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Results on Non-resonant Oscillations for some Nonlinear Vector Fourth Order Differential Systems

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Abstract

This paper presents vector versions of some existence results recently published for certain fourth order differential systems based on generalisations drawn from possibilities arising from the underlying auxiliary equation. The results obtained also extend some known works involving third order differential systems to the corresponding fourth order.

Key words: non-resonant oscillations, fourth order differential systems, Leray–Schauder continuation techniques

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1 Introduction

Studies into fourth order nonlinear differential equations of the form

$$x^{(iv)} + \phi(x''') + f(x'') + g(x') + h(x) = p(t, x, x', x'', x''') \tag{1.1}$$

have engaged several authors for over half a century, with a wide range of qualitative properties being investigated (See Reissig, Sansone and Conti [8] for the earlier results). Still there is much ground yet uncovered. Attention has recently focused on periodic boundary value problems of fourth order nonlinear differential equations and new results have appeared in this regard to point the way forward, particularly the papers by Ezeilo ([3, 4]) and Tejumola ([9, 10]). Credible solvability hypotheses have been evolved which were derived from techniques which include generalisations of the Routh-Hurwitz conditions, permutations involving roots of the underlying auxiliary equation, and an eigenvalue problem approach. The proofs were based on the application of one of several versions of the Leray-Schauder continuation technique in which the parameter dependent equations and the mode of computing the relevant *a priori* estimates may vary from one configuration of the given problem to the other depending on the hypotheses employed.

Vector results for boundary value problems of fourth order nonlinear differential systems appear not to have received much, if any, attention, since the theory for the scalar forms are currently still being developed themselves. Notwithstanding, it is important both from the mathematical and practical point of view, to attempt to provide vector versions to available results in the scalar case which appear on the horizon, which is by no means a trivial transition. This is our prime motivation for this work.

We shall be concerned with the T-periodic boundary value problems of forced fourth order differential systems of the form

$$X^{(IV)} + AX''' + BX'' + \frac{d}{dt}\mathcal{G}(X) + DX = P(t) \tag{1.2}$$

and

$$X^{(IV)} + AX''' + BX'' + G(t, X') + DX = P(t) \tag{1.3}$$

subject to the periodic boundary conditions

$$\mathcal{D}^{(r)}X(0) = \mathcal{D}^{(r)}X(T), \quad \left(\mathcal{D} = \frac{d}{dt}\right), \quad r = 0, 1, 2, 3 \tag{1.4}$$

on $[0, T]$ with $T > 0$.

Here, $X \equiv (x_i)_{1\le i\le n}\colon [0, T] \to \mathbb{R}^n$, the n-dimensional Euclidean Space, equipped with the usual norm $\|\cdot\|$ and scalar product $\langle\cdot,\cdot\rangle$, defined by $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$ for any pair $X, Y \in \mathbb{R}^n$, so that $\langle X, X\rangle = \|X\|^2$ is the usual Euclidean norm in $\mathbb{R}^n$. A, B and D are constant real symmetric $n \times n$ matrices, while G and P are n-vectors, which are T-periodic in t. Furthermore, $\mathcal{G} \in C^1$ while $G(t, Y)$ satisfies the Carathéodory conditions.

The classical spaces of k times continuously differentiable functions shall be denoted by $C^k([0, T], \mathbb{R}^n)$, $k \ge 0$ an integer, where $C^0 = C$ and $C^\infty = \bigcap_{k\ge 0} C^k$ with norms $\|X\|_{C^k}$ and $\|X\|_\infty$ respectively.

$L^p = L^p([0, T])$, $1 \le p \le \infty$, will denote the usual Lebesgue spaces, with their respective norms $\|X\|_{L^p}$.

Finally, $W_T^{k,p}([0, T], \mathbb{R}^n)$, will denote the Sobolev space of T-periodic functions of order k, defined by

$$\begin{aligned} W_T^{k,p} = \{X\colon [0,T] \to \mathbb{R}^n\colon X, X', \dots, X^{(k-1)} &\text{ are absolutely continuous on } [0,T], \\ X^{(k)} \in L^p(0,T) \text{ and } X^{(i)}(0) - X^{(i)}(T) = 0,\ i = 0, 1, 2, \dots, k-1,\ k \in \mathbb{R}\}& \end{aligned}$$

with corresponding norm $\|X\|_{W_T^{k,p}}$.

It is standard result that if M is a real $n \times n$ symmetric matrix, then for any $X \in \mathbb{R}^n$,

$$\delta_m \|X\|^2 \leq \langle MX, X\rangle \leq \Delta_m \|X\|^2, \tag{1.5}$$

where δ_m and Δ_m are positive real constants which represent respectively, the least and greatest eigenvalues of M. In general, $\lambda_i(M)$, $i = 1, \dots, n$, shall denote the eigenvalues of any matrix M; and $\|M\|$ denote the norm of M thought of as a linear operator in $\mathbb{R}^n$ (that is, spectral norm of M with respect to the inner product $\langle \cdot, \cdot \rangle$).

Furthermore, the following algebraic inequalities also hold:

Let M_1 and M_2 be any two real $n \times n$ commuting symmetric matrices. Then

(i) the eigenvalues $\lambda_i(M_1 + M_2)$, $i = 1, \dots, n$, of the sum of matrices M_1 and M_2 are all real and satisfy

$$\begin{aligned} \min\nolimits_{1\leq j\leq n} \lambda_j(M_1) + \min\nolimits_{1\leq k\leq n} \lambda_k(M_2) &\leq \lambda_i(M_1 + M_2) \\ &\leq \max\nolimits_{1\leq j\leq n} \lambda_j(M_1) + \max\nolimits_{1\leq k\leq n} \lambda_k(M_2) \end{aligned} \tag{1.6}$$

(ii) the eigenvalues $\lambda_i(M_1M_2)$, $i = 1, \dots, n$, of the product of matrices M_1 and M_2 are all real and satisfy

$$\min_{1\leq j,k\leq n} \{\lambda_j(M_1)\lambda_k(M_2)\} \leq \lambda_i(M_1M_2) \leq \max_{1\leq j,k\leq n} \{\lambda_j(M_1)\lambda_k(M_2)\}. \tag{1.7}$$

Our results complement existing results in the literature and also extend some known works (Afuwape et al. [1], Ezeilo and Nkashama [5], Ezeilo and Omari [6] and Ezeilo and Onyia [7]) involving third order differential systems to the corresponding fourth order.

2 Preliminary analysis and main results

Let $\boldsymbol{\mathcal{L}}\colon \mathrm{dom}\boldsymbol{\mathcal{L}} \subset W_T^{4,1} \to L^1$ be the linear differential operator defined by

$$\boldsymbol{\mathcal{L}}X := X^{(IV)} + AX''' + BX'' + CX' + DX. \tag{2.1}$$

By substituting $X(t) = I_n \exp(ik\omega t)$, with $I_n \in \mathbb{R}^n$ a unit vector, $i = \sqrt{-1}$, $\omega = \frac{2\pi}{T}$, $k \in \mathbb{N}$, we have

$$\boldsymbol{\mathcal{L}}_1 X \equiv \exp(ik\omega t)\left[k^2\omega^2(k^2\omega^2 I - B) + D + ik\omega(C - k^2\omega^2 A)\right], \tag{2.2}$$

where I is the $n \times n$ identity matrix.

If we set $\boldsymbol{\mathcal{L}}X \equiv 0$, then we have two situations, namely, the resonance case when $\ker \boldsymbol{\mathcal{L}} \neq \{0\}$, and the non-resonance case when $\ker \boldsymbol{\mathcal{L}} = \{0\}$.

The nonresonance situation corresponds to, amongst several possibilities, either (or both) of the following conditions holding:

$$\lambda_i(A^{-1}C) \neq k_i^2\omega^2, \quad \text{with } k_i = 0, 1, 2, \dots \tag{2.3}$$

or

$$\lambda_i(D) \neq k^2\omega^2(k^2\omega^2 - \lambda_i(B)), \quad \text{with } k = 1, 2, \dots, \ i = 1, \dots, n. \tag{2.4}$$

Condition (2.4) is also equivalent to

$$\lambda_i(D) \neq \frac{1}{4}\lambda_i^2(B)), \quad i = 1, \dots, n \tag{2.5}$$

or in the special case that $\lambda_i(B)) > 0$, but $\frac{1}{2}\lambda_i(B)) \neq \text{integer}^2\omega^2$, that is, $k^2\omega^2 < \frac{1}{2}\lambda_i(B)) < (k+1)^2\omega^2$, with $k = 1, 2, \dots$ (2.5) can be relaxed to

$$\lambda_i(D) \neq \frac{1}{4}\lambda_i^2(B)) - \eta^2, \quad i = 1, \dots, n, \tag{2.6}$$

where

$$\eta := \min\left\{\frac{1}{2}\lambda_i(B)) - k^2\omega^2,\ (k+1)^2\omega^2 - \frac{1}{2}\lambda_i(B))\right\} > 0,$$

with $k = 1, 2, \dots,\ i = 1, \dots, n$.

In this paper, we shall be concerned with the particular case of condition (2.3). Results relating to generalisations of conditions (2.5) and (2.6) will follow in a subsequent paper.

We deduce therefore that the homogeneous T-periodic boundary value problem $\mathcal{L}X = 0$ subject to (2.3) has no non-trivial solution. This in turn implies that the inhomogeneous T-periodic boundary value problem $\mathcal{L}X = P(t)$ subject to (2.3) has a solution for every $P \in L^1(0,T)$.

Our main results are as follows:

Theorem 2.1 *Let A and D be nonsingular matrices and suppose that $\mathcal{G}$ satisfies*

$$k^2\omega^2 + \alpha_1(\|X\|) \leq \frac{\langle A^{-1}\mathcal{G}(X), X\rangle}{\|X\|^2} \leq (k+1)^2\omega^2 - \alpha_2(\|X\|) \tag{2.7}$$

uniformly in $X \in \mathbb{R}^n$ with $\|X\| \geq r > 0$, and a.e. $[0,T]$, where $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, and $\alpha_i \colon \mathbb{R}_+^n \to \mathbb{R}$, $i = 1, 2$ are two functions which are such that

$$\lim_{\|X\| \to +\infty} \|X\|\alpha_i(\|X\|) = +\infty. \tag{2.8}$$

Then system (1.2)–(1.4) has at least one solution, for every $P \in L^1([0,T], \mathbb{R}^n)$ and all arbitrary matrix B.

Theorem 2.2 *Let A and D be nonsingular matrices and suppose that G satisfies*

$$k^2\omega^2 + \beta_1(\|X'\|) \leq \frac{\langle A^{-1}G(t,X'), X'\rangle}{\|X'\|^2} \leq (k+1)^2\omega^2 - \beta_2(\|X'\|) \tag{2.9}$$

uniformly in $X' \in \mathbb{R}^n$ *with* $\|X'\| \geq r > 0$, *and a.e.* $[0,T]$, *where* $k \in \mathbb{N}$, $\omega = \frac{2\pi}{T}$, *and* $\beta_i \colon \mathbb{R}^n_+ \to \mathbb{R}$, $i = 1, 2$ *are two functions which are such that*

$$\lim_{\|X'\| \to +\infty} \|X'\| \, \beta_i(\|X'\|) = +\infty. \tag{2.10}$$

Then system (1.3)–(1.4) *has at least one solution, for every* $P \in L^1([0,T], \mathbb{R}^n)$ *and all arbitrary matrix* B.

Let ν and β be constants defined by

$$\nu = \frac{1}{2}\left(k^2\omega^2 + (k+1)^2\omega^2\right), \qquad \beta = \frac{1}{2}\left((k+1)^2\omega^2 - k^2\omega^2\right).$$

Since $\nu \neq \text{integer}^2\omega^2$, the constant coefficient differential system

$$W^{(IV)} + AW^{'''} + BW^{''} + \nu AW' + DW = 0 \tag{2.11}$$

subject to the periodic boundary conditions (1.4) has only the trivial solution $W \equiv 0$. Consequently, for each $P \in L^1([0,T], \mathbb{R}^n)$, there exists exactly one function $W = \boldsymbol{\mathcal{K}} P \in W^{4,1}_T([0,T], \mathbb{R}^n)$, satisfying (1.4) and

$$W^{(IV)} + AW^{'''} + BW^{''} + \nu AW' + DW = P(\cdot) \tag{2.12}$$

by the Fredholm alternative, where $\boldsymbol{\mathcal{K}} \colon dom\, \boldsymbol{\mathcal{K}} \subset L^1 \to W^{4,1}_T$.

For ease of evaluation, we make the change of variable $Z = X - W$, with W given by (2.12), then from (1.2), (1.3) and (1.4) we obtain respectively

$$Z^{(IV)} + AZ''' + BZ'' + \frac{d}{dt}\Big(\mathcal{G}(Z+W) - \nu AW\Big) + DZ = 0 \tag{2.13}$$

and

$$Z^{(IV)} + AZ''' + BZ'' + G(t, Z' + W') - \nu AW' + DZ = 0 \tag{2.14}$$

subject to

$$\mathcal{D}^{(r)}Z(0) = \mathcal{D}^{(r)}Z(T), \quad \left(\mathcal{D} = \frac{d}{dt}\right), \quad r = 0,1,2,3. \tag{2.15}$$

For $\lambda \in [0,1]$, we shall embed (2.13) and (2.14) in the parameter (λ)-dependent systems

$$Z^{(IV)} + AZ''' + BZ'' + (1-\lambda)\nu AZ' + \lambda\frac{d}{dt}\Big(\mathcal{G}(Z+W) - \nu AW\Big) + DZ = 0 \tag{2.16}$$

and

$$Z^{(IV)} + AZ''' + BZ'' + (1-\lambda)\nu AZ^{'} + \lambda\big(G(t, Z'+W') - \nu AW'\big) + DZ = 0 \tag{2.17}$$

respectively.

Then observe that at $\lambda = 0$, both equations reduce to the constant coefficient differential system

$$Z^{(IV)} + AZ^{'''} + BZ^{''} + \nu AZ' + DZ = 0 \tag{2.18}$$

which subject to (2.15) has only the trivial solution $W \equiv 0$ as shown earlier. Similarly at $\lambda = 1$, the equations reduce to their original forms (1.2) and (1.3) respectively.

Thus the existence of solutions is then established by standard Leray–Schauder techniques by showing that there exists a constant $\Delta > 0$ independent of $\lambda \in [0, 1]$ such that

$$\|Z^{(r)}\|_C \leq \Delta, \quad \text{where } r = 0, 1, 2, 3 \tag{2.19}$$

for every possible solution of (2.16) and (2.17) respectively.

3 Proof of our main results

Proof of Theorem 2.1 In order to apply conditions (2.7) and (2.8), we shall re-write (2.16) as

$$Z''' + AZ'' + BZ' + (1-\lambda)\nu AZ + \lambda\big(\mathcal{G}(Z+W) - \nu AW\big) = -D\int_0^t Z(\tau)\, d\tau. \tag{3.1}$$

Then if we set $Z_1 = Z$, $Z_2 = Z_1'$, $Z_3 = Z_2'$, (3.1) reduces to the system

$$\left.\begin{aligned} &Z_1' = Z_2, \quad Z_2' = Z_3 \\ &Z_3' = Z_4 - AZ_3 - BZ_2 - \nu AZ_1 - \lambda\big(\mathcal{G}(Z_1 + W) - A\nu W - \nu AZ_1\big) \\ &Z_4' = -DZ_1 \end{aligned}\right\} \tag{3.2}$$

with the corresponding periodic boundary conditions

$$Z_r(0) = Z_r(T) \qquad r = 1, 2, 3, 4. \tag{3.3}$$

With this representation, the defining equation for Z_3' assumes prominence in our study, which we now write in the form

$$Z_3' = Z_4 - AZ_3 - BZ_2 - \widetilde{\mathcal{G}}_\lambda(t, Z_1) \tag{3.4}$$

where $\widetilde{\mathcal{G}}_\lambda(t, Z_1) = (1-\lambda)\nu AZ_1 + \lambda A\widetilde{\mathcal{G}}(t, Z_1)$, with $\widetilde{\mathcal{G}}(t, Z_1) = A^{-1}\mathcal{G}(Z_1 + W) - \nu W$.

Indeed with notation, we obtain by hypotheses (2.7) and (2.8) the estimate

$$\begin{aligned} \|\widetilde{\mathcal{G}}(t, Z_1) - \nu Z_1\| &= \|A^{-1}\mathcal{G}(Z_1 + W) - \nu W - \nu Z_1)\| \\ &\leq (\|Z_1\| + \|W\|_\infty)\left(\frac{\langle A^{-1}\mathcal{G}(Z_1), Z_1\rangle}{\|Z_1\|^2} - \nu\right) \\ &\leq (\|Z_1\| + \|W\|_\infty)(\beta - \alpha_2(\|Z_1\|)) \leq \beta\|Z_1\| - k_1, \end{aligned} \tag{3.5}$$

uniformly in $Z_1 \in \mathbb{R}^n$ and $t \in [0,T]$ with $\|Z_1\| \geq r_1$, for some constant $k_1 > 0$ and $r_1 > 0$ depending on k_1 and $\|W\|_\infty$.

This implies that for a suitable constant $k_2 > 0$,

$$\|\widetilde{\mathcal{G}}(t, Z_1) - \nu Z_1\|^2 \leq \beta^2 \|Z_1\|^2 - 2\beta k_1 \|Z_1\| + k_2 \tag{3.6}$$

for all $Z_1 \in \mathbb{R}^n$ and $t \in [0,T]$.

Now, proceeding as in [12], we multiply both sides of (3.4) scalarly by Z_3 and integrate over $[0,T]$ yielding

$$\int_0^T \big(\langle Z_3', Z_3\rangle - \langle Z_4, Z_3\rangle + \langle AZ_3, Z_3\rangle + \langle BZ_2, Z_3\rangle + \langle \widetilde{\mathcal{G}}_\lambda(t, Z_1), Z_3\rangle\big)\, dt = 0. \tag{3.7}$$

It is easily checked that $\int_0^T \langle Z_3', Z_3\rangle dt = 0$, $\int_0^T \langle Z_4, Z_3\rangle dt = 0$, $\int_0^T \langle BZ_2, Z_3\rangle dt = 0$, so that (3.7) reduces to

$$\int_0^T \big(\langle Z_3, Z_3\rangle + \langle A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1), Z_3\rangle\big)\, dt = 0. \tag{3.8}$$

The integrand in (3.8) can be reset as

$$\begin{aligned}
&\langle Z_3 + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1), Z_3\rangle = \langle Z_3 + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1), Z_3 + \nu Z_1 - \nu Z_1\rangle \\
&= \langle Z_3 + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1), Z_3 + \nu Z_1\rangle + \langle Z_3 + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1), -\nu Z_1\rangle \\
&= \frac{1}{2}\|Z_3 + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1)\|^2 + \frac{1}{2}\|Z_3 + \nu Z_1\|^2 - \frac{1}{2}\|A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1) - \nu Z_1\|^2 \\
&\quad - \nu\big(\langle Z_3, Z_1\rangle + \langle A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1), Z_3\rangle\big).
\end{aligned} \tag{3.9}$$

However, if we multiply both sides of (3.4) scalarly by Z_1 and integrate over $[0,T]$, we obtain

$$\int_0^T \big(\langle Z_3', Z_1\rangle - \langle Z_4, Z_1\rangle + \langle AZ_3, Z_1\rangle + \langle BZ_2, Z_1\rangle + \langle \widetilde{\mathcal{G}}_\lambda(t, Z_1), Z_1\rangle\big)\, dt = 0. \tag{3.10}$$

Again noting that $\int_0^T \langle Z_3', Z_1\rangle\, dt = 0$, $\int_0^T \langle Z_4, Z_1\rangle\, dt = 0$, $\int_0^T \langle BZ_2, Z_1\rangle\, dt = 0$, we obtain

$$\int_0^T \big(\langle Z_3, Z_1\rangle + \langle A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1), Z_3\rangle\big)\, dt = 0. \tag{3.11}$$

Thus the integral over $[0,T]$ of the last term in (3.9) vanishes leading from (3.8) to the identity

$$\begin{aligned}
&\int_0^T \|Z_3 + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1)\|^2 dt + \int_0^T \|Z_3 + \nu Z_1\|^2 dt \\
&\quad = \int_0^T \|A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1) - \nu Z_1\|^2 dt = \lambda^2 \int_0^T \|\widetilde{\mathcal{G}}(t, Z_1) - \nu Z_1\|^2 dt.
\end{aligned} \tag{3.12}$$

Using the inequality

$$\int_0^T \|Z_3 + \nu Z_1\|^2 dt \geq \beta^2 \int_0^T \|Z_1\|^2 dt \tag{3.13}$$

which is derived from Lemma 3.1 of [2] by setting $\nu := \|A^{-1} J_{\mathcal{G}}\|$, where $J_{\mathcal{G}}$ is the Jacobian matrix of $\mathcal{G}$, and $Z_3 = Z_1''$, and also by (3.6), (3.12) becomes

$$\begin{aligned}\beta^2 \int_0^T \|Z_1\|^2 dt + \int_0^T \|Z_1'' + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1)\|^2 dt \\ \leq \beta^2 \int_0^T \|Z_1\|^2 dt - 2\beta k_1 \int_0^T \|Z_1\| + k_2 T.\end{aligned} \tag{3.14}$$

This yields the estmates

$$\left(\int_0^T \|Z_1'' + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1)\|^2 \, dt\right)^{\frac{1}{2}} \leq k_3 := k_2 T \tag{3.15}$$

and

$$\int_0^T \|Z_1\| \, dt \leq k_4 := \frac{k_2 T}{2\beta k_1}. \tag{3.16}$$

Clearly by definition of $\widetilde{\mathcal{G}}_\lambda(t, Z_1)$ in (3.4), there exist constants $k_5 > 0$, $k_6 > 0$ such that

$$\|\widetilde{\mathcal{G}}_\lambda(t, Z_1)\| \leq k_5 \|Z_1\| + k_6 \tag{3.17}$$

so that by (3.16), we obtain for some constant $k_7 > 0$

$$\int_0^T \|\widetilde{\mathcal{G}}_\lambda(t, Z_1)\| dt \leq k_7. \tag{3.18}$$

Hence, we deduce from (3.15) that

$$\int_0^T \|Z_1'' + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1)\| \, dt \leq \left(T \int_0^T \|Z_1'' + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1)\|^2 \, dt\right)^{\frac{1}{2}} \leq k_8 := k_3 T^{\frac{1}{2}} \tag{3.19}$$

so that by (3.18), we obtain

$$\begin{aligned}\int_0^T \|Z_1''\| \, dt \leq \int_0^T \|Z_1'' + A^{-1}\widetilde{\mathcal{G}}_\lambda(t, Z_1)\| \, dt + \|A\|^{-1} \int_0^T \|\widetilde{\mathcal{G}}_\lambda(t, Z_1)\| \, dt \\ \leq k_9 := k_8 + \delta_A^{-1} k_7.\end{aligned} \tag{3.20}$$

Thus, applying (3.20) and the fact that $Z_1(0) = Z_1(T)$, we have

$$\|Z_1'\|_C \leq T^{-1} \left\| \int_0^T Z_1'(t) \, dt \right\| + \int_0^T \|Z_1''\| \, dt \leq k_9. \tag{3.21}$$

This in turn yields, by (3.16)

$$\|Z_1\|_C \leq T^{-1}\left\| \int_0^T Z_1(t)\,dt \right\| + \int_0^T \|Z_1'\|\,dt \leq k_{10} := k_4 T^{-1} + k_9 T \tag{3.22}$$

proving the case for $r = 0, 1$ in (2.19).

Next, we integrate both sides of (3.4) over $[0, T]$ using (3.3) to obtain

$$0 = \int_0^T Z_4 dt - \int_0^T \widetilde{\mathcal{G}}_\lambda(t, Z_1) dt \tag{3.23}$$

yielding, by (3.18)

$$\left\| \int_0^T Z_4 dt \right\| \leq k_7. \tag{3.24}$$

Also, since $Z_4' = -DZ_1$, we obtain

$$\begin{aligned} \|Z_4\|_C &\leq T^{-1}\left\| \int_0^T Z_4(t)\,dt \right\| + \int_0^T \|Z_4'\|\,dt \\ &\leq T^{-1}\left\| \int_0^T Z_4(t)\,dt \right\| + \|D\| \int_0^T \|Z_1\|\,dt \\ &\leq k_{11} := k_7 T^{-1} + \Delta_D k_4 T. \end{aligned} \tag{3.25}$$

This estimate (3.25) obtained for Z_4 will enable us to compute estimates for the outstanding cases $r = 2, 3$ in (2.19), and for these we reset (3.4) and (2.16) respectively as

$$Z_3' + AZ_3 = U, \quad \text{where } U = Z_4 - BZ_2 - \widetilde{\mathcal{G}}_\lambda(t, Z_1) \tag{3.26}$$

and

$$Z_1^{(IV)} + AZ_1''' = V, \text{ where } V = -(BZ_1'' + (1-\lambda)\nu AZ_1' + \lambda A \frac{d}{dt}\widetilde{\mathcal{G}}(t, Z_1) + DZ_1). \tag{3.27}$$

It follows from estimates already established for $\|Z_1\|_C$ and $\|Z_1'\|_C$ that for some constant $k_{12} > 0$

$$\|U\|_C \leq k_{12}. \tag{3.28}$$

Thus the linear equation (3.26) has solution $Z_3 = Z_1''$ which satisfies

$$\|Z_1''\|_C \leq k_{13} := k_{12} e^{\Delta_A T} T. \tag{3.29}$$

Similarly, we deduce from the estimates so far established for $\|Z_1\|_C$, $\|Z_1'\|_C$ and $\|Z_1''\|_C$ that for some constant $k_{14} > 0$

$$\|V\|_C \leq k_{14}. \tag{3.30}$$

Then multiplying (3.27) scalarly by $Z_1^{(IV)}$ and integrating over $[0,T]$ yields, in view of (3.27)

$$\int_0^T \|Z_1^{(IV)}\|^2\,dt \le \|V\|_C \int_0^T \|Z_1^{(IV)}\|\,dt \le k_{14}\Big(T\int_0^T \|Z_1^{(IV)}\|^2\,dt\Big)^{\frac{1}{2}} \tag{3.31}$$

from which we obtain

$$\Big(\int_0^T \|Z_1^{(IV)}\|^2\,dt\Big)^{\frac{1}{2}} \le k_{14}T^{\frac{1}{2}} \tag{3.32}$$

implying in turn that for some constant $k_{15} > 0$

$$\|Z_1'''\|_C \le k_{15} := k_{14}T \tag{3.33}$$

proving the cases for $r = 2, 3$ in (2.19), and concluding the proof of the theorem. □

Proof of Theorem 2.2 We shall proceed as in the proof of the preceding theorem, but here we shall deal with the given fourth order system (1.3) directly. Accordingly, effecting the usual transformation $Z = X - W$, we shall be concerned with the parameter (λ)-dependent system (2.17) which we now re-write as

$$A^{-1}Z^{(IV)} + Z''' + A^{-1}BZ'' + A^{-1}\widetilde{G}_\lambda(t, Z') + A^{-1}DZ = 0 \tag{3.34}$$

where

$$\widetilde{G}_\lambda(t, Z') = (1-\lambda)\nu AZ' + \lambda A\widetilde{G}(t, Z'),$$

with $\widetilde{G}(t, Z') = A^{-1}G(t, Z' + W') - \nu W'$.

It suffices to show that every possible solution $Z \in C^4([0,T], \mathbb{R}^n)$ of (3.34)–(2.15) satisfies (2.19) for all $\lambda \in [0,1]$.

Now, multiplying both sides of (3.34) scalarly by $Z''' + \nu Z'$ and integrating over $[0,T]$ using (2.15), observing that

$$\int_0^T \langle A^{-1}Z^{(IV)} + A^{-1}BZ'' + A^{-1}DZ, Z''' + \nu Z'\rangle\,dt = 0,$$

we obtain

$$\int_0^T \langle Z''' + A^{-1}\widetilde{G}_\lambda(t, Z'), Z''' + \nu Z'\rangle\,dt = 0. \tag{3.35}$$

As in the preceding case, it is easily verified that (3.35) can be written as

$$\begin{aligned}\int_0^T &\|Z''' + \nu Z'\|^2 dt + \int_0^T \|Z''' + A^{-1}\widetilde{G}_\lambda(t, Z')\|^2 dt \\ &= \int_0^T \|A^{-1}\widetilde{G}_\lambda(t, Z') - \nu Z'\|^2 dt = \lambda^2 \int_0^T \|\widetilde{G}(t, Z') - \nu Z'\|^2 dt.\end{aligned} \tag{3.36}$$

Again, applying hypotheses (2.9) and (2.10), we obtain as in (3.5) and (3.6) the estimate

$$\|\widetilde{G}(t,Z') - \nu Z'\|^2 \le \beta^2 \|Z'\|^2 - 2\beta k_{16}\|Z'\| + k_{17} \tag{3.37}$$

uniformly in $Z' \in \mathbb{R}^n$ and a.e. $t \in [0,T]$ with $\|Z'\| \ge r_1$, for some constants $k_{16} > 0$, $k_{17} > 0$, and $r_1 > 0$ depending on k_{16} and $\|W'\|_\infty$.

Hence using the inequality

$$\int_0^T \|Z''' + \nu Z'\|^2 dt \ge \beta^2 \int_0^T \|Z'\|^2 dt \tag{3.38}$$

which has been established in Lemma 3.2 of [11], (3.36) when combined with (3.37) becomes

$$\begin{aligned}\beta^2 \int_0^T \|Z'\|^2 dt + \int_0^T \|Z''' + A^{-1}\widetilde{G}_\lambda(t,Z')\|^2 dt \\ \le \beta^2 \int_0^T \|Z'\|^2 dt - 2\beta k_{16} \int_0^T \|Z'\| + k_{17}T\end{aligned} \tag{3.39}$$

yielding the estmates

$$\left(\int_0^T \|Z''' + A^{-1}\widetilde{G}_\lambda(t,Z')\|^2 \, dt\right)^{\frac{1}{2}} \le k_{18} := k_{17}T \tag{3.40}$$

and

$$\int_0^T \|Z'\| dt \le k_{19} := \frac{k_{17}T}{2\beta k_{16}}. \tag{3.41}$$

Moreover, by definition of $\widetilde{G}_\lambda(t,Z')$ in (3.34), there exist constants $k_{20} > 0$, $k_{21} > 0$ such that

$$\|\widetilde{G}_\lambda(t,Z')\| \le k_{20}\|Z'\| + k_{21} \tag{3.42}$$

so that by (3.41), we obtain for some constant $k_{22} > 0$

$$\int_0^T \|\widetilde{G}_\lambda(t,Z')\| \, dt \le k_{22}. \tag{3.43}$$

Applying the same arguments as in preceding proof after (3.18), we obtain

$$\int_0^T \|Z'''\| \, dt \le k_{23} \tag{3.44}$$

which yields by (3.44), since $Z'(0) = Z'(T)$

$$\|Z''\|_C \le k_{24} \tag{3.45}$$

and in turn implying, since $Z(0) = Z(T)$

$$\|Z'\|_C \le k_{25} \tag{3.46}$$

for some constant $k_{25} > 0$.

Again integrating (3.34) over $[0,T]$ using (2.15), and applying (3.43) yields

$$\int_0^T \|Z\|\,dt \leq k_{26} := k_{22}\delta_D^{-1} \tag{3.47}$$

leading to

$$\|Z\|_C \leq k_{27} := k_{26}T^{-1} + k_{25}T \tag{3.48}$$

for some constant $k_{27} > 0$.

Finally, combining (3.45), (3.46) and (3.48) yields

$$\|Z'''\|_C \leq k_{28} \tag{3.49}$$

using exactly the same approach as in the preceding proof. □

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