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Stability of Noor Iteration for a General Class of Functions in Banach Spaces

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Abstract

In this paper, we prove the stability of Noor iteration considered in Banach spaces by employing the notion of a general class of functions introduced by Bosede and Rhoades [6]. We also establish similar result on Ishikawa iteration as a special case. Our results improve and unify some of the known stability results in literature.

Key words: stability, Noor and Ishikawa iterations

2000 Mathematics Subject Classification: 47J25, 47H10, 54H25

1 Introduction

Many stability results have been obtained by various authors using different contractive definitions.

Let (E, d) be a complete metric space, $T\colon E \to E$ a selfmap of E; and $F_T = \{p \in E : Tp = p\}$ the set of fixed points of T in E.

For example, Harder and Hicks [10] considered the following concept to obtain various stability results:

Let $\{x_n\}_{n=0}^{\infty} \subset E$ be the sequence generated by an iteration procedure involving the operator T, that is,

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \dots, \tag{1}$$

where $x_0 \in E$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T in E. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and set

$$\epsilon_n = d(y_{n+1}, f(T, y_n)), \quad n = 0, 1, 2, \dots \tag{2}$$

Then, the iteration procedure (1) is said to be T*-stable or stable with respect to* T if $\lim_{n\to\infty}\epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

By observing that metric is induced by the norm, (2) becomes

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\|, \quad n = 0, 1, 2, \ldots, \tag{3}$$

whenever E is a normed linear space or a Banach space.

If in (1),

$$f(T, x_n) = Tx_n, \quad n = 0, 1, 2, \ldots,$$

then, we have the Picard iteration process. Also, if in (1),

$$f(T, x_n) = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n = 0, 1, 2, \ldots,$$

with $\{\alpha_n\}_{n=0}^{\infty}$ a sequence of real numbers in $[0, 1]$, then we have the Mann iteration process.

For any $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tu_n \\ u_n &= (1 - \beta_n)x_n + \beta_n Tx_n, \end{aligned} \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1].

For arbitrary $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Tq_n \\ q_n &= (1 - \beta_n)x_n + \beta_n Tr_n \\ r_n &= (1 - \gamma_n)x_n + \gamma_n Tx_n, \end{aligned} \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0, 1]$.

2 Preliminaries

Several researchers in literature including Rhoades [23] and Osilike [20] obtained a lot of stability results for some iteration procedures using various contractive definitions. For example, Osilike [20] considered the following contractive definition: there exist $L \geq 0$, $a \in [0, 1)$ such that for each $x, y \in E$,

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y). \tag{6}$$

Later, Imoru and Olatinwo [12] extended the results of Osilike [20] and proved some stability results for Picard and Mann iteration processes using the following contractive condition: there exist $b \in [0, 1)$ and a monotone increasing function $\varphi\colon \Re^+ \to \Re^+$ with $\varphi(0) = 0$ such that for each $x, y \in E$,

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + bd(x, y). \tag{7}$$

Recently, Bosede and Rhoades [6] observed that the process of "generalizing" (6) could continue ad infinitum. As a result of this observation, Bosede and

Rhoades [6] introduced the notion of a general class of functions to prove the stability of Picard and Mann iterations. (For Example, See Bosede and Rhoades [6]).

Our aim in this paper is to prove the stability of Noor iteration for a general class of functions as well as establish similar result on Ishikawa iteration as a special case.

We shall employ the following contractive definition: Let $(E, \|.\|)$ be a Banach space, $T : E \to E$ a selfmap of E, with a fixed point p such that for each $y \in E$ and $0 \leq a < 1$, we have

$$\|p - Ty\| \leq a \|p - y\| . \tag{8}$$

Remark 1 The contractive condition (8) is more general than those considered by Imoru and Olatinwo [12], Osilike [20] and several others in the following sense: By replacing L in (6) with more complicated expressions, the process of "generalizing" (6) could continue ad infinitum. In this paper, we make an obvious assumption implied by (6), and one which renders all generalizations of the form (7) *unnecessary.*

Also, the condition "$\varphi(0) = 0$" usually imposed by Imoru and Olatinwo [12] in the contractive definition (7) is **no longer necessary** in our contraction condition (8) and this is a further improvement to several known stability results in literature.

In the sequel, we shall use the following Lemma which is contained in Berinde [2].

Lemma 1 *Let δ be a real number satisfying $0 \leq \delta < 1$, and $\{\epsilon_n\}$ a positive sequence satisfying $\lim_{n\to\infty} \epsilon_n = 0$. Then, for any positive sequence $\{u_n\}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n,$$

it follows that $\lim_{n\to\infty} u_n = 0$.

3 Main results

Theorem 1 *Let $(E, \|.\|)$ be a Banach space, $T\colon E \to E$ a selfmap of E with a fixed point p, satisfying the contractive condition (8). For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Noor iteration process defined by (5) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1] such that*

$$0 < \alpha \leq \alpha_n, \quad 0 < \beta \leq \beta_n, \quad 0 < \gamma \leq \beta_n, \quad \textit{for all } n. \tag{9}$$

Then, Noor iteration process is T-stable.

Proof Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to p. Suppose also that $\{y_n\}_{n=0}^{\infty} \subset E$ is an arbitrary sequence in E. Define

$$\epsilon_n = \|y_{n+1} - (1 - \alpha_n)y_n - \alpha_n T q_n\| , \quad n = 0, 1, \ldots,$$

where $q_n = (1-\beta_n)y_n + \beta_n Tr_n$ and $r_n = (1-\gamma_n)y_n + \beta_n Ty_n$.

Assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then, using the contractive condition (8) and the triangle inequality, we shall prove that $\lim_{n\to\infty} y_n = p$ as follows:

$$\begin{aligned}
\|y_{n+1} - p\| &\leq \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n Tq_n\| + \|(1-\alpha_n)y_n + \alpha_n Tq_n - p\| \\
&= \epsilon_n + \|(1-\alpha_n)y_n + \alpha_n Tq_n - ((1-\alpha_n)+\alpha_n)p\| \\
&= \epsilon_n + \|(1-\alpha_n)(y_n - p) + \alpha_n(Tq_n - p)\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n \|Tq_n - p\| \\
&= \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n \|p - Tq_n\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a \|p - q_n\| \\
&= \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a \|q_n - p\|. \qquad (10)
\end{aligned}$$

For the estimate of $\|q_n - p\|$ in (10), we get

$$\begin{aligned}
\|q_n - p\| &= \|(1-\beta_n)y_n + \beta_n Tr_n - p\| \\
&= \|(1-\beta_n)y_n + \beta_n Tr_n - ((1-\beta_n)+\beta_n)p\| \\
&= \|(1-\beta_n)(y_n - p) + \beta_n(Tr_n - p)\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n \|Tr_n - p\| \\
&= (1-\beta_n)\|y_n - p\| + \beta_n \|p - Tr_n\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n a \|p - r_n\| \\
&= (1-\beta_n)\|y_n - p\| + \beta_n a \|r_n - p\|. \qquad (11)
\end{aligned}$$

Substitute (11) into (10) gives

$$\|y_{n+1} - p\| \leq \epsilon_n + \Big(1 - (1-a)\alpha_n - \alpha_n\beta_n a\Big)\|y_n - p\| + \alpha_n\beta_n a^2 \|r_n - p\|. \qquad (12)$$

For $\|r_n - p\|$ in (12), we have

$$\begin{aligned}
\|r_n - p\| &= \|(1-\gamma_n)y_n + \gamma_n Ty_n - p\| \\
&= \|(1-\gamma_n)y_n + \gamma_n Ty_n - ((1-\gamma_n)+\gamma_n)p\| \\
&= \|(1-\gamma_n)(y_n - p) + \gamma_n(Ty_n - p)\| \\
&\leq (1-\gamma_n)\|y_n - p\| + \gamma_n \|Ty_n - p\| \\
&= (1-\gamma_n)\|y_n - p\| + \gamma_n \|p - Ty_n\| \\
&\leq (1-\gamma_n)\|y_n - p\| + \gamma_n a \|p - y_n\| \\
&= (1 - \gamma_n + \gamma_n a)\|y_n - p\|. \qquad (13)
\end{aligned}$$

Substituting (13) into (12) and using (9), we get

$$\begin{aligned}
\|y_{n+1} - p\| &\leq \epsilon_n + \Big(1 - (1-a)\alpha_n - \alpha_n\beta_n a\Big)\|y_n - p\| \\
&\quad + \alpha_n\beta_n a^2(1 - \gamma_n + \gamma_n a)\|y_n - p\| \\
&= \epsilon_n + \Big(1 - (1-a)\alpha_n - (1-a)\alpha_n\beta_n a - (1-a)\alpha_n\beta_n\gamma_n a^2\Big)\|y_n - p\| \\
&\leq \Big(1 - (1-a)\alpha - (1-a)\alpha\beta a - (1-a)\alpha\beta\gamma a^2\Big)\|y_n - p\| + \epsilon_n. \qquad (14)
\end{aligned}$$

Observe that

$$0 \leq \Big(1 - (1-a)\alpha - (1-a)\alpha\beta a - (1-a)\alpha\beta\gamma a^2\Big) < 1.$$

Therefore, taking the limit as $n \to \infty$ of both sides of the inequality (14), and using Lemma 1, we get $\lim_{n\to\infty} \|y_n - p\| = 0$, that is,

$$\lim_{n\to\infty} y_n = p.$$

This completes the proof. □

Theorem 2 *Let $(E, \|.\|)$ be a Banach space, $T\colon E \to E$ a selfmap of E with a fixed point p, satisfying the contractive condition (8). For $x_0 \in E$, let $\{x_n\}_{n=0}^{\infty}$ be the Ishikawa iteration process defined by (4) converging to p, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in $[0,1]$ such that*

$$0 < \alpha \leq \alpha_n \ \text{ and } \ 0 < \beta \leq \beta_n, \quad \text{for all } n. \tag{15}$$

Then, Ishikawa iteration process is T-stable.

Proof Assume that $\{x_n\}_{n=0}^{\infty}$ converges to p. Assume also that $\{y_n\}_{n=0}^{\infty} \subset E$ is an arbitrary sequence in E. Set

$$\epsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T u_n\|, \quad n = 0, 1, \dots,$$

where $u_n = (1-\beta_n)y_n + \beta_n T y_n$. Assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall prove that $\lim_{n\to\infty} y_n = p$.

Using the contractive condition (8) and the triangle inequality, we have

$$\begin{aligned}
\|y_{n+1} - p\| &\leq \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T u_n\| \\
&\quad + \|(1-\alpha_n)y_n + \alpha_n T u_n - p\| \\
&= \epsilon_n + \|(1-\alpha_n)y_n + \alpha_n T u_n - ((1-\alpha_n) + \alpha_n)p\| \\
&= \epsilon_n + \|(1-\alpha_n)(y_n - p) + \alpha_n(T u_n - p)\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n \|T u_n - p\| \\
&= \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n \|p - T u_n\| \\
&\leq \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a \|p - u_n\| \\
&= \epsilon_n + (1-\alpha_n)\|y_n - p\| + \alpha_n a \|u_n - p\|.
\end{aligned} \tag{16}$$

We estimate $\|u_n - p\|$ in (16) as follows:

$$\begin{aligned}
\|u_n - p\| &= \|(1-\beta_n)y_n + \beta_n T y_n - p\| \\
&= \|(1-\beta_n)y_n + \beta_n T y_n - ((1-\beta_n) + \beta_n)p\| \\
&= \|(1-\beta_n)(y_n - p) + \beta_n(T y_n - p)\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n \|T y_n - p\| \\
&= (1-\beta_n)\|y_n - p\| + \beta_n \|p - T y_n\| \\
&\leq (1-\beta_n)\|y_n - p\| + \beta_n a \|p - y_n\| \\
&= (1-\beta_n + \beta_n a)\|y_n - p\|.
\end{aligned} \tag{17}$$

Substituting (17) into (16) and using (15), we have

$$\begin{aligned}\|y_{n+1} - p\| &\leq \epsilon_n + \Big(1 - (1-a)\alpha_n - (1-a)a\alpha_n\beta_n\Big)\|y_n - p\| \\ &\leq \Big(1 - (1-a)\alpha - (1-a)a\alpha\beta\Big)\|y_n - p\| + \epsilon_n. \qquad (18)\end{aligned}$$

Since

$$0 \leq \Big(1 - (1-a)\alpha - (1-a)a\alpha\beta\Big) < 1,$$

then taking the limit as $n \to \infty$ of both sides of (18), and using Lemma 1, we have

$$\lim_{n\to\infty} \|y_n - p\| = 0,$$

that is,

$$\lim_{n\to\infty} y_n = p.$$

This completes the proof. □

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