Abd El-Mohsen Badawy; Daniela Guffová; Miroslav Haviar
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Triple Constructions of Decomposable *MS*-Algebras

Abd El-Mohsen BADAWY [1], Daniela GUFFOVÁ [2a], Miroslav HAVIAR [2b]

[1] *Department of Mathematics, Faculty of Science, Tanta University, Egypt*
e-mail: mohsen_badawy@ymail.com

[2] *Department of Mathematics, Matej Bel University*
Tajovského 40, 974 01 Banská Bystrica, Slovakia
[a] *e-mail: Daniela.Guffova@umb.sk*
[b] *e-mail: Miroslav.Haviar@umb.sk*

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Abstract

A simple triple construction of principal MS-algebras is given which is parallel to the construction of principal p-algebras from principal triples presented by the third author in [7]. It is shown that there exists a one-to-one correspondence between principal MS-algebras and principal MS-triples. Further, a triple construction of a class of decomposable MS-algebras that includes the class of principal MS-algebras is given. It is a modification of the quadruple constructions by T. S. Blyth and J. C. Varlet [1], [2] and T. Katriňák and K. Mikula [10]; instead of Kleene algebras and the filters $L^\vee$ used in their quadruples, de Morgan algebras and the filters $D(L)$, respectively, are used in our triples.

Key words: principal MS-algebra, principal MS-triple, decomposable MS-algebra, decomposable MS-triple, de Morgan algebra, filter

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1 Introduction

In 1980 T. S. Blyth and J. C. Varlet presented the first triple construction of MS-algebras from the subvariety K_2 by means of Kleene algebras and distributive lattices [3]. In [4] this construction was improved via the language of quadruples. It was independently done by T. Katriňák and K. Mikula (in an unpublished paper) who then compared both approaches in [10].

Later, the third author [6] proved that there exists a one-to-one correspondence between the class of locally bounded K_2-algebras and the class of decomposable K_2-quadruples. In his work he assumed that the filter $L^\vee$ of an MS-algebra L was principal which allowed him to simplify the previous constructions and work with pairs of elements only. A year later in [7] he presented a similar triple construction of principal p-algebras.

In Section 3 of this paper we present a simple triple construction of principal MS-algebras similar to that of [7] and we show that there is a one-to-one correspondence between principal MS-algebras and so-called principal MS-triples.

We also introduce a class of so-called decomposable MS-algebras containing the class of principal MS-algebras and we present a triple construction of decomposable MS-algebras generalising that in Section 3. It is a modification of the quadruple constructions by T. S. Blyth and J. C. Varlet [3], [4] and T. Katriňák and K. Mikula [10].

Firstly, we use de Morgan algebras instead of Kleene algebras in our triples and secondly, the filter chosen for our construction is different. Instead of the filter $L^\vee$ used in the constructions in [3], [4], [10] and [6], in our constructions in Sections 3 and 4 we consider the set $D(L)$ of dense elements of an MS-algebra L. As $D(L)$ is a filter for any MS-algebra L we do not need a quadruple to construct an MS-algebra. It is sufficient to use the triple construction, because we do not need to use the modal operator used in the constructions by T. S. Blyth and J. C. Varlet [3], [4] and T. Katriňák and K. Mikula [10] or the congruence used by the third author [6].

2 Preliminaries

An *MS-algebra* is an algebra $(L; \vee, \wedge, ^0, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and ^{0} is a unary operation such that for all $x, y \in L$

(1) $x \leq x^{00}$;

(2) $(x \wedge y)^0 = x^0 \vee y^0$;

(3) $1^0 = 0$.

The class of all MS-algebras is equational. A *de Morgan algebra* is an MS-algebra satisfying the additional identity

(4) $x = x^{00}$.

A de Morgan algebra satisfying the identity

(5) $(x \wedge x^0) \vee y \vee y^0 = y \vee y^0$

is called a *Kleene algebra*.

Let L be an MS-algebra. Then

(i) $L^{00} = \{x \in L \mid x = x^{00}\}$ is a de Morgan algebra and a subalgebra of L (as $x^{00} \vee y^{00} = (x \vee y)^{00}$ and $x^{00} \wedge y^{00} = (x \wedge y)^{00}$);

(ii) $D(L) = \{x \in L \mid x^0 = 0\}$ is a filter (of dense elements) of L.

The following definition mimics the one in [7].

Definition 2.1 An *MS*-algebra $(L; \vee, \wedge, ^0, 0, 1)$ is called a *principal MS-algebra* if it satisfies the following conditions:

(i) The filter $D(L)$ is principal, i.e. there exists an element $d_L \in L$ such that $D(L) = [d_L)$;

(ii) $x = x^{00} \wedge (x \vee d_L)$ for any $x \in L$.

Now we introduce a new concept of a decomposable *MS*-algebra generalising the concept of a principal *MS*-algebra.

Definition 2.2 An *MS*-algebra $(L; \vee, \wedge, ^0, 0, 1)$ will be called a *decomposable MS-algebra* if for every $x \in L$ there exists $d \in D(L)$ such that $x = x^{00} \wedge d$.

Let L be a principal *MS*-algebra with $D(L) = [d_L)$ and for $x \in L$ let $d := x \vee d_L$. Then $d \in [d_L)$ and Definition 2.1 gives us $x = x^{00} \wedge d$. Thus Definition 2.2 is satisfied for any principal *MS*-algebra.

3 Principal *MS*-algebras

In this section we give a construction of principal *MS*-algebras which works with pairs of elements only and is similar to the construction of principal p-algebras from [7].

Definition 3.1 An (abstract) *principal MS-triple* is (M, D, φ), where

(i) M is a de Morgan algebra;

(ii) D is a bounded distributive lattice;

(iii) φ is a $(0, 1)$-lattice homomorphism from M into D.

Theorem 3.2 *Let (M, D, φ) be a principal MS-triple. Then*

$$L = \{(x, y) \mid x \in M, y \in D, y \leq \varphi(x)\}$$

is a principal MS-algebra if we define

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$$
$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$$
$$(x, y)^0 = (x^0, \varphi(x^0))$$
$$1_L = (1_M, 1_D)$$
$$0_L = (0_M, 0_D).$$

Moreover, $L^{00} \cong M$ and $D(L) \cong D$.

Proof One can easily prove that L is a sublattice of $M \times D$. Obviously, $0_D = \varphi(0_M)$ and $1_D = \varphi(1_M)$. Hence, L is a bounded distributive lattice. Clearly,

$$(x, y) \wedge (x, y)^{00} = (x \wedge x^{00}, y \wedge \varphi(x^{00})) = (x, y),$$

so the identity (1) holds in L. We can verify the identities (2) and (3) similarly.

Now,

$$\begin{aligned} D(L) &= \{(x, y) \in L \mid (x, y)^0 = (0_M, 0_D)\} \\ &= \{(x, y) \in L \mid (x^0, \varphi(x^0)) = (0_M, 0_D)\} \\ &= \{(1_M, y) \mid y \in D\} \\ &\cong D. \end{aligned}$$

Evidently, an element $d_L = (1_M, 0_D)$ is the smallest dense element of L and the filter $D(L)$ is principal.

Also, for any $(x, y) \in L$,

$$\begin{aligned} (x, y)^{00} \wedge ((x, y) \vee (1_M, 0_D)) &= (x^{00}, \varphi(x^{00})) \wedge (x \vee 1_M, y \vee 0_D) \\ &= (x, \varphi(x)) \wedge (1_M, y) = (x, y). \end{aligned}$$

Hence L is a principal *MS*-algebra.

It remains to prove that $L^{00} \cong M$. We have

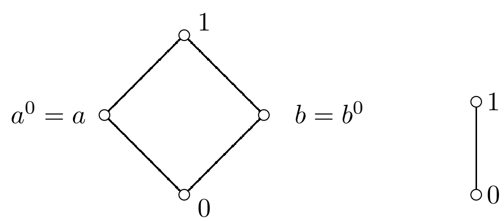
$$\begin{aligned} L^{00} &= \{(x, y) \in L \mid (x, y)^{00} = (x, y)\} \\ &= \{(x, y) \in L \mid (x^{00}, \varphi(x^{00})) = (x, y)\} \\ &= \{(x, y) \mid x \in M, y \in D, y = \varphi(x)\} \\ &= \{(x, \varphi(x)) \mid x \in M\}, \end{aligned}$$

which is obviously isomorphic to M. The proof is complete. □

We shall say that the principal *MS*-algebra L from Theorem 3.2 is *associated* with the principal *MS*-triple (M, D, φ) and the construction of L described in Theorem 3.2 will be called a *principal MS-construction.*

We illustrate *the principal MS-construction* on the following example.

Example 3.3 Let M be the four-element subdirectly irreducible de Morgan algebra and let D be the two-element lattice (see Fig. 1).



M D

Figure 1

Define a lattice homomorphism $\varphi\colon M \to D$ by the rule

$$\varphi(0) = \varphi(a) = 0, \qquad \varphi(b) = \varphi(1) = 1.$$

Then (M, D, φ) is a principal MS-triple and by the principal MS-construction we obtain a principal MS-algebra L such that

$$L = \{(0,0), (a,0), (b,0), (b,1), (1,0), (1,1)\}$$

and

$$(0,0)^0 = (1,1), (a,0)^0 = (a,0),$$
$$(b,0)^0 = (b,1)^0 = (b,1), (1,0)^0 = (1,1)^0 = (0,0).$$

The algebra L is represented in Figure 2. The shaded elements form a de Morgan algebra L^{00} which is obviously isomorphic to M. One can also observe that the filter $D(L)$ is isomorphic to the given lattice D. Moreover, the mapping $\varphi(L)\colon L^{00} \to D(L)$ defined by $\varphi(L)(x,y) = (x,y) \vee (1,0)$ is a $(0,1)$-lattice homomorphism. Hence the triple $(L^{00}, D(L), \varphi(L))$ is a principal MS-triple.

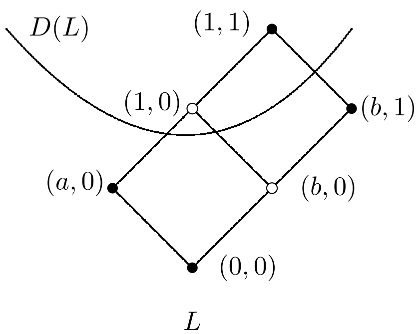


Figure 2

Let L be a principal MS-algebra and let d_L be the smallest dense element of L. Define a mapping $\varphi(L)\colon L^{00} \to D(L)$ by $\varphi(L)(a) = a \vee d_L$. It is obvious that $\varphi(L)$ is a $(0,1)$-lattice homomorphism.

We say that $(L^{00}, D(L), \varphi(L))$ is the principal MS-triple *associated* with L.

The following theorem states that every principal MS-algebra can be obtained by the principal MS-construction.

Theorem 3.4 *Let L be a principal MS-algebra. Let $(L^{00}, D(L), \varphi(L))$ be the principal MS-triple associated with L. Then the principal MS-algebra L_1 associated with $(L^{00}, D(L), \varphi(L))$ is isomorphic to L.*

Proof Let $D(L) = [d_L)$. We shall prove that the mapping $f\colon L \to L_1$ defined by

$$f(a) = (a^{00}, a \vee d_L)$$

is the desired isomorphism. It is obvious that $f(a) \in L_1$, as

$$a \vee d_L \leq \varphi(L)(a^{00}) = a^{00} \vee d_L.$$

It is easy to prove that f is a lattice homomorphism and that $f(0) = (0, d_L)$ and $f(1) = (1, 1)$. Moreover, we have

$$f(a^0) = (a^{000}, a^0 \vee d_L) = (a^0, \varphi(L)(a^0)) = f(a)^0,$$

so f is a homomorphism of MS-algebras.

Now we will prove the injectivity. Assume that $f(a_1) = f(a_2)$. Then we have $a_1^{00} = a_2^{00}$ and $a_1 \vee d_L = a_2 \vee d_L$ and we immediately obtain

$$a_1 = a_1^{00} \wedge (a_1 \vee d_L) = a_2^{00} \wedge (a_2 \vee d_L) = a_2.$$

To prove the surjectivity of f, let $(x, y) \in L_1$. Set $a = x \wedge y$. Using the facts that $x \in L^{00}$, $y \in D(L)$ and $y \leq \varphi(L)(x)$, we get

$$\begin{aligned} f(a) &= ((x \wedge y)^{00}, (x \wedge y) \vee d_L) \\ &= (x^{00} \wedge y^{00}, (x \vee d_L) \wedge (y \vee d_L)) \\ &= (x \wedge 1_L, (x \vee d_L) \wedge y) \\ &= (x, \varphi(L)(x) \wedge y) \\ &= (x, y). \end{aligned}$$

The proof is complete. □

Now we shall show that the principal MS-algebras are represented by the principal MS-triples uniquely.

Definition 3.5 An isomorphism of principal MS-triples (M, D, φ) and (M_1, D_1, φ_1) is a pair (f, g) where f is an isomorphism of M and M_1, g is an isomorphism of D and D_1 and the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "f"'] & D \arrow[d, "g"] \\ M_1 \arrow[r, "\varphi_1"] & D_1 \end{tikzcd}$$

is commutative.

Theorem 3.6 *Two principal MS-algebras are isomorphic if and only if their associated principal MS-triples are isomorphic.*

Proof Let $h\colon L_1 \to L_2$ be an isomorphism of MS-algebras. Then the pair of restrictions $h\restriction L_1^{00}$ and $h\restriction D(L_1)$ is the required isomorphism of their associated principal MS-triples.

Conversely, let (M_1, D_1, φ_1) and (M_2, D_2, φ_2) be the principal MS-triples associated to principal MS-algebras L_1 and L_2 and let

$$(f, g)\colon (M_1, D_1, \varphi_1) \to (M_2, D_2, \varphi_2)$$

be an isomorphism of principal MS-triples. Let us denote by L'_1 and L'_2 the principal MS-algebras associated to the principal MS-triples (M_1, D_1, φ_1) and (M_2, D_2, φ_2), respectively. Consider the mapping $h\colon L'_1 \to L'_2$ defined by the rule $h(a, x) = (f(a), g(x))$. It is clear that h is a $(0, 1)$-lattice isomorphism. Moreover, we have

$$\begin{aligned} h((a,x)^0) &= h(a^0, \varphi_1(a^0)) = (f(a^0), g(\varphi_1(a^0))) \\ &= (f(a^0), \varphi_2(f(a^0))) = (f(a)^0, \varphi_2(f(a)^0)) = (f(a), g(x))^0 = (h(a,x))^0. \end{aligned}$$

Hence h is an isomorphism of MS-algebras. □

The next theorem together with the previous two theorems show that there is a one-to-one correspondence between principal MS-algebras and principal MS-triples.

Theorem 3.7 *Let (M, D, φ) be a principal MS-triple and let L be its associated principal MS-algebra. Then*

$$(L^{00}, D(L), \varphi(L)) \cong (M, D, \varphi).$$

Proof By Theorem 3.2 the mappings $f\colon L^{00} \to M$ and $g\colon D(L) \to D$ such that $f(a, \varphi(a)) = a$ and $g(1_M, x) = x$ are isomorphisms. It remains to prove that the diagram

$$\begin{tikzcd} L^{00} \arrow[r, "\varphi(L)"] \arrow[d, "f"'] & D(L) \arrow[d, "g"] \\ M \arrow[r, "\varphi"] & D \end{tikzcd}$$

is commutative. Let $u \in L^{00}$. Then $u = (a, \varphi(a))$ for some $a \in M$ and we have

$$\begin{aligned} g(\varphi(L)(u)) &= g((a, \varphi(a)) \vee (1_M, 0_D)) \\ &= g(a \vee 1_M, \varphi(a) \vee 0_D) = g(1_M, \varphi(a)) = \varphi(a) = \varphi(f(a, \varphi(a))), \end{aligned}$$

as required. The proof is complete. □

Hence, here the situation is different from [6], where it was possible to construct an MS-algebra from the subvariety K_2 (of algebras abstracting Stone and Kleene algebras, cf. [6, p. 72] or [2]) from two non-isomorphic K_2-quadruples.

Example 3.8 Let K be the three-element subdirectly irreducible Kleene algebra and let D be the two-element lattice. Define two homomorphisms $\varphi_1, \varphi_2\colon K \to D$, by the rules

$$\varphi_1(0) = \varphi_1(a) = 0, \quad \varphi_1(1) = 1$$

and

$$\varphi_2(0) = 0, \quad \varphi_2(a) = \varphi_2(1) = 1$$

(see Figure 3).

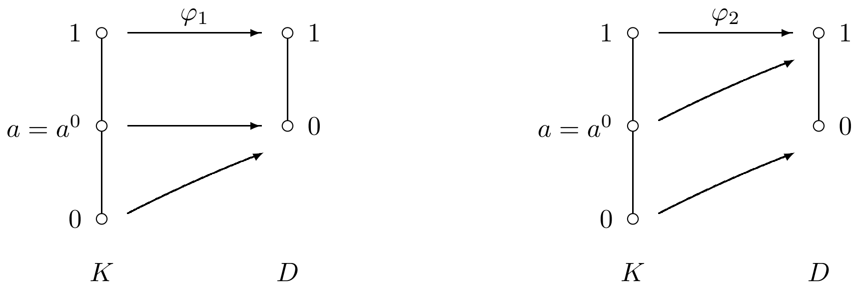


Figure 3

By the principal MS-constructions, from the principal MS-triples (K, D, φ_1) and (K, D, φ_2) we obtain the non-isomorphic principal MS-algebras L_1 resp. L_2 depicted in Figure 4.

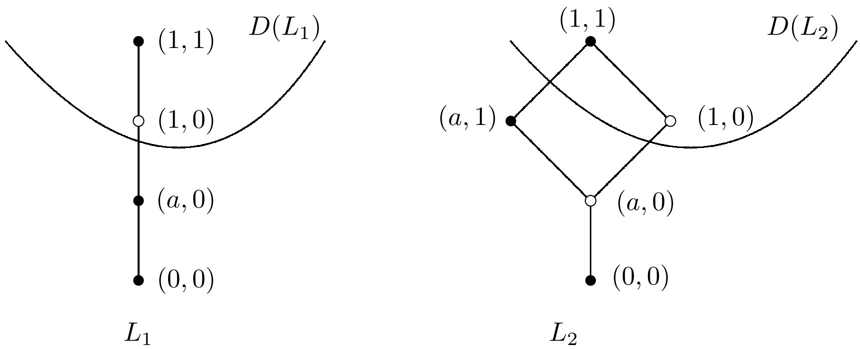


Figure 4

One can easily observe that $L_1^{00} \cong L_2^{00}$ (Kleene algebras L_1^{00}, L_2^{00} are shaded) and $D(L_1) \cong D(L_2)$, but $\varphi(L_1) \neq \varphi(L_2)$. So taking two different $(0,1)$-homomorphisms between a de Morgan algebra and a bounded distributive lattice can lead to obtaining two non-isomorphic principal MS-algebras by the principal MS-construction.

4 Decomposable MS-algebras

In this section we present a construction of decomposable MS-algebras. As the class of decomposable MS-algebras includes the class of principal MS-algebras, the construction given in this section generalises the one given in Theorem 3.2.

Our construction is similar to those by T. S. Blyth and J. C. Varlet [3], [4] and T. Katriňák and K. Mikula [10]. However, working with the filter $D(L)$

instead of the filter $L^{\vee} = \{x \vee x^0 \mid x \in L\}$, which they used, enables us to use the triple construction only. Also we use de Morgan algebras instead of Kleene algebras in our triples. Consequently, we construct decomposable MS-algebras not only from the subvariety K_2.

For a distributive lattice D we will use the notation $F(D)$ for the lattice of all filters of D ordered by inclusion and the notation $F_d(D)$ for the dual lattice of the lattice $F(D)$.

We consider the mapping $\varphi(L)\colon L^{00} \to F(D(L))$ defined by

$$\varphi(L)(a) = \{x \in D(L) \mid x \geq a^0\} = \big[a^0\big) \cap D(L), \quad a \in L^{00}.$$

Obviously, for a decomposable MS-algebra L the mapping $\varphi(L)$ defined above is a $(0,1)$-homomorphism from L^{00} into $F(D(L))$ and $\varphi(L)(a) \cap [y)$ is a principal filter of $D(L)$ for every $a \in L^{00}$ and for every $y \in D(L)$.

Definition 4.1 A *decomposable MS-triple* is (M, D, φ), where

(i) M is a de Morgan algebra;

(ii) D is a distributive lattice with 1;

(iii) φ is a $(0,1)$-lattice homomorphism from M into $F(D)$ such that for every element $a \in M$ and for every $y \in D$ there exists an element $t \in D$ with $\varphi(a) \cap [y) = [t)$.

In the following theorem we present a triple construction for decomposable MS-algebras.

Theorem 4.2 *Let (M, D, φ) be a decomposable MS-triple. Then*

$$L = \{(a, \varphi(a^0) \vee [x)) \mid a \in M, x \in D\}$$

is a decomposable MS-algebra if we define

$$(a, \varphi(a^0) \vee [x)) \vee (b, \varphi(b^0) \vee [y)) = (a \vee b, (\varphi(a^0) \vee [x)) \cap (\varphi(b^0) \vee [y))),$$
$$(a, \varphi(a^0) \vee [x)) \wedge (b, \varphi(b^0) \vee [y)) = (a \wedge b, (\varphi(a^0) \vee [x)) \vee (\varphi(b^0) \vee [y))),$$
$$(a, \varphi(a^0) \vee [x))^0 = (a^0, \varphi(a)),$$
$$1_L = (1, [1)),$$
$$0_L = (0, D).$$

Conversely, every decomposable MS-algebra L can be constructed in this way from its associated decomposable MS-triple $(L^{00}, D(L), \varphi(L))$, where $\varphi(L)(a) = \big[a^0\big) \cap D(L)$.

Proof Let $(a, \varphi(a^0) \vee [x)), (b, \varphi(b^0) \vee [y)) \in L$. As φ is a $(0,1)$-lattice homomorphism, we have

$$(a, \varphi(a^0) \vee [x)) \wedge (b, \varphi(b^0) \vee [y)) = (a \wedge b, \varphi((a \wedge b)^0) \vee [x \wedge y)),$$

and

$$\begin{aligned}(a,\varphi(a^0)\vee[x))\vee(b,\varphi(b^0)\vee[y)) &= (a\vee b,(\varphi(a^0)\vee[x))\cap(\varphi(b^0)\vee[y)))\\ &= (a\vee b,\varphi((a\vee b)^0)\vee[t)),\quad t\in D,\end{aligned}$$

because

$$\begin{aligned}&(\varphi(a^0)\vee[x))\cap(\varphi(b^0)\vee[y))\\ &=(\varphi(a^0)\cap\varphi(b^0))\vee(\varphi(a^0)\cap[y))\vee(\varphi(b^0)\cap[x))\vee([x)\cap[y))\\ &=\varphi((a\vee b)^0)\vee[t)\,,\quad t\in D,\end{aligned}$$

where $[t) = [q)\vee[p)\vee[x\vee y) = [q\wedge p\wedge(x\vee y))$ and $\varphi(a^0)\cap[y) = [q)$ and $\varphi(b^0)\cap[x) = [p)$, $p,q\in D$. This implies that L is a sublattice of $M\times F_d(D)$.

Now we shall prove that L is an MS-algebra. Clearly,

$$(a,\varphi(a^0)\vee[x))^{00} = (a^0,\varphi(a))^0 = (a,\varphi(a^0))\geq(a,\varphi(a^0)\vee[x)),$$

so the identity (1) holds in L. Moreover, we have

$$\begin{aligned}\big[(a,\varphi(a^0)\vee[x))\wedge(b,\varphi(b^0)\vee[y))\big]^0 &= (a\wedge b,\varphi((a\wedge b)^0)\vee[x\wedge y))^0\\ =((a\wedge b)^0,\varphi(a\wedge b)) &= (a^0\vee b^0,\varphi(a)\cap\varphi(b)) = (a^0,\varphi(a))\vee(b^0,\varphi(b))\\ &= (a,\varphi(a^0)\vee[x))^0\vee(b,\varphi(b^0)\vee[y))^0\end{aligned}$$

and $(1,[1))^0 = (0,D)$, thus the identities (2), (3) are satisfied in L.

It remains to prove that L is decomposable. For every $(a,\varphi(a^0)\vee[x))\in L$ we have

$$(a,\varphi(a^0)\vee[x)) = (a,\varphi(a^0))\wedge(1,[x)) = (a,\varphi(a^0)\vee[x))^{00}\wedge(1,[x)),$$

where $(1,[x))\in D(L)$. We have proved that L is a decomposable MS-algebra.

Conversely, let L be a decomposable MS-algebra. Then L^{00} is a de Morgan algebra and $D(L)$ is a filter of L which is indeed a distributive lattice with 1. Let us consider the mapping $\varphi(L)\colon L^{00}\to F(D(L))$ defined by

$$\varphi(L)(a) = \big[a^0\big)\cap D(L).$$

Obviously, $\varphi(L)$ is a $(0,1)$-homomorphism from L^{00} into $F(D(L))$ and $\varphi(L)(a)\cap[y)$ is a principal filter of $D(L)$ for every $a\in L^{00}$ and for every $y\in D(L)$. Hence $(L^{00},D(L),\varphi(L))$ is a decomposable MS-triple.

Now denote by L_1 the decomposable MS-algebra constructed from the decomposable MS-triple $(L^{00},D(L),\varphi(L))$ by the previous construction. Let us consider the mapping $\alpha\colon L\to L_1$ defined by $\alpha(x) = (x^{00},[x)\cap D(L))$. Since $x = x^{00}\wedge d$, we have

$$\varphi(L)(x^0)\vee[d) = (\big[x^{00}\big)\cap D(L))\vee[d) = \big[x^{00}\wedge d\big)\cap D(L) = [x)\cap D(L).$$

Now for every $(x^{00}, \varphi(L)(x^0) \vee [d)) \in L_1$ we get

$$(x^{00}, \varphi(L)(x^0) \vee [d)) = (x^{00}, [x) \cap D(L)) = \alpha(x),$$

so α is surjective.

To prove that α is injective, let $\alpha(x) = \alpha(y)$ for some $x, y \in L$. Then the equality $(x^{00}, [x) \cap D(L)) = (y^{00}, [y) \cap D(L))$ implies that $x^{00} = y^{00}$ and $[x) \cap D(L) = [y) \cap D(L)$. Since L is a decomposable MS-algebra, we have $x = x^{00} \wedge d_1$ and $y = y^{00} \wedge d_2$ for some $d_1, d_2 \in D(L)$. Then we obtain

$$\begin{aligned}
[x) \vee D(L) &= \big[x^{00} \wedge d_1\big) \vee D(L) \\
&= \big[x^{00}\big) \vee [d_1) \vee D(L) = \big[x^{00}\big) \vee D(L) \\
&= \big[y^{00}\big) \vee D(L) = \big[y^{00}\big) \vee [d_2) \vee D(L) \\
&= \big[y^{00} \wedge d_2\big) \vee D(L) = [y) \vee D(L).
\end{aligned}$$

By distributivity we get $([x) \cap D(L)) \vee [x) = ([y) \cap D(L)) \vee [y)$, which implies $x = y$, as required.

Finally, we have

$$\alpha(x)^0 = (x^{00}, [x) \cap D(L))^0 = (x^0, \big[x^0\big) \cap D(L)) = \alpha(x^0)$$

and also

$$\begin{aligned}
\alpha(x \wedge y) &= ((x \wedge y)^{00}, [x \wedge y) \cap D(L)) \\
&= (x^{00} \wedge y^{00}, ([x) \vee [y)) \cap D(L)) \\
&= (x^{00} \wedge y^{00}, ([x) \cap D(L)) \vee ([y) \cap D(L))) \\
&= (x^{00}, [x) \cap D(L)) \wedge (y^{00}, [y) \cap D(L)) = \alpha(x) \wedge \alpha(y)
\end{aligned}$$

and

$$\begin{aligned}
\alpha(x \vee y) &= ((x \vee y)^{00}, [x \vee y) \cap D(L)) \\
&= (x^{00} \vee y^{00}, ([x) \cap [y)) \cap D(L) \\
&= (x^{00} \vee y^{00}, ([x) \cap D(L)) \cap ([y) \cap D(L))) \\
&= (x^{00}, [x) \cap D(L)) \vee (y^{00}, [y) \cap D(L)) = \alpha(x) \vee \alpha(y).
\end{aligned}$$

Hence α is the desired isomorphism. □

We shall say that the decomposable MS-algebra constructed in Theorem 4.2 is *associated* with the decomposable MS-triple (M, D, φ) and the construction of L described in Theorem 4.2 will be called a *decomposable MS-construction.*

Lemma 4.3 *Let L be a decomposable MS-algebra associated with the decomposable triple (M, D, φ). Then*

(i) $L^{00} = \{(a, \varphi(a^0)) \mid a \in M\}$;

(ii) $D(L) = \{(1, [x)) \mid x \in D\}$;

(iii) $D \cong D(L), M \cong L^{00}$.

Proof (i) As $(a, \varphi(a^0) \vee [x))^{00} = (a^0, \varphi(a))^0 = (a, \varphi(a^0))$ for every $a \in M$, we have $L^{00} = \{(a, \varphi(a^0)) \mid a \in M\}$.

(ii) For every $x \in D$ $(1, [x))^0 = (1, \varphi(1^0) \vee [x))^0 = (0, \varphi(1)) = (0, D)$ holds. Hence $D(L) = \{(1, [x)) \mid x \in D\}$.

(iii) It is easy to check that $\psi\colon a \mapsto (a, \varphi(a^0))$ and $\chi\colon d \mapsto (1, [d))$ are desired isomorphisms of M and L^{00}, and of D and $D(L)$, respectively. $\square$

Definition 4.4 An isomorphism of decomposable *MS*-triples (M, D, φ) and (M_1, D_1, φ_1) is a pair (α, β) where α is an isomorphism of M and M_1, β is an isomorphism of D and D_1 and the diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & F(D) \arrow[d, "F(\beta)"] \\ M_1 \arrow[r, "\varphi_1"] & F(D_1) \end{tikzcd}$$

commutes. ($F(\beta)$ is the isomorphism of $F(D)$ and $F(D_1)$ induced by β.)

Theorem 4.5 *Two decomposable MS-algebras are isomorphic if and only if their associated decomposable MS-triples are isomorphic.*

Proof Let L_1, L_2 be decomposable *MS*-algebras and let $\tau\colon L_1 \to L_2$ be an isomorphism. Let us consider the isomorphisms

$$\alpha\colon L_1^{00} \to L_2^{00} \quad \text{and} \quad F(\beta)\colon F(D(L_1)) \to F(D(L_2))$$

such that α is defined by $\alpha(x) = \tau(x)$ and $F(\beta)$ is defined by

$$F(\beta)(A) = \{\tau(a) \mid a \in A\}$$

for $A \in F(D(L_1))$. Then we have

$$\varphi(L_2)(\alpha(x)) = \varphi(L_2)(\tau(x)) = \big[(\tau(x))^0\big) \cap D(L_2)$$

and

$$\begin{aligned} F(\beta)(\varphi(L_1)(x)) &= F(\beta)(\big[x^0\big) \cap D(L_1)) \\ &= \{\tau(y) \mid y \in \big[x^0\big) \cap D(L_1)\} = \big[(\tau(x))^0\big) \cap D(L_2), \end{aligned}$$

for every $x \in L_1^{00}$. So (α, β) is an isomorphism of decomposable triples $(L_1^{00}, D(L_1), \varphi(L_1))$ and $(L_2^{00}, D(L_2), \varphi(L_2))$.

Conversely, assume that the triples $(L_1^{00}, D(L_1), \varphi(L_1))$ and $(L_2^{00}, D(L_2), \varphi(L_2))$ are isomorphic. Let us consider the mapping $g\colon L_1 \to L_2$ defined by

$$g(a, \varphi(L_1)(a^0) \vee [x)) = (\alpha(a), F(\beta)([a) \cap D(L_1)) \vee [\beta(x))).$$

Now let $(a, \varphi(L_1)(a^0) \vee [x)) = (b, \varphi(L_1)(b^0) \vee [y))$. Then we have $a = b$ and $\varphi(L_1)(a^0) \vee [x) = \varphi(L_1)(b^0) \vee [y)$ and we immediately get $\alpha(a) = \alpha(b)$ and $([a) \cap D(L_1)) \vee [x) = ([b) \cap D(L_1)) \vee [y)$. Using $F(\beta)$ we obtain

$$(\alpha(a), F(\beta)([a) \cap D(L_1)) \vee [\beta(x))) = (\alpha(b), F(\beta)([b) \cap D(L_1)) \vee [\beta(y))).$$

Thus g is well-defined. One can also verify that g is a lattice isomorphism. From

$$\begin{aligned} g((a, \varphi(L_1)(a^0) \vee [x))^0) &= g(a^0, \varphi(L_1)(a)) \\ &= (\alpha(a^0), \varphi(L_2)(\alpha(a))) \\ &= (\alpha(a), \varphi(L_2)(\alpha(a^0)) \vee [\beta(x)))^0 \\ &= (g(a, \varphi(L_1)(a^0) \vee [x)))^0 \end{aligned}$$

it follows that g is an MS-isomorphism and the proof is complete. □

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References

[1] Blyth, T., Varlet, J.: *On a common abstraction of de Morgan algebras and Stone algebras*. Proc. Roy. Soc. Edinburgh. **94A** (1983), 301–308.

[2] Blyth, T., Varlet, J.: *Subvarieties of the class of* MS-*algebras*. Proc. Roy. Soc. Edinburgh **95A** (1983), 157–169.

[3] Blyth, T., Varlet, J.: *Sur la construction de certaines* MS-*algebres*. Portugaliae Math. **39** (1980), 489–496.

[4] Blyth, T., Varlet, J.: *Corrigendum sur la construction de certaines* MS-*algebres*. Portugaliae Math. **42** (1983), 469–471.

[5] Chen, C. C.: *Stone lattice I, Construction theorems*. Cond. J. Math. **21** (1969), 884–894.

[6] Haviar, M.: *On certain construction of* MS-*algebras.*. Portugaliae Math. **51** (1994), 71–83.

[7] Haviar, M.: *Construction and affine completeness of principal p-algebras*. Tatra Mountains Math. **5** (1995), 217–228.

[8] Katriňák, T.: *A new proof of the construction theorem for Stone algebras*. Proc. Amer. Math. Soc. **40** (1973), 75–78.

[9] Katriňák,T., Mederly, P.: *Construction of p-algebras*. Algebra Universalis **17** (1983), 288–316.

[10] Katriňák, T., Mikula, K.: *On a construction of* MS-*algebras*. Portugaliae Math. **45** (1988), 157–163.