1Y there exists a 2-form α such that

$$\pi_{2,1}^*\alpha = E + F, \tag{9}$$

where F is a 2-contact 2-form. The α is not unique. In fibered coordinates

$$\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot{q}^\nu + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu, \tag{10}$$

where $F_{\sigma\nu}(t, q^\lambda, \dot{q}^\lambda)$ are arbitrary functions, skew-symmetric in the indices.

With help of α equations for paths of E (8) take the form

$$J^1\gamma^* i_\xi \alpha = 0 \quad \text{for every vertical vector field } \xi \text{ on } J^1Y \tag{11}$$

of equations for holonomic integral sections of a local Pfaffian system on J^1Y. It is to be stressed that the set of solutions of equations (11) does not depend upon a choice of the 2-form F, and that (for any F) equations (11) are *locally equivalent* with equations of paths of E (8).

We denote the family of all the local 2-forms on J^1Y associated with E as above by $[\alpha]$ and call it the *Lepage class of* E. Note that forms belonging to the Lepage class of E satisfy

$$\alpha_1 - \alpha_2 \text{ is a 2-contact 2-form}$$

(on the intersection of their domains) and

$$p_1\alpha = E.$$

A dynamical form E is called *locally variational* if in a neighborhood of every point in J^2Y there exists a Lagrangian such that E coincides with its Euler--Lagrange form. It is known that if such a Lagrangian exists, there exists also an equivalent local first-order Lagrangian $\lambda = Ldt$ such that (7) coincide with the Euler-Lagrange expressions of λ

$$E_\sigma \equiv A_\sigma + B_{\sigma\nu}\ddot{q}^\nu = \frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^\sigma}. \tag{12}$$

Equations for paths of a locally variational form are Euler-Lagrange equations. For the Lepage class of a locally variational form we have $[\alpha] = [d\theta_\lambda]$ where θ_λ is the Cartan form of λ.

The following theorem shows the importance of the properties of the Lepage class for variationality of dynamical forms (see [4]).

Theorem 1. *A dynamical form E is locally variational if and only if the corresponding Lepage class $[\alpha]$ contains a closed representative. In this case, moreover, the closed 2-form $\alpha_E \in [\alpha]$ is unique and global (defined on J^1Y).*

The form α_E is called *Lepage equivalent of E* and the corresponding mechanical system is called *Lagrangian system.*

A direct calculation of $d\alpha$ for a representative of the class $[\alpha]$ leads to the famous *Helmholtz conditions* (necessary and sufficient conditions of variationality).

Theorem 2. *A dynamical form E is locally variational if and only if in fibered coordinates the following conditions hold:*

$$\begin{aligned} &(B_{\sigma\nu})_{\mathrm{alt}(\sigma\nu)} = 0\,, && \Big(\frac{\partial B_{\sigma\nu}}{\partial \dot q^\lambda}\Big)_{\mathrm{alt}(\nu\lambda)} = 0\,, \\ &\Big(-\frac{\partial A_\sigma}{\partial \dot q^\nu} + \frac{d' B_{\sigma\nu}}{dt}\Big)_{\mathrm{sym}(\sigma\nu)} = 0\,, && \Big(\frac{\partial A_\sigma}{\partial q^\nu} - \frac{1}{2}\frac{d'}{dt}\Big(\frac{\partial A_\sigma}{\partial \dot q^\nu}\Big)\Big)_{\mathrm{alt}(\sigma\nu)} = 0\,, \end{aligned} \tag{13}$$

where sym *and* alt *means symmetrization and skew-symmetrization respectively, and*

$$\frac{d'}{dt} = \frac{\partial}{\partial t} + \dot q^\sigma \frac{\partial}{\partial q^\sigma}\,.$$

Note that for a locally variational form E be globally variational (i.e. to arise as an Euler-Lagrange form from a global first-order Lagrangian) it is necessary and sufficient that the Lepage equivalent α_E of E is exact.

3 Constrained mechanical systems

Our approach to the inverse variational problem for nonholonomically constrained systems is based on the model representing nonholonomic constraints as a submanifold Q in J^1Y, naturally endowed with a nonintegrable distribution, and a constrained system as a dynamical form (an exterior differential system) defined on the constraint submanifold [4], [5]; here we follow the exposition of the survey article [7].

In what follows, greek indices σ, ν etc. run over $1, 2, \dots, m$ as above, and the latin indices a, b, i, j (respectively l, s) run over $1, 2, \dots, k = \operatorname{codim} Q$ (respectively $1, 2, \dots, m-k$). Summation over repeated indices is understood.

Let us consider a submanifold $Q \subset J^1Y$ of codimension k, $1 \le k \le m-1$, fibred over Y, called a *constraint submanifold*. We denote by $\iota\colon Q \to J^1Y$ the canonical embedding. Locally, Q is given by k independent equations

$$f^a(t, q^\sigma, \dot q^\sigma) = 0\,, \quad 1 \le a \le k\,, \tag{14}$$

or, in normal form,

$$\dot q^{m-k+a} = g^a(t, q^\sigma, \dot q^l)\,, \quad 1 \le a \le k\,, \tag{15}$$

where $l = 1, 2, \dots, m-k$.

We shall consider also the first prolongation $\hat Q$ of the constraint Q, that is a submanifold in J^2Y, consisting of all points $J^2_x\gamma$ such that $J^1_x\gamma \in Q$, $x \in \mathbb{R}$. Locally $\hat Q$ is defined by the equations of the constraint and their derivatives:

$$f^a = 0\,, \quad \frac{df^a}{dt} = 0\,, \tag{16}$$

respectively, in normal form,

$$\dot q^{m-k+a} = g^a, \quad \ddot q^{m-k+a} = \frac{dg^a}{dt}\,. \tag{17}$$

We denote by $\hat\iota\colon \hat Q \to J^2Y$ the corresponding canonical embedding. The manifold $\hat Q$ is fibred over Q, Y and $\mathbb{R}$, the fibred projections are simply restrictions of the corresponding canonical projections of the underlying fibred manifolds. We write $\bar{\pi}_2 \colon \hat{Q} \to \mathbb{R}$, $\bar{\pi}_{2,1} \colon \hat{Q} \to Q$, $\bar{\pi}_{2,0} \colon \hat{Q} \to Y$, and $\bar{\pi}_1 \colon Q \to \mathbb{R}$, $\bar{\pi}_{1,0} \colon Q \to Y$. Usually we shall use on Q adapted coordinates $(t, q^\sigma, \dot{q}^s)$, and on $\hat{Q}$ associated coordinates $(t, q^\sigma, \dot{q}^s, \ddot{q}^s)$, where $1 \le \sigma \le m$, $1 \le s \le m-k$.

Similarly as in the unconstrained case, for every q-form η on Q one has a *unique decomposition* into a sum of a $\bar{\pi}_2$-horizontal form and i-contact forms, $i = 1, 2, \dots q$, on $\hat{Q}$ [6]; we write

$$\bar{\pi}_{2,1}^* \eta = \bar{h}\eta + \bar{p}_1\eta + \dots + \bar{p}_q\eta \,. \tag{18}$$

In particular, we get an invariant *splitting of the exterior derivative* d to the horizontal and contact part, $\bar{\pi}_{2,1}^* d = \bar{h}d + \bar{p}_1 d$. The operator $\bar{h}d$ (the constraint total derivative) has the component

$$\frac{d_{\mathrm{c}}}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^a \frac{\partial}{\partial q^{m-k+a}} + \ddot{q}^s \frac{\partial}{\partial \dot{q}^s} \,. \tag{19}$$

For convenience of notations we also put

$$\frac{d'_{\mathrm{c}}}{dt} = \frac{\partial}{\partial t} + \dot{q}^s \frac{\partial}{\partial q^s} + g^a \frac{\partial}{\partial q^{m-k+a}} \,. \tag{20}$$

Over every nonholonomic constraint there naturally arises a bundle, called the *canonical distribution* [4] or *Chetaev bundle* [11], giving a geometric meaning to *virtual displacements* in the space of positions and velocities, and to the concept of *reactive (Chetaev) forces*. It is a corank k distribution $\mathcal{C}$ *on the manifold* Q, locally annihilated by the system of k linearly independent 1-forms

$$\varphi^a = \left(\frac{\partial f^a}{\partial \dot{q}^\sigma} \circ \iota\right) \bar{\omega}^\sigma = \bar{\omega}^{m-k+a} - \frac{\partial g^a}{\partial \dot{q}^s} \bar{\omega}^s, \tag{21}$$

where

$$\bar{\omega}^\sigma = \iota^* \omega^\sigma, \tag{22}$$

or, equivalently, locally spanned by the following system of $2(m-k)+1$ independent vector fields

$$\begin{aligned} &\frac{\partial_{\mathrm{c}}}{\partial t} \equiv \frac{\partial}{\partial t} + \left(g^a - \frac{\partial g^a}{\partial \dot{q}^l} \dot{q}^l\right) \frac{\partial}{\partial q^{m-k+a}} \,, \\ &\frac{\partial_{\mathrm{c}}}{\partial q^s} \equiv \frac{\partial}{\partial q^s} + \frac{\partial g^a}{\partial \dot{q}^s} \frac{\partial}{\partial q^{m-k+a}} \,, \\ &\frac{\partial}{\partial \dot{q}^s} \,. \end{aligned} \tag{23}$$

Vector fields belonging to the canonical distribution are called *Chetaev vector fields*.

The annihilator of $\mathcal{C}$ is denoted by $\mathcal{C}^0$.

The ideal in the exterior algebra on Q locally generated by the 1-forms φ^a, $1 \le a \le k$, is called the *constraint ideal*, and denoted by $\mathcal{I}(\mathcal{C}^0)$. Differential forms belonging to the constraint ideal are called *constraint forms*.

Let us recall the following theorem [4]:

Theorem 3. *The constraint Q is given by equations affine in the first derivatives if and only if the canonical distribution $\mathcal{C}$ on Q is $\bar{\pi}_{1,0}$-projectable (i.e. the projection of $\mathcal{C}$ is* a distribution on *Y).*

A nonholonomic constraint Q is called *semiholonomic* if its canonical distribution $\mathcal{C}$ is completely integrable.

The canonical distribution is naturally lifted to the distribution $\hat{\mathcal{C}}$ on $\hat{Q}$, defined with help of its annihilator by $\hat{\mathcal{C}}^0 = \bar{\pi}_{2,1}^* \mathcal{C}^0$.

Now, let E be a dynamical form on J^2Y and $[\alpha]$ its Lepage class as above. According to [4], a *constrained mechanical system* associated with $[\alpha]$ is the class

$$[\bar{\alpha}] = \iota^*[\alpha] \mod \mathcal{I}(\mathcal{C}^0) . \tag{24}$$

This means that $[\bar{\alpha}]$ is defined on the constraint Q and consists of all (possibly local) 2-forms on Q such that

$$\bar{\alpha} = \bar{A}_l \omega^l \wedge dt + \bar{B}_{ls} \omega^l \wedge d\dot{q}^s + F + \varphi, \tag{25}$$

where F is a 2-contact and φ is a constraint 2-form on Q, and

$$\begin{aligned} \bar{A}_l &= \Big(A_l + A_{m-k+b} \frac{\partial g^b}{\partial \dot{q}^l} + \Big(B_{l,m-k+a} + B_{m-k+b,m-k+a} \frac{\partial g^b}{\partial \dot{q}^l} \Big) \frac{d' g^a}{dt} \Big) \circ \iota, \\ \bar{B}_{ls} &= \Big(B_{ls} + B_{l,m-k+a} \frac{\partial g^a}{\partial \dot{q}^s} + B_{m-k+a,s} \frac{\partial g^a}{\partial \dot{q}^l} + B_{m-k+b,m-k+a} \frac{\partial g^b}{\partial \dot{q}^l} \frac{\partial g^a}{\partial \dot{q}^s} \Big) \circ \iota. \end{aligned} \tag{26}$$

In place of a single dynamical form $E = p_1\alpha$, for the constrained system we get the class $[\bar{E}]$ on $\hat{Q}$,

$$\bar{E} = \bar{p}_1 \bar{\alpha} = \hat{\iota}^* E + \varphi^a \wedge \nu_a \tag{27}$$

where φ^a are the canonical constraint 1-forms defined above and ν_a are horizontal forms. Putting $\bar{E}^{\mathrm{c}} = (\hat{\iota}^* E)|_{\hat{\mathcal{C}}}$ we get an element of $\Lambda^2(\hat{\mathcal{C}})$, a 2-form along the canonical distribution, called *constrained dynamical form*; $\bar{E}^{\mathrm{c}}$ is the same for all $\bar{E} \in [\bar{E}]$. In coordinates

$$\bar{E}^{\mathrm{c}} = (\bar{A}_s + \bar{B}_{sr} \ddot{q}^r) \bar{\omega}^s \wedge dt . \tag{28}$$

By a *constrained section* of π we shall mean a section $\gamma : I \to Y$, $I \subset \mathbb{R}$, such that $J^1\gamma(I) \subset Q$. Hence, constrained sections satisfy the first order ODE's of the constraint (14) resp. (15). In particular, constrained sections are integral sections of the canonical distribution $\mathcal{C}$.

We have the following theorem [4] providing equations of motion of nonholonomically constrained systems in both intrinsic and coordinate form:

Theorem 4. *Let $\gamma\colon I \to Y$ be a constrained section. The following conditions are equivalent:*

(1) γ is a path of $\bar{E}^{\mathrm{c}}$, i.e. it satisfies

$$\bar{E}^{\mathrm{c}} \circ J^2\gamma = 0 . \tag{29}$$

(2) For every $\bar{\pi}_1$-vertical Chetaev vector field Z on Q

$$J^1\gamma^* i_Z \bar{\alpha} = 0 \tag{30}$$

where $\bar{\alpha}$ is any representative of the class $[\bar{\alpha}]$.

(3) Along $J^2\gamma$,

$$\bar{A}_s + \bar{B}_{sr}\ddot{q}^r = 0\,, \quad 1 \le s \le m-k\,. \tag{31}$$

The above equations are called *reduced nonholonomic equations* [4]. Remarkably, reduced equations do not contain Lagrange multipliers.

4 The nonholonomic variational principle

We shall briefly recall a variational principle proposed in [6], providing reduced nonholonomic equations as equations for extremals.

Consider a Lagrangian λ on J^1Y, let θ_λ be its Cartan form. Let $\iota\colon Q \to J^1Y$ be a nonholonomic constraint, $\mathcal{C}$ the canonical distribution. Denote by $\mathcal{S}_{[a,b]}(\bar{\pi}_1)$ the set of sections of $\bar{\pi}_1$, defined around an interval $[a,b] \subset \mathbb{R}$, $a < b$. By *constrained action* we mean the function

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \to \int_a^b \delta^*\iota^*\theta_\lambda \in \mathbb{R}\,. \tag{32}$$

Given a $\bar{\pi}_1$-projectable vector field $Z \in \mathcal{C}$, denote by ϕ and ϕ_0 the flows of Z and its projection Z_0, respectively. The one-parameter family $\{\delta_u\}$ of sections of $\bar{\pi}_1$, where $\delta_u = \phi_u\,\delta\,\phi_{0u}^{-1}$, is called *constrained variation of* δ induced by Z. The function

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \to \left(\frac{d}{du}\int_{\phi_{0u}([a,b])} \delta_u^*\,\iota^*\theta_\lambda\right)_{u=0} = \int_a^b \delta^*\,\mathcal{L}_Z\,\iota^*\theta_\lambda \in \mathbb{R} \tag{33}$$

is then the *first constrained variation* of the action function of λ over $[a,b]$, *induced by* Z. Restricting the domain of definition $\mathcal{S}_{[a,b]}(\bar{\pi}_1)$ of the function (33) to the subset $\mathcal{S}^h_{[a,b]}(\bar{\pi}_1)$ of *holonomic* sections of the projection $\bar{\pi}_1$, i.e. $\delta = J^1\gamma$ where $\gamma \in \mathcal{S}_{[a,b]}(\pi)$, one can regard the first constrained variation (33) as a function

$$\mathcal{S}_{[a,b],Q}(\pi) \ni \gamma \to \int_a^b J^1\gamma^*\,\mathcal{L}_Z\,\iota^*\theta_\lambda \in \mathbb{R} \tag{34}$$

defined on a *subset of sections of the projection* $\pi\colon Y \to \mathbb{R}$. Applying to (34) Cartan's formula for the decomposition of Lie derivative we obtain the *nonholonomic first variation formula*

$$\int_a^b J^1\gamma^*\,\mathcal{L}_Z\iota^*\theta_\lambda = \int_a^b J^1\gamma^*\,i_Z\iota^*d\theta_\lambda + \int_a^b J^1\gamma^*\,di_Z\iota^*\theta_\lambda\,, \tag{35}$$

giving us the splitting of the first constrained variation to a “constrained Euler-Lagrange term” and a boundary term.

A section γ of π is called a *constrained extremal* of λ on $[a,b]$ if $\operatorname{Im} J^1\gamma \subset Q$, and if the first constraint variation of the action on the interval $[a,b]$ vanishes for every “fixed endpoints” variation Z over $[a,b]$. γ is called a *constrained extremal of* λ if it is its constrained extremal on every interval $[a,b] \subset \operatorname{Dom}\gamma$.

Theorem 5. *Consider a Lagrangian* λ *on* J^1Y *and a nonholonomic constraint. Let* $\gamma\colon I \to Y$ *be a constrained section. The following conditions are equivalent:*

(1) γ is a constrained extremal of λ.

(2) For every $\bar\pi_1$-vertical Chetaev vector field Z on Q

$$J^1\gamma^* i_Z \iota^* d\theta_\lambda = 0. \tag{36}$$

(3) Along $J^2\gamma$,

$$\frac{\partial_c \bar L}{\partial q^s} - \frac{d_c}{dt}\frac{\partial \bar L}{\partial \dot q^s} - \bar L_a\Big(\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt}\frac{\partial g^a}{\partial \dot q^s}\Big) = 0\,, \quad 1 \le s \le m-k\,, \tag{37}$$

where $\bar L = L \circ \iota$ and

$$\bar L_a = \frac{\partial L}{\partial \dot q^{m-k+a}} \circ \iota\,. \tag{38}$$

The proof uses the same techniques as the proof of the similar assertion in the unconstrained case. Keeping the above notations, we can see that for a Lagrangian system the corresponding constrained system is

$$[\bar\alpha] = [\iota^* d\theta_\lambda], \tag{39}$$

the constrained dynamical form is

$$\bar E^c_\lambda = \Big(\frac{\partial_c \bar L}{\partial q^s} - \frac{d_c}{dt}\frac{\partial \bar L}{\partial \dot q^s} - \bar L_a\Big(\frac{\partial_c g^a}{\partial q^s} - \frac{d_c}{dt}\frac{\partial g^a}{\partial \dot q^s}\Big)\Big)\bar\omega^s \wedge dt, \tag{40}$$

and

$$\bar A_s = \frac{\partial_c \bar L}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial \bar L}{\partial \dot q^s} - \bar L_a\Big(\frac{\partial_c g^a}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial g^a}{\partial \dot q^s}\Big) \tag{41}$$

$$\bar B_{sr} = -\frac{\partial^2 \bar L}{\partial \dot q^r \partial \dot q^s} + \bar L_a \frac{\partial^2 g^a}{\partial \dot q^r \partial \dot q^s}. \tag{42}$$

We call equations (36) or (37) *constrained Euler-Lagrange equations*, $\bar E^c_\lambda$ the *constrained Euler-Lagrange form*, and its components *constrained Euler-Lagrange expressions*.

In what follows, we use the following notations:

$$\varepsilon_s = \frac{\partial_c}{\partial q^s} - \frac{d_c}{dt}\frac{\partial}{\partial \dot q^s}\,, \qquad \varepsilon'_s = \frac{\partial_c}{\partial q^s} - \frac{d'_c}{dt}\frac{\partial}{\partial \dot q^s}\,. \tag{43}$$

Finally, let us recall the fundamental relation between well-known Chetaev equations (with Lagrange multipliers) [2] and reduced equations (without Lagrange multipliers) [4], [13]:

Theorem 6. *A constrained section γ of π is a solution of constrained Euler-Lagrange equations (36) or (37) if and only if it is a solution of Chetaev equations*

$$\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt}\frac{\partial L}{\partial \dot q^\sigma} = \lambda_a \frac{\partial f^a}{\partial \dot q^\sigma}\,. \tag{44}$$

It is worth note that for *semiholonomic constraints* one has $\varepsilon_s(g^a) = 0$ [5], so that the constrained Euler-Lagrange equations simplify to

$$\frac{\partial_c \bar L}{\partial q^s} - \frac{d_c}{dt}\frac{\partial \bar L}{\partial \dot q^s} = 0\,. \tag{45}$$

5 The local inverse variational problem for nonholonomic systems

Now we are prepared to generalize the inverse variational problem to nonholonomic mechanics. In what follows we consider a constraint $\iota : Q \to J^1Y$. Given a system of second order differential equations on Q, (31), the question is if the equations are *constraint variational*, i.e. if they come from a constrained variational functional as equations for constrained extremals. Similarly as in the unconstrained case, the problem has several different formulations: local and global, direct (covariant) and contravariant (variational multipliers). We shall deal with the *local inverse problem in covariant form* (for equations "as they stand"), so that in what follows, $Y = \mathbb{R} \times \mathbb{R}^m$ and $J^1Y = \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m$.

More precisely, consider a system of mixed first order and second order ODE's

$$\begin{aligned} \dot{q}^{m-k+a} - g^a(t, q^\sigma, \dot{q}^l) = 0\,, \quad & 1 \le a \le k\,, \\ \bar{A}_s(t, q^\sigma, \dot{q}^l) + \bar{B}_{sr}(t, q^\sigma, \dot{q}^l)\ddot{q}^r = 0\,, \quad & 1 \le s \le m-k \end{aligned} \tag{46}$$

for sections $\gamma\colon I \to Y$. The equations give rise to a nonholonomic constraint $Q \subset J^1Y$ of corank k, with the canonical distribution $\mathcal{C}$, and a constrained system, represented either by a class of first order 2-forms

$$\bar{\alpha} = \bar{A}_s\bar{\omega}^s \wedge dt + \bar{B}_{sr}\bar{\omega}^s \wedge d\dot{q}^r + F + \nu \tag{47}$$

where F is a 2-contact and ν is a constraint form on Q, or, by a constrained dynamical form

$$\bar{E}^{\mathrm{c}} = (\bar{A}_s + \bar{B}_{sr}\ddot{q}^r)\bar{\omega}^s \wedge dt \tag{48}$$

on $\hat{Q}$.

Definition 1. A constrained dynamical form $\bar{E}^c$ on Q will be called *constraint variational* if there exist $m+1$ functions $\bar{L}$, $\bar{L}_a$ such that

$$\bar{A}_s + \bar{B}_{sr}\ddot{q}^r = \frac{\partial_{\mathrm{c}}\bar{L}}{\partial q^s} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial\bar{L}}{\partial\dot{q}^s} - \bar{L}_a\Big(\frac{\partial_{\mathrm{c}} g^a}{\partial q^s} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial g^a}{\partial\dot{q}^s}\Big). \tag{49}$$

A system of equations (46) is called *constraint variational* (as it stands) if the corresponding constrained dynamical form $\bar{E}^c = (\bar{A}_s + \bar{B}_{sr}\ddot{q}^r)\bar{\omega}^s \wedge dt$ is constraint variational.

Note that if a system of equations (a constrained dynamical form) is constraint variational, and $\bar{L}$, $\bar{L}_a$ are the corresponding "constraint Lagrange functions" then the *constraint Lagrangian* takes the form

$$\lambda_{\mathrm{c}} = \bar{L}\,dt + \bar{L}_a\varphi^a, \tag{50}$$

and the action is

$$\mathcal{S}_{[a,b]}(\bar{\pi}_1) \ni \delta \to \int_a^b \delta^*\theta_{\lambda_{\mathrm{c}}} \in \mathbb{R}\,, \tag{51}$$

where $\theta_{\lambda_{\mathrm{c}}}$ is the constraint Lepage equivalent of λ_{c} (constraint Cartan form) as introduced in [10]; in coordinates,

$$\theta_{\lambda_{\mathrm{c}}} = \bar{L}\,dt + \frac{\partial\bar{L}}{\partial\dot{q}^s}\bar{\omega}^s + \bar{L}_a\varphi^a. \tag{52}$$

Immediately from the definition we can see that *given an unconstrained Lagrangian system, the corresponding constrained system is constraint variational for any nonholonomic constraint.* Indeed, in this case,

$$\bar{L} = L \circ \iota, \quad \bar{L}_a = \frac{\partial L}{\partial \dot{q}^{m-k+a}} \circ \iota, \tag{53}$$

and

$$\theta_{\lambda_c} = \iota^* \theta_\lambda, \tag{54}$$

where $\lambda = L\,dt$ is a first order Lagrangian for the given variational dynamical form.

On the other hand, as we shall see below, a nonholonomic system which is constraint variational may arise from a *non-variational* unconstrained system on J^1Y. Moreover, such an unconstrained system need not be unique.

We have the following main theorem on constraint variationality of reduced equations on nonholonomic manifolds:

Theorem 7. *Let $\bar{E}^c$ be a constrained dynamical form, $[\bar{\alpha}]$ the corresponding class of 2-forms. $\bar{E}^c$ is constraint variational if and only if in a neighborhood of every point in Q the class $[\bar{\alpha}]$ has a closed representative.*

Proof. If $\bar{E}^c$ is constraint variational, we have Lagrange functions $\bar{L}$, $\bar{L}_a$ such that

$$\bar{E}^c = \Big(\bar{A}_s + \bar{B}_{sr}\ddot{q}^r\Big)\bar{\omega}^s \wedge dt, \tag{55}$$

with

$$\bar{A}_s = \varepsilon'_s(\bar{L}) - \bar{L}_a \varepsilon'_s(g^a), \quad \bar{B}_{sr} = -\frac{\partial^2 \bar{L}}{\partial \dot{q}^s \partial \dot{q}^r} + \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^s \partial \dot{q}^r}\,. \tag{56}$$

Putting

$$\rho = \bar{L}\,dt + \frac{\partial \bar{L}}{\partial \dot{q}^s}\bar{\omega}^s + \bar{L}_a \varphi^a \tag{57}$$

we obtain

$$d\rho \sim \Big(\varepsilon'_s(\bar{L}) - \bar{L}_a\,\varepsilon'_s(g^a)\Big)\bar{\omega}^s \wedge dt + \Big(\bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s} - \frac{\partial^2 \bar{L}}{\partial \dot{q}^r \partial \dot{q}^s}\Big)\bar{\omega}^s \wedge d\dot{q}^r \in [\bar{\alpha}] \tag{58}$$

as desired.

Let us show the converse. Given $\bar{E}^c = (\bar{A}_s + \bar{B}_{sr}\ddot{q}^r)\bar{\omega}^s \wedge dt$, let

$$\begin{aligned}\bar{\alpha} = {} & \bar{A}_s \bar{\omega}^s \wedge dt + \bar{B}_{rs}\bar{\omega}^r \wedge d\dot{q}^s + \bar{F}_{rs}\bar{\omega}^r \wedge \bar{\omega}^s \\ & + \varphi^a \wedge (b_a dt + b_{as}\bar{\omega}^s + c_{as} d\dot{q}^s) + \gamma_{ab}\varphi^a \wedge \varphi^b\end{aligned} \tag{59}$$

where $\bar{F}_{rs} = -\bar{F}_{sr}$ and $\gamma_{ab} = -\gamma_{ba}$, be a 2-form belonging to the class $[\bar{\alpha}]$ of $\bar{E}^c$, and assume that it is closed. Then $\bar{\alpha} = d\rho$ where ρ is a local 1-form on Q, i.e. in coordinates it reads as follows:

$$\rho = \rho^0 dt + \rho^1_s \bar{\omega}^s + \rho^2_a \varphi^a + \rho^3_s d\dot{q}^s. \tag{60}$$

Computing $d\rho$ and equating its components with those of (59) we can immediately see that the term $d\dot{q}^r \wedge d\dot{q}^s$ is missing in $\bar{\alpha}$. Hence

$$\frac{\partial \rho^3_s}{\partial \dot{q}^r} = \frac{\partial \rho^3_r}{\partial \dot{q}^s}, \tag{61}$$

i.e.

$$\rho^3_s = \frac{\partial h}{\partial \dot{q}^s} + h_s(t, q^\sigma), \tag{62}$$

meaning that ρ is of the form

$$\begin{aligned} \rho = &\Big(\rho^0 - \frac{d'_{\mathrm{c}} h}{dt} - \dot{q}^s \frac{d'_{\mathrm{c}} h_s}{dt}\Big) dt + \Big(\rho^1_s - \frac{\partial_{\mathrm{c}} h}{\partial q^s} - \dot{q}^r \frac{\partial_{\mathrm{c}} h_r}{\partial q^s}\Big)\bar{\omega}^s \\ &+ \Big(\rho^2_a - \frac{\partial h}{\partial q^{m-k+a}} - \dot{q}^s \frac{\partial h_s}{\partial q^{m-k+a}}\Big)\varphi^a + d(h + h_s \dot{q}^s)\,. \end{aligned} \tag{63}$$

We conclude that without loss of generality we may assume $\bar{\alpha} = d\bar{\rho}$ where

$$\bar{\rho} = \bar{L}\, dt + f_s \bar{\omega}^s + \bar{L}_a \varphi^a. \tag{64}$$

Comparing now $d\bar{\rho}$ with $\bar{\alpha}$ and accounting that

$$\begin{aligned} d\varphi^a = &-\varepsilon'_s(g^a)\bar{\omega}^s \wedge dt + \Big(\frac{\partial_{\mathrm{c}}}{\partial q^r}\frac{\partial g^a}{\partial \dot{q}^s}\Big)\bar{\omega}^s \wedge \bar{\omega}^r + \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s}\bar{\omega}^s \wedge d\dot{q}^r \\ &- \frac{\partial g^a}{\partial q^{m-k+b}}\varphi^b \wedge dt - \Big(\frac{\partial}{\partial q^{m-k+b}}\frac{\partial g^a}{\partial \dot{q}^s}\Big)\varphi^b \wedge \bar{\omega}^s \end{aligned} \tag{65}$$

we obtain:

$$f_s = \frac{\partial \bar{L}}{\partial \dot{q}^s}\,, \tag{66}$$

and

$$\bar{A}_s = \frac{\partial_{\mathrm{c}} \bar{L}}{\partial q^s} - \frac{d'_{\mathrm{c}} f_s}{dt} - \bar{L}_a \varepsilon'_s(g^a), \quad \bar{B}_{rs} = -\frac{\partial f_r}{\partial \dot{q}^s} + \bar{L}_a \frac{\partial^2 g^a}{\partial \dot{q}^r \partial \dot{q}^s} \tag{67}$$

proving that $\bar{E}^{\mathrm{c}}$ is constraint variational. Moreover, we find expressions for the other components of $\bar{\alpha}$ by means of $\bar{L}$ and $\bar{L}_a$ as follows:

$$\bar{F}_{rs} = \frac{1}{2}\Bigg(\Big(\frac{\partial_{\mathrm{c}} f_s}{\partial q^r} - \frac{\partial_{\mathrm{c}} f_r}{\partial q^s}\Big) - \bar{L}_a\Big(\frac{\partial_{\mathrm{c}}}{\partial q^r}\frac{\partial g^a}{\partial \dot{q}^s} - \frac{\partial_{\mathrm{c}}}{\partial q^s}\Big(\frac{\partial g^a}{\partial \dot{q}^r}\Big)\Big)\Bigg) \tag{68}$$

and

$$\begin{aligned} b_a &= \frac{\partial \bar{L}}{\partial q^{m-k+a}} - \frac{d'_{\mathrm{c}} \bar{L}_a}{dt} - \bar{L}_b \frac{\partial g^b}{\partial q^{m-k+a}} \\ b_{as} &= \frac{\partial f_s}{\partial q^{m-k+a}} - \frac{\partial_{\mathrm{c}} \bar{L}_a}{\partial q^s} - \bar{L}_b \frac{\partial}{\partial q^{m-k+a}}\Big(\frac{\partial g^b}{\partial \dot{q}^s}\Big) \\ c_{as} &= -\frac{\partial \bar{L}_a}{\partial \dot{q}^s} \\ \gamma_{ab} &= \frac{1}{2}\Big(\frac{\partial \bar{L}_b}{\partial q^{m-k+a}} - \frac{\partial \bar{L}_a}{\partial q^{m-k+b}}\Big) \end{aligned} \tag{69}$$

□

Notice that in the class $[\bar\alpha]$ we have three distinguished representatives: $\bar\alpha_1 = d\bar\rho$ with components as above,

$$\bar\alpha_2 = \Big(\varepsilon'_s(\bar L) - \bar L_a \varepsilon'_s(g^a)\Big)\bar\omega^s \wedge dt - \Big(\frac{\partial^2 \bar L}{\partial \dot q^r \partial \dot q^s} - \bar L_a \frac{\partial^2 g^a}{\partial \dot q^r \partial \dot q^s}\Big)\bar\omega^r \wedge d\dot q^s \tag{70}$$

and

$$\begin{aligned}\bar\alpha_3 = &\Big(\varepsilon'_s(\bar L) - \bar L_a \varepsilon'_s(g^a)\Big)\bar\omega^s \wedge dt + \Big(\frac{\partial_{\mathrm c}}{\partial q^r}\frac{\partial \bar L}{\partial \dot q^s} - \bar L_a\Big(\frac{\partial_{\mathrm c}}{\partial q^r}\frac{\partial g^a}{\partial \dot q^s}\Big)\Big)\bar\omega^r \wedge \bar\omega^s \\ &- \Big(\frac{\partial^2 \bar L}{\partial \dot q^r \partial \dot q^s} - \bar L_a \frac{\partial^2 g^a}{\partial \dot q^r \partial \dot q^s}\Big)\bar\omega^r \wedge d\dot q^s\,.\end{aligned} \tag{71}$$

The following theorem provides variationality conditions of reduced equations, called *constraint Helmholtz conditions*, first obtained in [9].

Theorem 8. *Let $\bar E^{\mathrm c}$ be a constrained dynamical form, $[\bar\alpha]$ the corresponding class of 2-forms. $\bar E^{\mathrm c}$ is constraint variational if and only if (locally) there exist functions b_a, c_{as} and γ_{ab} on Q (i.e. functions of variables $(t, q^\sigma, \dot q^l)$) such that $\gamma_{ab} = -\gamma_{ba}$, the γ's are solutions of the equations*

$$\Big(\frac{d'_{\mathrm c}\gamma_{ab}}{dt} - 2\gamma_{bc}\frac{\partial g^c}{\partial q^{m-k+a}} - \frac{\partial b_a}{\partial q^{m-k+b}}\Big)_{\mathrm{alt}(ab)} = 0\,, \tag{72}$$

and the following conditions hold

$$\begin{aligned}(\bar B_{ls})_{\mathrm{alt}(ls)} &= 0\\ \Big(\frac{\partial \bar B_{ls}}{\partial \dot q^r} - \frac{\partial^2 g^a}{\partial \dot q^l \partial \dot q^r}c_{as}\Big)_{\mathrm{alt}(sr)} &= 0\\ \Big(\frac{\partial \bar A_l}{\partial \dot q^s} - \varepsilon'_l(g^a)c_{as} - \frac{d'_{\mathrm c}\bar B_{ls}}{dt} - \frac{\partial^2 g^a}{\partial \dot q^l \partial \dot q^s}b_a\Big)_{\mathrm{sym}(ls)} &= 0\\ \Big(-\frac{\partial_{\mathrm c}\bar A_l}{\partial q^s} + \varepsilon'_l(g^a)b_{as} + \frac{1}{2}\frac{d'_{\mathrm c}}{dt}\Big(\frac{\partial \bar A_l}{\partial \dot q^s} - \varepsilon'_l(g^a)c_{as}\Big) + b_a\frac{\partial_{\mathrm c}}{\partial q^s}\Big(\frac{\partial g^a}{\partial \dot q^l}\Big)\Big)_{\mathrm{alt}(ls)} &= 0\\ \frac{\partial \bar A_l}{\partial q^{m-k-a}} + 2\gamma_{ac}\varepsilon'_l(g^c) - \frac{\partial_{\mathrm c} b_a}{\partial q^l} - b_c\frac{\partial^2 g^c}{\partial \dot q^l \partial q^{m-k+a}} + \frac{d'_{\mathrm c} b_{al}}{dt} + \frac{\partial g^c}{\partial q^{m-k+a}}b_{cl} &= 0\\ \frac{\partial \bar B_{ls}}{\partial q^{m-k+a}} - 2\gamma_{ab}\frac{\partial^2 g^b}{\partial \dot q^l \partial \dot q^s} + \frac{\partial b_{al}}{\partial \dot q^s} - \frac{\partial_{\mathrm c} c_{as}}{\partial q^l} - \frac{\partial^2 g^b}{\partial \dot q^l \partial q^{m-k+a}}c_{bs} &= 0\end{aligned} \tag{73}$$

where

$$b_{as} = \frac{\partial b_a}{\partial \dot q^s} - \frac{d'_{\mathrm c} c_{as}}{dt} - \frac{\partial g^b}{\partial q^{m-k+a}}c_{bs}\,. \tag{74}$$

Proof. By the preceding theorem the result comes from the condition $d\bar\alpha = 0$ where $\bar\alpha$ is given by (59). □

Notice that by the above computation we obtain for components of the 2-form F the following formula

$$\bar F_{rs} = \frac{1}{4}\Big(\Big(\frac{\partial \bar A_r}{\partial \dot q^s} - \frac{\partial \bar A_s}{\partial \dot q^r}\Big) - \big(\varepsilon'_r(g^a)c_{as} - \varepsilon'_s(g^a)c_{ar}\big)\Big)\,, \tag{75}$$

which is just another expression of (68).

Compared with Helmholtz conditions, the constraint Helmholtz conditions have a rather surprising form. While the former are *identities* to be fulfilled by the components of a dynamical form (i.e. by the functions on the left-hand sides of the corresponding equations), the latter are rather *equations* for unknown functions b_a, c_{as} and γ_{ab}. This means that for a system of equations (46), if the answer to the question on constraint variationality is affirmative, the corresponding constraint Lagrangian form need not be unique. This is closely related with the yet unsolved problem on the structure of constraint null Lagrangians.

6 Examples: Planar motions

In this section we shall study examples of various simple mechanical systems, namely planar systems subject to one nonholonomic constraint. This means that we have one reduced equation of motion in this case. In the notation used so far, $m = 2$, $k = 1$, $Y = \mathbb{R} \times \mathbb{R}^2$; coordinates in the plane will be denoted by (x, y).

The unconstrained equations of motion are of the form

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -F_1\,, \quad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = -F_2\,, \tag{76}$$

where the force on the right-hand side generally is not assumed variational. The functions $B_{\sigma\nu}$ are the same for any (variational and non-variational) force, obstructions to variationality may enter only through additional terms to A_σ, i.e. $A_\sigma \to \mathcal{A}_\sigma = A_\sigma + F_\sigma$.

A nonholonomic constraint in $J^1(\mathbb{R} \times \mathbb{R}^2)$ is given by equation

$$\dot{y} = g(t,\, x,\, y,\, \dot{x})\,, \tag{77}$$

so that

$$\varphi^1 = dy - \frac{\partial g}{\partial \dot{x}}\, dx - \Big(g - \dot{x}\frac{\partial g}{\partial \dot{x}}\Big) dt\,, \tag{78}$$

and the reduced equation of motion takes the form (37) modified by Φ, i.e.

$$\frac{\partial_{\mathrm{c}} \bar{L}}{\partial x} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial \bar{L}}{\partial \dot{x}} - \bar{L}_1\Big(\frac{\partial_{\mathrm{c}} g}{\partial x} - \frac{d_{\mathrm{c}}}{dt}\frac{\partial g}{\partial \dot{x}}\Big) = -\bar{\Phi}, \tag{79}$$

where

$$\bar{\Phi} = \bar{F}_1 + \bar{F}_2\frac{\partial g}{\partial \dot{x}}, \quad \bar{F}_\sigma = F_\sigma \circ \iota. \tag{80}$$

The constraint Helmholtz conditions (73) reduce to the following equations for functions b_1 and c_{11} (due to skew symmetry, $\gamma_{11} = 0$):

$$\begin{gathered}
\frac{\partial \bar{A}_1}{\partial \dot{x}} - \varepsilon'_1(g) c_{11} - \frac{d'_{\mathrm{c}}\bar{B}_{11}}{dt} - \frac{\partial^2 g}{\partial \dot{x}^2} b_1 = 0\\
\frac{\partial \bar{A}_1}{\partial y} - \frac{\partial_{\mathrm{c}} b_1}{\partial x} - \frac{\partial^2 g}{\partial \dot{x}\partial y} b_1 + \frac{d'_{\mathrm{c}} b_{11}}{dt} + \frac{\partial g}{\partial y} b_{11} = 0\\
\frac{\partial \bar{B}_{11}}{\partial y} + \frac{\partial b_1}{\partial \dot{x}} - \frac{\partial_{\mathrm{c}} c_{11}}{\partial x} - \frac{\partial^2 g}{\partial \dot{x}\partial y} c_{11} = 0
\end{gathered} \tag{81}$$

where

$$b_{11} = \frac{\partial b_1}{\partial \dot{x}} - \frac{d'_c c_{11}}{dt} - \frac{\partial g}{\partial y} c_{11} \,. \tag{82}$$

Recall that conditions (81) are fulfilled for every constrained system arising from an unconstrained Lagrangian one. Adding the force (F_1, F_2) to equations of motion, the reduced equation changes by $\bar{\Phi}$, and the first two conditions (81) by $\frac{\partial \Phi}{\partial \dot{x}}$ and $\frac{\partial \Phi}{\partial y}$, respectively. Hence for a Lagrangian system in a force field (F_1, F_2) the constraint Helmholtz conditions are fulfilled trivially for every constraint satisfying the following compatibility condition:

$$\bar{\Phi} = \quad \bar{F}_1 + \bar{F}_2 \frac{\partial g}{\partial \dot{x}} \quad = \quad \chi(t, x), \tag{83}$$

where $\chi(t, x)$ is an arbitrary function. For such a case equations (81) retain the same solution (b_1, b_{11}, c_{11}) as in the case without additional forces. Moreover, if $\chi(t, x) = 0$, the "free" Lagrangian system (i.e. with $F_1 = F_2 = 0$) and that (essentially different!) moving in a constraint-compatible force field $(F_1, F_2) \neq 0$ have *the same* reduced motion equation.

6.1 Motion in a homogeneous field

Let us consider the motion of a mass particle m in a homogeneous field, for concreteness e.g. in the gravitational field $\vec{G}$. Such a particle moves in a plane xOy along a parabolic trajectory (so called *parabolic* or *projectile motion*),

$$x(t) = vt\cos\alpha, \quad y(t) = vt\sin\alpha - \frac{1}{2}Gt^2,$$

where $\vec{v} = (v\cos\alpha, v\sin\alpha)$ is the initial velocity. The unconstrained system is variational, with the Lagrangian

$$\lambda = L\,dt, \quad L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - mGy, \tag{84}$$

the corresponding dynamical form is

$$E_\lambda = -m\ddot{x}\,dx \wedge dt - m(\ddot{y} + G)dy \wedge dt. \tag{85}$$

Consider a constraint (77). Then

$$\bar{B}_{11} = -m\left(1 + \left(\frac{\partial g}{\partial \dot{x}}\right)^2\right), \quad \bar{A}_1 = -m\frac{\partial g}{\partial \dot{x}}\left(G + \frac{d'_c g}{dt}\right), \tag{86}$$

so that the reduced equation is of the form

$$-m\ddot{x}\left(1 + \left(\frac{\partial g}{\partial \dot{x}}\right)^2\right) - m\frac{\partial g}{\partial \dot{x}}\left(G + \frac{d'_c g}{dt}\right) \quad = \quad 0. \tag{87}$$

Since the unconstrained system is Lagrangian, the arising constrained system is constraint variational for any nonholonomic constraint. This means, of course, that the constraint Helmholtz conditions have a solution (certain functions b_1, b_{11}, c_{11}) for every fixed constraint (77).

Now, let us consider the question on constraint variationality of equation (87) from the other side: Let us try to find a solution of the inverse problem directly, by solving the constraint Helmholtz conditions as equations for b_1 and c_{11}.

Accounting commutation relations for constraint derivative operators, the constraint Helmholtz conditions take the form

$$-\frac{\partial^2 g}{\partial \dot{x}^2}\left(m\left(G+\frac{d'_{\mathrm{c}}g}{dt}\right)+b_1\right)-\varepsilon'_1(g)\left(c_{11}+m\frac{\partial g}{\partial \dot{x}}\right)=0\,, \tag{88}$$

$$-\frac{\partial^2 g}{\partial \dot{x}\partial y}\left(m\left(G+\frac{d'_{\mathrm{c}}g}{dt}\right)+b_1\right)-m\frac{\partial g}{\partial \dot{x}}\frac{\partial}{\partial y}\left(\frac{d'_{\mathrm{c}}g}{dt}\right)-\frac{\partial_{\mathrm{c}}b_1}{\partial x}+\frac{d'_{\mathrm{c}}b_{11}}{dt}+b_{11}\frac{\partial g}{\partial y}=0\,, \tag{89}$$

$$-\frac{\partial^2 g}{\partial \dot{x}\partial y}\left(c_{11}+2m\frac{\partial g}{\partial \dot{x}}\right)+\frac{\partial b_{11}}{\partial \dot{x}}-\frac{\partial_{\mathrm{c}}c_{11}}{\partial x}=0\,, \tag{90}$$

with

$$b_{11}=\frac{\partial b_1}{\partial \dot{x}}-\frac{d'_{\mathrm{c}}c_{11}}{dt}-\frac{\partial g}{\partial y}c_{11}\,. \tag{91}$$

Condition (88) can be fulfilled e.g. for functions b_1 and c_{11} of the form

$$b_1=-m\left(G+\frac{d'_{\mathrm{c}}g}{dt}\right), \tag{92}$$

$$c_{11}=-m\frac{\partial g}{\partial \dot{x}}. \tag{93}$$

Then

$$b_{11}=-m\frac{\partial g}{\partial x}\,. \tag{94}$$

It can be verified by a direct calculation that with the above choice of functions b_1, c_{11} and b_{11} the remaining two constraint Helmholtz conditions (89) and (90) are satisfied. In this way we have obtained that the reduced equation (87) is indeed constraint variational.

We can ask the question if the above solution to the constraint Helmholtz conditions is in correspondence with the original (unconstrained) system, since, in principle, the obtained b_1, b_{11}, and c_{11} could correspond to a different unconstrained Lagrangian system having the same reduced equation of motion. To this end let us compute the corresponding functions related with the Lagrangian (84); let us use notations $b_1(L)$, $c_{11}(L)$, and $b_{11}(L)$ to distinguish them from the b_1, c_{11}, and b_{11} above.

We have

$$\bar{L}=\frac{1}{2}m\dot{x}^2+\frac{1}{2}mg^2-mGy\,,\quad \bar{L}_1=mg\,, \tag{95}$$

hence, by (69)

$$b_1(L) = \frac{\partial \bar{L}}{\partial y} - \frac{d'_c \bar{L}_1}{dt} - \bar{L}_1 \frac{\partial g}{\partial y} = -m\left(G + \frac{d'_c g}{dt}\right) = b_1,$$
$$b_{11}(L) = \frac{\partial^2 \bar{L}}{\partial y \partial \dot{x}} - \frac{\partial_c \bar{L}_1}{\partial x} - \bar{L}_1 \frac{\partial^2 g}{\partial y \partial \dot{x}} = -m\frac{\partial g}{\partial x} = b_{11},$$
$$c_{11}(L) = -\frac{\partial \bar{L}_1}{\partial \dot{x}} = -m\frac{\partial g}{\partial \dot{x}} = c_{11}.$$

Finally, the constraint Lagrangian and the constraint Cartan form read

$$\lambda_{\rm c} = \left(\frac{1}{2}m(\dot{x}^2 + g^2) - mGy\right) dt + mg\varphi^1,$$
$$\bar{\rho} = \lambda_{\rm c} + m\left(\dot{x} + g\frac{\partial g}{\partial \dot{x}}\right)(dx - \dot{x}\, dt)\,. \tag{96}$$

Remark 1. An interesting constraint for the Lagrangian system (84) was considered in [14], namely

$$\dot{y} = \sqrt{v^2 - \dot{x}^2}. \tag{97}$$

In this case the reduced equation has the form

$$-\frac{mv^2}{v^2 - \dot{x}^2}\ddot{x} + \frac{mG\dot{x}}{\sqrt{v^2 - \dot{x}^2}} = 0 \quad \Longrightarrow \tag{98}$$
$$\ddot{x} - \frac{G}{v^2}\dot{x}\sqrt{v^2 - \dot{x}^2} = 0$$

and it can be solved analytically (see [14] for the solution and conservation laws).

The functions b_1, c_{11} and b_{11} given by (92), (93) and (94).take the form

$$b_1 = -mG, \quad c_{11} = \frac{m\dot{x}}{\sqrt{v^2 - \dot{x}^2}}, \quad b_{11} = 0,$$

and a constraint Lagrangian is

$$\lambda_{\rm c} = -mGy\, dt + m\sqrt{v^2 - \dot{x}^2}\varphi^1.$$

One can easily verify that, indeed, $\varepsilon_1(\bar{L}) - \bar{L}_1\varepsilon_1(g)$ is the left-hand-side of the reduced equation (98).

There are, however, also other solutions b_1, b_{11} and c_{11} of the constraint Helmholtz conditions. One of them is $b_1 = -mG$, $b_{11} = 0$, $c_{11} = 0$ as can be easily verified substituting into (81). A corresponding constraint Lagrangian, leading to the same reduced equation (98), is then

$$\lambda'_{\rm c} = \bar{L}'\, dt\,,$$

where

$$\bar{L}' = -mGy + L_0, \quad L_0 = \frac{1}{2}mv\left((v + \dot{x})\ln(v + \dot{x}) + (v - \dot{x})\ln(v - \dot{x})\right).$$

The 1-form

$$\tau = {\lambda_c}' - \lambda_c = L_0 dt - m\sqrt{v^2 - \dot{x}^2}\,\varphi^1$$

leads to identically zero left-hand side of the reduced equation and thus it is a null constraint Lagrangian.

We can see that the constraint Lagrangian $\bar{L}'\,dt$ can be extended e.g. to the Lagrangian $\left(\bar{L}' + L_0'\right)dt$, defined on $J^1(\mathbb{R}\times\mathbb{R}^2)$, where L_0' is a polynomial of at least second degree in the variable $\dot{y} - \sqrt{v^2 - \dot{x}^2}$, for example, one can take simply $L_0' = \frac{1}{2}m\left(\dot{y} - \sqrt{v^2 - \dot{x}^2}\right)^2$. For such an additional Lagrangian it holds $L_0' \circ \iota = 0$ and $\frac{\partial L_0'}{\partial \dot{y}} \circ \iota = 0$. (In general, for a constraint $\dot{y} = g(t, x, y, \dot{x})$ the same is fulfilled for a polynomial of at least second degree in the variable $\dot{y} - g$.) If $L_0' = \frac{1}{2}m\left(\dot{y} - \sqrt{v^2 - \dot{x}^2}\right)^2$ then the corresponding unconstrained equations of motion of $\tilde{\lambda} = (\bar{L}' + L_0')dt$ take the form

$$\begin{aligned} -\frac{mv^2}{v^2 - \dot{x}^2}\left(\frac{\dot{x}^2}{v^2} + \frac{\dot{y}}{\sqrt{v^2 - \dot{x}^2}}\right)\ddot{x} - \frac{m\dot{x}}{\sqrt{v^2 - \dot{x}^2}}\ddot{y} = 0,\\ -mG - \frac{m\dot{x}}{\sqrt{v^2 - \dot{x}^2}}\ddot{x} - m\ddot{y} = 0 \end{aligned} \tag{99}$$

and apparently they are not equivalent with the motion equations of the Lagrangian (84).

6.2 Damped motion in a homogeneous field

Let us turn to the case when the unconstrained system is not variational.

Consider the same Lagrangian (84) as above, but now suppose that additionally the motion is damped by Stokes force $\vec{F} = -\beta\vec{v}$, i.e. $(F_\sigma) = (-\beta\dot{x}, -\beta\dot{y})$, where β is a positive constant. (The trajectory of the particle is the well-known ballistic curve.)

The dynamical form

$$E = -(m\ddot{x} + \beta\dot{x})dx \wedge dt - (m\ddot{y} + mG + \beta\dot{y})dy \wedge dt \tag{100}$$

is not variational. Denote

$$\mathcal{A}_\sigma = A_\sigma + F_\sigma \tag{101}$$

where A_σ corresponds to the undamped (variational) system above.

Given a nonholonomic constraint (77) we obtain

$$\begin{aligned} \bar{B}_{11} &= -m\left(1 + \left(\frac{\partial g}{\partial \dot{x}}\right)^2\right)\\ \bar{\mathcal{A}}_1 &= -m\frac{\partial g}{\partial \dot{x}}\left(G + \frac{d_c' g}{dt}\right) - \beta\left(\dot{x} + g\frac{\partial g}{\partial \dot{x}}\right) \end{aligned} \tag{102}$$

yielding the reduced equation

$$-m\ddot{x}\left(1 + \left(\frac{\partial g}{\partial \dot{x}}\right)^2\right) - m\frac{\partial g}{\partial \dot{x}}\left(G + \frac{d_c' g}{dt}\right) - \beta\left(\dot{x} + g\frac{\partial g}{\partial \dot{x}}\right) = 0 \tag{103}$$

which differs from the preceding (constraint variational) motion equation by an additional force term

$$\bar{\Phi} = -\beta\left(\dot{x} + g\frac{\partial g}{\partial \dot{x}}\right). \tag{104}$$

We shall be interested under what conditions equation (103) is *constraint variational.*

For additional (non-variational) forces F_1 and F_2 it is necessary to add to constraint Helmholtz conditions (89) and (90) additional terms $\frac{\partial \bar{F}_1}{\partial \dot{x}}$ and $\frac{\partial \bar{F}_1}{\partial y}$, respectively. Condition (88) remains unchanged. Then there is a possibility to fulfill the constraint Helmholtz conditions by a simple way, namely to find such a constraint g for which equation (83) is satisfied. Integrating this equation we obtain

$$g = \sqrt{\phi(t,\, x,\, y) + \dot{x}\chi(t,\, x) - \dot{x}^2}\,, \tag{105}$$

where $\phi(t,\, x,\, y)$ and $\chi(t,\, x)$ are arbitrary functions of indicated variables. For every constraint of this type the constraint Helmholtz conditions are the same as for the undamped case. Let us emphasize that the family of solutions b_1, b_{11} and c_{11} of constraint Helmholtz conditions remains unchanged as well. One of these solutions is thus again given by (92), (93) and (94). A corresponding constraint Lagrangian is then, accordingly

$$\lambda_c = \left(\frac{1}{2}m(\phi + \dot{x}\chi) - mGy)\right)dt + m\sqrt{\phi + \dot{x}\chi - \dot{x}^2}\,\varphi^1\,.$$

An interesting case occurs for $\phi = 2Gy$, $\chi = 0$. Then $\bar{L} = 0$, hence

$$\lambda_c = m\sqrt{2Gy - \dot{x}^2}\,\varphi^1.$$

So, we can see that there is a possibility to choose a constraint Lagrangian for which $\bar{L} = 0$: this Lagrangian belongs to the constraint ideal. Note that on the other hand, there is no possibility to get $\bar{L}_1 = 0$, i.e. λ_c of a form $\bar{L}dt$. The corresponding reduced equation reads

$$\frac{2mG\dot{x}}{\sqrt{2Gy - \dot{x}^2}} - \frac{2mGy}{2Gy - \dot{x}^2}\ddot{x} = 0.$$

Remark 2. It is worth note that condition $\bar{\Phi} = 0$ yields the same reduced equation, hence the same constraint dynamics (which, moreover is constraint variational) for essentially different unconstrained systems. In our example this concerns a variational system given by Lagrangian (84) and a non variational one, given by the same Lagrangian and a non-potential Stokes force. Recall that this happens subject a constraint

$$g = \sqrt{\phi - \dot{x}^2}\,. \tag{106}$$

7 Example: Relativistic particle

A physically highly interesting example of a constrained system subject to a nonlinear nonholomic costraint is a massive particle in the special relativity theory. It was studied in detail in [8]. It can be modeled with help of an initially variational unconstrained system on $Y = \mathbb{R} \times \mathbb{R}^4$ ($m = 4$, coordinates $(s, q^\sigma, \dot{q}^\sigma), 1 \leq \sigma \leq 4$) defined by the following Lagrangian

$$L = -\frac{1}{2} m_0 \sqrt{(\dot{q}^4)^2 - \sum_{l=1}^{3} (\dot{q}^l)^2} + \dot{q}^\sigma \phi_\sigma - \psi, \tag{107}$$

where $\phi(q^\sigma)$ and $\psi(q^\sigma)$ are functions on Y. The corresponding Euler-Lagrange form reads

$$E_\lambda = \varepsilon_l(L)\, dq^l \wedge dt + \varepsilon_4(L) dq^4 \wedge dt, \quad 1 \leq l \leq 3\,,$$
$$\varepsilon_l(L) = B_{ls}\ddot{q}^s + A_l = -m_0 \ddot{q}^l + \dot{q}^\sigma \left(\frac{\partial \phi_\sigma}{\partial q^l} - \frac{\partial \phi_l}{\partial q^\sigma} \right) - \frac{\partial \psi}{\partial q^l}\,,$$
$$\varepsilon_4(L) = B_{4s}\ddot{q}^s + A_4 = m_0 \ddot{q}^4 + \dot{q}^\sigma \left(\frac{\partial \phi_\sigma}{\partial q^4} - \frac{\partial \phi_4}{\partial q^\sigma} \right) - \frac{\partial \psi}{\partial q^4}.$$

The constraint is given by the standard condition for 4-velocity,

$$(\dot{q}^4)^2 - \sum_{p=1}^{3} (\dot{q}^p)^2 = 1 \implies \dot{q}^4 = \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}. \tag{108}$$

For coefficients of reduced equations we obtain (see (26))

$$\bar{A}_l = \dot{q}^a \left(\frac{\partial \phi_a}{\partial q^l} - \frac{\partial \phi_l}{\partial q^a} \right) - \frac{\partial \psi}{\partial q^l} + \left(\dot{q}^a \left(\frac{\partial \phi_a}{\partial q^4} - \frac{\partial \phi_4}{\partial q^a} \right) - \frac{\partial \psi}{\partial q^4} \right) \frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}$$
$$+ \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} \left(\frac{\partial \phi_4}{\partial q^l} - \frac{\partial \phi_l}{\partial q^4} \right), \tag{109}$$

$$\bar{B}_{ls} = -m_0 \left(\delta_{ls} - \frac{\dot{q}^l \dot{q}^s}{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} \right). \tag{110}$$

Our aim is to find a solution of constraint Helmholtz conditions for the corresponding reduced equations of motion

$$\bar{A}_l + \bar{B}_{ls} \ddot{q}^s = 0.$$

The first of conditions (73) is fulfilled because $\bar{B}_{ls}$ are symmetric. As for the second of conditions (73), it holds

$$\frac{\partial \bar{B}_{ls}}{\partial \dot{q}^r} - \frac{\partial \bar{B}_{lr}}{\partial \dot{q}^s} = m_0 \frac{\delta^l_r \dot{q}^s - \delta^l_s \dot{q}^r}{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}. \tag{111}$$

On the other hand, we have

$$c_{1s} \frac{\partial^2 g}{\partial \dot{q}^l \partial \dot{q}^r} - c_{1r} \frac{\partial^2 g}{\partial \dot{q}^l \partial \dot{s}^s} = \frac{c_{1s} \delta^l_r - c_{1r} \delta^l_s}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}} - \frac{c_{1s} \dot{q}^l \dot{q}^r - c_{1r} \dot{q}^l \dot{q}^s}{(1 + \sum_{p=1}^{3} (\dot{q}^p)^2)^{3/2}}. \tag{112}$$

Comparing (111) and (112) we find a solution

$$c_{1l} = \frac{m_0 \dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}}, \quad 1 \leq l \leq 3. \tag{113}$$

Substituting c_{1l} into the third condition in (73) and taking into account that $\varepsilon'_l(g) = 0$ (g depends on $\dot{q}^l$, $1 \leq l \leq 3$, only), we obtain, after some calculations,

$$b_1 = \dot{q}^l \left(\frac{\partial \phi_l}{\partial q^4} - \frac{\partial \phi_4}{\partial q^l} \right) - \frac{\partial \psi}{\partial q^4}. \tag{114}$$

Finally, using (74), we get

$$b_{1l} = \frac{\partial \phi_l}{\partial q^4} - \frac{\partial \phi_4}{\partial q^l}. \tag{115}$$

The remaining constraint Helmholtz conditions of (73) are then fulfilled.

It can be easily verified that functions (113), (114) and (115) are the same as those calculated from Lagrangian (107) using (69).

The constraint Lagrangian is

$$\lambda_{\mathrm{c}} = \bar{L}\, ds + \bar{L}_1 \varphi^1, \quad \varphi^1 = -\frac{\dot{q}^l}{\sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2}} \omega^l + \iota^* \omega^4,$$

where

$$\bar{L} = L \circ \iota = -\frac{1}{2} m_0 + \dot{q}^l \phi_l + \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2 \phi_4} - \psi,$$

$$\bar{L}_1 = -m_0 \sqrt{1 + \sum_{p=1}^{3} (\dot{q}^p)^2} + \phi_4.$$

In coordinates (t, q^l, v^l), adapted to the fibration $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, i.e. such that

$$\dot{q}^l = v^l \dot{q}^4, \quad \dot{q}^4 = \frac{1}{\sqrt{1 - v^2}},$$

and with the notation $(\phi_l) = \vec{A}$, $\phi_4 = -V$ we obtain

$$\bar{L} = -\frac{1}{2} m_0 + \frac{1}{\sqrt{1 - v^2}} (\vec{v}\vec{A} - V) - \psi, \quad \bar{L}_1 = -\frac{m_0}{\sqrt{1 - v^2}} - V.$$

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