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ON A KIND OF GENERALIZED LEHMER PROBLEM

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Abstract. For $1 \leqslant c \leqslant p-1$, let $E_1, E_2, \ldots, E_m$ be fixed numbers of the set $\{0,1\}$, and let $a_1, a_2, \ldots, a_m$ $(1 \leqslant a_i \leqslant p,\ i = 1, 2, \ldots, m)$ be of opposite parity with $E_1, E_2, \ldots, E_m$ respectively such that $a_1 a_2 \ldots a_m \equiv c \pmod p$. Let

$$N(c,m,p) = \frac{1}{2^{m-1}} \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{\substack{a_m=1 \\ a_1 a_2 \ldots a_m \equiv c \ (\mathrm{mod}\ p)}}^{p-1} (1-(-1)^{a_1+E_1})(1-(-1)^{a_2+E_2}) \ldots (1-(-1)^{a_m+E_m}).$$

We are interested in the mean value of the sums

$$\sum_{c=1}^{p-1} E^2(c,m,p),$$

where $E(c,m,p) = N(c,m,p) - ((p-1)^{m-1})/(2^{m-1})$ for the odd prime p and any integers $m \geqslant 2$. When $m = 2$, $c = 1$, it is the Lehmer problem. In this paper, we generalize the Lehmer problem and use analytic method to give an interesting asymptotic formula of the generalized Lehmer problem.

Keywords: Lehmer problem, character sum, Dirichlet L-function, asymptotic formula

MSC 2010: 11N37, 11M06

1. Introduction

Let p be an odd prime. For each integer a with $1 \leqslant a \leqslant p-1$, we know that there exists one and only one b with $1 \leqslant b \leqslant p-1$ such that $ab \equiv 1 \pmod p$. Lehmer [2] asks us to find the number of (a,b) in which a and b are of opposite parity with the

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conditions above. For any fixed integer c $(1 \leqslant c \leqslant p-1)$, let $N(c,p)$ be the number of solutions of the congruent equation $ab \equiv c \pmod p$ for $1 \leqslant a, b \leqslant p-1$ in which a and b are of opposite parity; this can be expressed by

$$N(c,p) = \sum_{\substack{a=1 \\ ab\equiv c \pmod p \\ 2\nmid a+b}}^{p-1} \sum_{b=1}^{p-1} 1.$$

Wenpeng Zhang [7] has studied the problem and has got

$$N(c,p) = \frac{1}{2} \sum_{\substack{a=1 \\ ab\equiv c \pmod p}}^{p-1} \sum_{b=1}^{p-1} (1-(-1)^{a+b}) = \frac{p-1}{2} + O(p^{1/2}\log^2 p).$$

It is also known that Wenpeng Zhang [8] has proved the formula

$$\sum_{c=1}^{p-1} \Big(N(c,p) - \frac{p-1}{2}\Big)^2 = \frac{3}{4}p^2 + O(p^{1+\varepsilon}),$$

from which it can be concluded that the error term of $N(c,p)$ may be the best estimate. There are many other results about the problem by a lot of scholars ([6], [5], [3], [4]).

Let $E_1, E_2, \ldots, E_m$ be fixed numbers of the set $\{0,1\}$, and let $a_1, a_2, \ldots, a_m$ $(1 \leqslant a_i \leqslant p,\ i = 1, 2, \ldots, m)$ be of opposite parity with $E_1, E_2, \ldots, E_m$ respectively such that $a_1 a_2 \ldots a_m \equiv c \pmod p$. As a general case of $N(c,p)$, we define

$$\begin{aligned} &N(c,m,p) \\ &= \frac{1}{2^{m-1}} \sum_{\substack{a_1=1 \\ a_1 a_2 \ldots a_m \equiv c \pmod p}}^{p-1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_m=1}^{p-1} (1-(-1)^{a_1+E_1})(1-(-1)^{a_2+E_2})\ldots(1-(-1)^{a_m+E_m}), \end{aligned}$$

and denote

$$E(c,m,p) = N(c,m,p) - \frac{(p-1)^{m-1}}{2^{m-1}}.$$

In this paper, we will study the mean value of the sums defined above, that is

$$\sum_{c=1}^{p-1} E^2(c,m,p),$$

where $E(c,m,p) = N(c,m,p) - ((p-1)^{m-1})/(2^{m-1})$ for the odd prime p and any integers $m \geqslant 2$, and get an interesting asymptotic formula. That is, we will prove the following theorem.

Theorem. *Let p be an odd prime, let $E_1, E_2, \ldots, E_m$ be fixed numbers of the set $\{0,1\}$, let $E(c,m,p)$ be defined as above. Then for any integers $m \geqslant 2$ and $1 \leqslant c \leqslant p-1$ we have the asymptotic formula*

$$\sum_{c=1}^{p-1} E^2(c,m,p) = \frac{1}{2} K(m) \Big(\frac{-5p}{\pi^2}\Big)^m \zeta^{2m-1}(2) \prod_{\substack{p_0 \neq p \\ p_0 \neq 2}} A(m,p_0,2) + O(p^{m-1+\varepsilon}),$$

where

$$K(m) = \sum_{i=0}^{m} \binom{m}{i} \Big(-\frac{2}{5}\Big)^i \sum_{j=0}^{i} \binom{i}{j} (-1)^j \frac{1}{2^{|i-2j|}} \Bigg(3\binom{m+|i-2j|-1}{|i-2j|} + \binom{2m-2}{m-1}\Bigg),$$

$\zeta(s)$ is the Riemann zeta function, $\prod_{\substack{p_0 \neq p \\ p_0 \neq 2}}$ denotes the product for all primes except p and 2 and $A(m,p,s) = \sum\limits_{r=0}^{2m-2} 1/p^{rs} \sum\limits_{t=0}^{r} (-1)^t \binom{2m-1}{t} \binom{m+r-t-1}{r-t}^2$.

If $m = 2$, $c = 1$, and E_1, E_2 are of opposite parity (such as $E_1 = 0$, $E_2 = 1$ or $E_1 = 1$, $E_2 = 0$), this is the Lehmer problem, namely $N(1,2,p) = N(1,p)$. If $m = 3$, we have the following corollary.

Corollary. *Let p be an odd prime, and $E(c,3,p) = N(c,3,p) - \frac{1}{4}(p-1)^2$. Then we have the asymptotic formula*

$$\sum_{c=1}^{p-1} E^2(c,3,p) = \frac{765}{2\pi^6} p^3 \zeta^5(2) \prod_{\substack{p_0 \neq p \\ p_0 \neq 2}} \Big(1 + \frac{4}{p_0^2} + \frac{1}{p_0^4}\Big) + O(p^{2+\varepsilon}).$$

2. Some lemmas

To complete the proof of the above theorems, we need the several lemmas.

Lemma 1. *Let p be an odd prime, and let χ denote the Dirichlet character modulo p. Then we have the identities*

$$\sum_{a=1}^{p-1} (-1)^a \chi(a) = \begin{cases} 0, & \chi(-1) = 1; \\ \dfrac{\mathrm{i}}{\pi} 2\chi(2)(\overline{\chi}(2) - 2)\tau(\chi) L(1,\overline{\chi}), & \chi(-1) = -1, \end{cases}$$

where $\tau(\chi) = \sum\limits_{n=1}^{p-1} \chi(n) \mathrm{e}\big(\frac{n}{p}\big)$ is the Gauss sum, $L(1,\chi)$ is the Dirichlet L-function and i is the imaginary unit.

P r o o f. See [7]. □

Lemma 2. *Let p be an odd prime, let $E_1, E_2, \ldots, E_m$ be fixed numbers of the set $\{0,1\}$. For any integer c $(1 \leqslant c \leqslant p-1)$ and $m \geqslant 2$, define*

$$N(c,m,p) = \frac{1}{2^{m-1}} \sum_{a_1=1}^{p-1} \sum_{a_2=1}^{p-1} \ldots \times \sum_{\substack{a_m=1 \\ a_1 a_2 \ldots a_m \equiv c \ (\mathrm{mod}\ p)}}^{p-1} \bigl(1-(-1)^{a_1+E_1}\bigr)\bigl(1-(-1)^{a_2+E_2}\bigr)\ldots\bigl(1-(-1)^{a_m+E_m}\bigr).$$

Then we have

$$N(c,m,p) = \frac{(p-1)^{m-1}}{2^{m-1}} + \frac{2}{(p-1)}\Bigl(\frac{\mathrm{i}}{\pi}\Bigr)^m (-1)^{E_1+E_2+\ldots+E_m+m} \times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(c)\chi(2^m)(\overline{\chi}(2)-2)^m \tau^m(\chi) L^m(1,\overline{\chi}),$$

where χ is the Dirichlet character modulo p, $\tau(\chi)$ is the Gauss sum, $L(1,\chi)$ is the Dirichlet L-function and i *is the imaginary unit.*

P r o o f. From the definition of $N(c,m,p)$ and the orthogonality of character sums we get

$$\begin{aligned} N(c,m,p) &= \frac{1}{2^{m-1}} \sum_{a_1=1}^{p-1} \ldots \sum_{\substack{a_m=1 \\ a_1 a_2 \ldots a_m \equiv c \ (\mathrm{mod}\ p)}}^{p-1} (1-(-1)^{a_1+E_1})(1-(-1)^{a_2+E_2})\ldots(1-(-1)^{a_m+E_m}) \\ &= \frac{1}{2^{m-1}} \sum_{a_1=1}^{p-1} \ldots \sum_{\substack{a_m=1 \\ a_1 a_2 \ldots a_m \equiv c \ (\mathrm{mod}\ p)}}^{p-1} 1 - \frac{1}{2^{m-1}} \sum_{a_1=1}^{p-1} \ldots \sum_{\substack{a_m=1 \\ a_1 a_2 \ldots a_m \equiv c \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{a_1+E_1} + \ldots \\ &\quad + \frac{1}{2^{m-1}} \sum_{a_1=1}^{p-1} \ldots \sum_{\substack{a_m=1 \\ a_1 a_2 \ldots a_m \equiv c \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{a_1+a_2+\ldots+a_m+E_1+E_2+\ldots+E_m+m} \\ &= \frac{1}{2^{m-1}} \sum_{a_1=1}^{p-1} \ldots \sum_{a_{m-1}=1}^{p-1} 1 - \frac{1}{2^{m-1}} \sum_{a_1=1}^{p-1} \ldots \sum_{a_{m-1}=1}^{p-1} (-1)^{a_1+E_1} + \ldots \\ &\quad + \frac{1}{2^{m-1}} (-1)^{E_1+E_2+\ldots+E_m+m} \sum_{a_1=1}^{p-1} \ldots \sum_{\substack{a_m=1 \\ a_1 a_2 \ldots a_m \equiv c \ (\mathrm{mod}\ p)}}^{p-1} (-1)^{a_1+a_2+\ldots+a_m} \end{aligned}$$

$$= \frac{(p-1)^{m-1}}{2^{m-1}} - \frac{1}{2^{m-1}}(-1)^{E_1} \sum_{a_2=1}^{p-1} \cdots \sum_{a_{m-1}=1}^{p-1} \sum_{a_1=1}^{p-1} (-1)^{a_1} + \ldots$$

$$+ \frac{(-1)^{E_1+E_2+\ldots+E_m+m}}{2^{m-1}(p-1)} \sum_{\chi \bmod p} \overline{\chi}(c) \sum_{a_1=1}^{p-1} \cdots \sum_{a_m=1}^{p-1} (-1)^{a_1+a_2+\ldots+a_m} \chi(a_1 \ldots a_m)$$

$$= \frac{(p-1)^{m-1}}{2^{m-1}} + \frac{(-1)^{E_1+E_2+\ldots+E_m+m}}{2^{m-1}(p-1)} \sum_{\chi \bmod p} \overline{\chi}(c) \left(\sum_{a=1}^{p-1} (-1)^a \chi(a) \right)^m$$

where we have used mathematical induction to prove that all of the sums are zero except the first and the last. Therefore according to Lemma 1, we have

$$N(c,m,p) = \frac{(p-1)^{m-1}}{2^{m-1}} + \frac{(-1)^{E_1+E_2+\ldots+E_m+m}}{2^{m-1}(p-1)}$$
$$\times \left(\sum_{\substack{\chi \bmod p \\ \chi(-1)=1}} \overline{\chi}(c) \left(\sum_{a=1}^{p-1} (-1)^a \chi(a) \right)^m + \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(c) \left(\sum_{a=1}^{p-1} (-1)^a \chi(a) \right)^m \right)$$
$$= \frac{(p-1)^{m-1}}{2^{m-1}} + \frac{(-1)^{E_1+E_2+\ldots+E_m+m}}{2^{m-1}(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(c) \left(\sum_{a=1}^{p-1} (-1)^a \chi(a) \right)^m$$
$$= \frac{(p-1)^{m-1}}{2^{m-1}} + \frac{2}{(p-1)} \left(\frac{\mathrm{i}}{\pi} \right)^m (-1)^{E_1+E_2+\ldots+E_m+m}$$
$$\times \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \overline{\chi}(c) \chi(2^m) (\overline{\chi}(2) - 2)^m \tau^m(\chi) L^m(1, \overline{\chi}).$$

This proves Lemma 2. □

Lemma 3. *Let q be an odd integer, $d_m(n)$ the m divisor function. Then for any positive integer $m \geqslant 2$ and any integer $k \geqslant 0$, we have*

$$\sum_{n=1}^{+\infty}{}' \frac{d_m(2^k n) d_m(n)}{n^2}$$
$$= \left(\frac{3\binom{m+k-1}{k} + \binom{2m-2}{m-1}}{4} \right) \zeta^{2m-1}(2) \prod_{p \mid q} \left(1 - \frac{1}{p^2} \right)^{2m-1} \prod_{p \dagger 2q} A(m, p_0, 2),$$

where $\zeta(s)$ is the Riemann zeta function, $\sum_n'$ denotes the summation over n except $(n,q) = 1$, $\prod_p$ denotes the product over all primes p such that the conditions are satisfied, and $A(m,p,s) = \sum_{r=0}^{2m-2} 1/p^{rs} \sum_{t=0}^{r} (-1)^t \binom{2m-1}{t} \binom{m+r-t-1}{r-t}^2$.

P r o o f. See Lemma 9 in [5]. □

Lemma 4. *Let p be an odd prime, let χ denote the Dirichlet character modulo p and $L(1,\chi)$ the Dirichlet L-function. Then for any integer k $(=0,\pm1,\pm2,\ldots)$ and $m\geqslant 2$ we have*

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)|L(1,\chi)|^{2m} = \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \overline{\chi}(2^k)|L(1,\chi)|^{2m}$$
$$= \frac{p-1}{2^{k+3}}\left(3\binom{m+k-1}{k}+\binom{2m-2}{m-1}\right)\zeta^{2m-1}(2)\prod_{\substack{p_0\neq p\\ p_0\neq 2}} A(m,p_0,2)+O(p^{\varepsilon}),$$

where $\prod_{\substack{p_0\neq p\\ p_0\neq 2}}$ denotes the product over all primes p_0 except 2 and p, $A(m,p,s)=\sum\limits_{r=0}^{2m-2}1/p^{rs}\sum\limits_{t=0}^{r}(-1)^t\binom{2m-1}{t}\binom{m+r-t-1}{r-t}^2$, and ε is any fixed positive number.

P r o o f. First, for $\mathrm{Re}(s)>1$, the series $L(s,\chi)$ is absolutely convergent, so applying Abel's identity (see [1]) we have

$$L^m(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)d_m(n)}{n^s}$$
$$= \sum_{n=1}^{p/2^k}\frac{\chi(n)d_m(n)}{n^s} + s\int_{p/2^k}^{+\infty}\frac{\sum_{p/2^k<n\leqslant y}\chi(n)d_m(n)}{y^{s+1}}\,\mathrm{d}y$$
$$= \sum_{l=1}^{p}\frac{\chi(l)d_m(l)}{l^s} + s\int_{p}^{+\infty}\frac{\sum_{p<l\leqslant z}\chi(l)d_m(l)}{z^{s+1}}\,\mathrm{d}z.$$

Obviously the above formula also holds for $s=1$ and $\chi(-1)=-1$. Hence according to the definition of the Dirichlet L-function, for any integer k we have

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)|L(1,\chi)|^{2m} = \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)\left(\sum_{n=1}^{\infty}\frac{\chi(n)d_m(n)}{n}\right)\left(\sum_{l=1}^{\infty}\frac{\overline{\chi}(l)d_m(l)}{l}\right)$$
$$= \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)\left(\sum_{n=1}^{p/2^k}\frac{\chi(n)d_m(n)}{n} + \int_{p/2^k}^{+\infty}\frac{\sum_{p/2^k<n\leqslant y}\chi(n)d_m(n)}{y^2}\,\mathrm{d}y\right)$$
$$\times\left(\sum_{l=1}^{p}\frac{\overline{\chi}(l)d_m(l)}{l} + \int_{p}^{+\infty}\frac{\sum_{p<l\leqslant z}\overline{\chi}(l)d_m(l)}{z^2}\,\mathrm{d}z\right)$$

$$
\begin{aligned}
&= \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)\Big(\sum_{n=1}^{p/2^k} \frac{\chi(n)d_m(n)}{n}\Big)\Big(\sum_{l=1}^{p} \frac{\overline{\chi}(l)d_m(l)}{l}\Big)\\
&\quad + \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)\Big(\sum_{n=1}^{p/2^k} \frac{\chi(n)d_m(n)}{n}\Big)\Big(\int_p^{+\infty} \frac{\sum_{p<l\leqslant z}\overline{\chi}(l)d_m(l)}{z^2}\,\mathrm{d}z\Big)\\
&\quad + \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)\Big(\sum_{l=1}^{p} \frac{\overline{\chi}(l)d_m(l)}{l}\Big)\Big(\int_{p/2^k}^{+\infty} \frac{\sum_{p/2^k<n\leqslant y}\chi(n)d_m(n)}{y^2}\,\mathrm{d}y\Big)\\
&\quad + \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)\Big(\int_{p/2^k}^{+\infty} \frac{\sum_{p/2^k<n\leqslant y}\chi(n)d_m(n)}{y^2}\,\mathrm{d}y\Big)\Big(\int_p^{+\infty} \frac{\sum_{p<l\leqslant z}\overline{\chi}(l)d_m(l)}{z^2}\,\mathrm{d}z\Big)\\
&\equiv M_1 + M_2 + M_3 + M_4 \quad \text{(say)}.
\end{aligned}
$$

Now we will estimate each term of the above.

(i) From the orthogonality relation for character sums modulo p, we know that for $(p, nl) = 1$ we have the identity (see [1])

$$
\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(n)\overline{\chi}(l) = \begin{cases} \dfrac{p-1}{2}, & \text{if } n \equiv l \bmod p;\\[2mm] -\dfrac{p-1}{2}, & \text{if } n \equiv -l \bmod p;\\[2mm] 0, & \text{otherwise.} \end{cases}
$$

Then according to Lemma 3, we can easily get

$$
\begin{aligned}
M_1 &= \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k)\Big(\sum_{n=1}^{p/2^k} \frac{\chi(n)d_m(n)}{n}\Big)\Big(\sum_{l=1}^{p} \frac{\overline{\chi}(l)d_m(l)}{l}\Big)\\
&= \sideset{}{'}\sum_{n=1}^{p/2^k} \sideset{}{'}\sum_{l=1}^{p} \frac{d_m(n)d_m(l)}{nl} \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k n)\overline{\chi}(l)\\
&= \frac{p-1}{2} \sideset{}{'}\sum_{\substack{n=1\\ 2^k n\equiv l\ (\mathrm{mod}\ p)}}^{p/2^k} \sideset{}{'}\sum_{l=1}^{p} \frac{d_m(n)d_m(l)}{nl} - \frac{p-1}{2} \sideset{}{'}\sum_{\substack{n=1\\ 2^k n\equiv -l\ (\mathrm{mod}\ p)}}^{p/2^k} \sideset{}{'}\sum_{l=1}^{p} \frac{d_m(n)d_m(l)}{nl}\\
&= \frac{p-1}{2} \sideset{}{'}\sum_{n=1}^{p/2^k} \frac{d_m(2^k n)d_m(n)}{2^k n^2} - \frac{p-1}{2} \sideset{}{'}\sum_{\substack{n=1\\ 2^k n+l\equiv 0\ (\mathrm{mod}\ p)}}^{p/2^k} \sideset{}{'}\sum_{l=1}^{p} \frac{d_m(n)d_m(l)}{nl}
\end{aligned}
$$

$$= \frac{p-1}{2}\sum_{n=1}^{+\infty}{}' \frac{d_m(2^kn)d_m(n)}{2^kn^2} + O(p^{\varepsilon}) + O\Bigg(p^{\varepsilon}\sum_{\substack{n=1\\2^kn+l=p}}^{p/2^k}\sum_{l=1}^{p}\frac{2^k}{l} + p^{\varepsilon}\sum_{\substack{n=1\\2^kn+l=p}}^{p/2^k}\sum_{l=1}^{p}\frac{1}{n}\Bigg)$$

$$= \frac{p-1}{2^{k+3}}\left(3\binom{m+k-1}{k} + \binom{2m-2}{m-1}\right)\zeta^{2m-1}(2)\prod_{\substack{p_0\neq p\\p_0\neq 2}} A(m,p_0,2) + O(p^{\varepsilon}),$$

where $\sum_n'$ indicates the summation over n such that $(n,p)=1$, $\prod_{\substack{p_0\neq p\\p_0\neq 2}}$ denotes the product over all primes p_0 except 2 and p and

$$A(m,p,s) = \sum_{r=0}^{2m-2}\frac{1}{p^{rs}}\sum_{t=0}^{r}(-1)^t\binom{2m-1}{t}\binom{m+r-t-1}{r-t}^2.$$

(ii) According to the method of (i) and making use of some properties of characters, we have

$$M_2 = \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^k)\Bigg(\sum_{n=1}^{p/2^k}\frac{\chi(n)d_m(n)}{n}\Bigg)\Bigg(\int_p^{+\infty}\frac{\sum_{p<l\leqslant z}\overline{\chi}(l)d_m(l)}{z^2}\,\mathrm{d}z\Bigg)$$

$$= \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^k)\Bigg(\sum_{n=1}^{p/2^k}\frac{\chi(n)d_m(n)}{n}\Bigg)\Bigg(\int_p^{p^{3(2^m-2)}}\frac{\sum_{p<l\leqslant z}\bar{\chi}(l)d_m(l)}{z^2}\,\mathrm{d}z\Bigg)$$

$$+ \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^k)\Bigg(\sum_{n=1}^{p/2^k}\frac{\chi(n)d_m(n)}{n}\Bigg)\Bigg(\int_{p^{3(2^m-2)}}^{+\infty}\frac{\sum_{p<l\leqslant z}\bar{\chi}(l)d_m(l)}{z^2}\,\mathrm{d}z\Bigg)$$

$$\leqslant \int_p^{p^{3(2^m-2)}}\frac{1}{z^2}\Bigg|\sum_{n=1}^{p/2^k}{}'\sum_{p<l\leqslant z}{}'\frac{d_m(n)d_m(l)}{n}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^kn)\bar{\chi}(l)\Bigg|\,\mathrm{d}z$$

$$+ \int_{p^{3(2^m-2)}}^{+\infty}\frac{1}{z^2}\Bigg|\sum_{n=1}^{p/2^k}{}'\frac{d_m(n)}{n}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^kn)\sum_{p<l\leqslant z}\bar{\chi}(l)d_m(l)\Bigg|\,\mathrm{d}z$$

$$\ll \int_p^{p^{3(2^m-2)}}\frac{p}{z^2}\Bigg|\sum_{\substack{n=1\\2^kn\equiv l(\bmod p)}}^{p/2^k}{}'\sum_{p<l\leqslant z}{}'\frac{d_m(n)d_m(l)}{n}\Bigg|\,\mathrm{d}z + \int_p^{p^{3(2^m-2)}}\frac{p}{z^2}\Bigg|\sum_{\substack{n=1\\2^kn\equiv -l \pmod p}}^{p/2^k}{}'\sum_{p<l\leqslant z}{}'\frac{d_m(n)d_m(l)}{n}\Bigg|\,\mathrm{d}z$$

$$+ p^{\varepsilon}\int_{p^{3(2^m-2)}}^{+\infty}\frac{1}{z^2}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\Bigg|\sum_{p<l\leqslant z}\bar{\chi}(l)d_m(l)\Bigg|\,\mathrm{d}z.$$

According to the properties of $d_m(n)$ we have

$$\mathop{{\sum}'}_{\substack{n=1\\ 2^k n\equiv l \pmod p}}^{p/2^k} \mathop{{\sum}'}_{p<l\leqslant z} \frac{d_m(n)d_m(l)}{n} \ll p^{\varepsilon} \sum_{n=1}^{p/2^k} \sum_{1<l\leqslant z/p} \frac{1}{n} \ll p^{-1+\varepsilon} z;$$

$$\mathop{{\sum}'}_{\substack{n=1\\ 2^k n\equiv -l \pmod p}}^{p/2^k} \mathop{{\sum}'}_{p<l\leqslant z} \frac{d_m(n)d_m(l)}{n} \ll p^{\varepsilon} \sum_{n=1}^{p/2^k} \sum_{1<l\leqslant z/p} \frac{1}{n} \ll p^{-1+\varepsilon} z.$$

Applying the Cauchy inequality and Lemma 4 in [8] we can easily get

$$\begin{aligned}&\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left| \sum_{p<l\leqslant z} \bar{\chi}(l) d_m(l) \right| \\ &\qquad\leqslant \left(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} 1^2 \right)^{1/2} \left(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left| \sum_{p<l\leqslant z} \bar{\chi}(l) d_m(l) \right|^2 \right)^{1/2} \ll p^{1/2} z^{1-2/2^m+\varepsilon}.\end{aligned}$$

Therefore, we have

$$M_2 \ll p^{\varepsilon} \int_p^{p^{3(2^m-2)}} \frac{1}{z}\,\mathrm{d}z + p^{1/2+\varepsilon} \int_{p^{3(2^m-2)}}^{+\infty} z^{-1-2/2^m+\varepsilon}\,\mathrm{d}z \ll p^{\varepsilon}.$$

(iii) Similarly, we also have

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k) \left(\sum_{l=1}^{p} \frac{\overline{\chi}(l) d_m(l)}{l} \right) \left(\int_{p/2^k}^{+\infty} \frac{\sum_{p/2^k<n\leqslant y} \chi(n) d_m(n)}{y^2}\,\mathrm{d}y \right) \ll p^{\varepsilon}.$$

(iv) Using the method of (ii), we have

$$\begin{aligned}&\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k) \left(\int_{p/2^k}^{+\infty} \frac{\sum_{p/2^k<n\leqslant y} \chi(n) d_m(n)}{y^2}\,\mathrm{d}y \right) \left(\int_p^{+\infty} \frac{\sum_{p<l\leqslant z} \overline{\chi}(l) d_m(l)}{z^2}\,\mathrm{d}z \right) \\ &= \int_{p/2^k}^{+\infty} \int_p^{+\infty} \frac{1}{y^2 z^2} \left(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^k) \sum_{p/2^k<n\leqslant y} \chi(n) d_m(n) \sum_{p<l\leqslant z} \overline{\chi}(l) d_m(l) \right) \mathrm{d}y\,\mathrm{d}z \\ &\ll \int_{p/2^k}^{+\infty} \int_p^{+\infty} \frac{1}{y^2 z^2} \left| \sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \left| \sum_{p/2^k<n\leqslant y} \chi(n) d_m(n) \right| \left| \sum_{p<l\leqslant z} \overline{\chi}(l) d_m(l) \right| \right| \mathrm{d}y\,\mathrm{d}z\end{aligned}$$

$$\leqslant \int_{p/2^k}^{+\infty}\int_{p}^{+\infty}\frac{1}{y^2z^2}\Bigg|\Bigg(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\Bigg|\sum_{p/2^k<n\leqslant y}\chi(n)d_m(n)\Bigg|^2\Bigg)^{1/2}$$

$$\times\Bigg(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\Bigg|\sum_{p<l\leqslant z}\overline{\chi}(l)d_m(l)\Bigg|^2\Bigg)^{1/2}\Bigg|\,\mathrm{d}y\,\mathrm{d}z$$

$$\ll \int_{p/2^k}^{+\infty}\int_{p}^{+\infty}\frac{1}{y^2z^2}\big|y^{1-2/2^m+\varepsilon}z^{1-2/2^m+\varepsilon}\big|\,\mathrm{d}y\,\mathrm{d}z$$

$$\ll \int_{p/2^k}^{+\infty}\int_{p}^{+\infty}y^{-1-2/2^m+\varepsilon}z^{-1-2/2^m+\varepsilon}\,\mathrm{d}y\,\mathrm{d}z \ll p^{-4/2^m+\varepsilon}.$$

Combining the estimates of (i), (ii), (iii) and (iv), we immediately obtain

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2^k)|L(1,\chi)|^{2m}$$

$$=\frac{p-1}{2^{k+3}}\left(3\binom{m+k-1}{k}+\binom{2m-2}{m-1}\right)\zeta^{2m-1}(2)\prod_{\substack{p_0\neq p\\ p_0\neq 2}}A(m,p_0,2)+O(p^{\varepsilon}),$$

where $\prod_{\substack{p_0\neq p\\ p_0\neq 2}}$ denotes the product over all primes p_0 except 2 and p, $A(m,p,s) = \sum\limits_{r=0}^{2m-2}1/p^{rs}\sum\limits_{t=0}^{r}(-1)^t\binom{2m-1}{t}\binom{m+r-t-1}{r-t}^2$, and ε is any fixed positive number. This completes the proof of Lemma 4. □

3. Proof of Theorem

In this section, we complete the proof of the theorem. First, from the definition of $E(c,m,p)$ and the orthogonality of character sums we have

$$\sum_{c=1}^{p-1}E^2(c,m,p)$$

$$=\frac{4}{(p-1)^2}\Big(\frac{\mathrm{i}}{\pi}\Big)^{2m}\sum_{c=1}^{p-1}\Bigg(\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\overline{\chi}(c)\chi(2^m)(\overline{\chi}(2)-2)^m\tau^m(\chi)L^m(1,\overline{\chi})\Bigg)^2$$

$$= \frac{4}{(p-1)^2}\Big(\frac{\mathrm{i}}{\pi}\Big)^{2m} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \chi_1(2^m)(\overline{\chi}_1(2)-2)^m \tau^m(\chi_1) L^m(1,\overline{\chi}_1)$$

$$\times \sum_{\substack{\chi_2 \bmod p \\ \chi_2(-1)=-1}} \chi_2(2^m)(\overline{\chi}_2(2)-2)^m \tau^m(\chi_2) L^m(1,\overline{\chi}_2) \sum_{c=1}^{p-1} \overline{\chi}_1(c)\overline{\chi}_2(c)$$

$$= \frac{4}{p-1}\Big(\frac{\mathrm{i}}{\pi}\Big)^{2m} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1}} \sum_{\substack{\chi_1 \bmod p \\ \chi_1(-1)=-1 \\ \overline{\chi_1\chi_2}=\chi_0}}$$

$$(1-2\chi_1(2))^m(1-2\chi_2(2))^m \tau^m(\chi_1)\tau^m(\chi_2) L^m(1,\overline{\chi}_1) L^m(1,\overline{\chi}_1)$$

$$= \frac{4}{p-1}\Big(\frac{\mathrm{i}}{\pi}\Big)^{2m} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} |1-2\chi(2)|^{2m} |\tau(\chi)|^{2m} |L(1,\chi)|^{2m}$$

$$= \frac{4}{p-1}\Big(\frac{\mathrm{i}}{\pi}\Big)^{2m} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (5-2\chi(2)-2\overline{\chi}(2))^m |\tau(\chi)|^{2m} |L(1,\chi)|^{2m}$$

$$= \frac{4(5p)^m}{p-1}\Big(\frac{\mathrm{i}}{\pi}\Big)^{2m} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \Big(1-\frac{2}{5}(\chi(2)-\overline{\chi}(2))\Big)^m |L(1,\chi)|^{2m}$$

$$= \frac{4}{p-1}\Big(\frac{-5p}{\pi^2}\Big)^m \sum_{i=0}^{m} \binom{m}{i}\Big(-\frac{2}{5}\Big)^i \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} (\chi(2)-\overline{\chi}(2))^i |L(1,\chi)|^{2m}$$

$$= \frac{4}{p-1}\Big(\frac{-5p}{\pi^2}\Big)^m \sum_{i=0}^{m} \binom{m}{i}\Big(-\frac{2}{5}\Big)^i \sum_{j=0}^{i}\binom{i}{j}(-1)^j \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi^{i-2j}(2)|L(1,\chi)|^{2m},$$

where χ^{-1} means $\overline{\chi}$. From Lemma 4 we have

$$\sum_{c=1}^{p-1} E^2(c,m,p) = \frac{1}{2}K(m)\Big(\frac{-5p}{\pi^2}\Big)^m \zeta^{2m-1}(2) \prod_{\substack{p_0\neq p \\ p_0\neq 2}} A(m,p_0,2) + O(p^{m-1+\varepsilon}),$$

where

$$K(m) = \sum_{i=0}^{m}\binom{m}{i}\Big(-\frac{2}{5}\Big)^i \sum_{j=0}^{i}\binom{i}{j}(-1)^j \frac{1}{2^{|i-2j|}} \times \left(3\binom{m+|i-2j|-1}{|i-2j|} + \binom{2m-2}{m-1}\right)$$

and

$$A(m,p,s) = \sum_{r=0}^{2m-2} \frac{1}{p^{rs}} \sum_{t=0}^{r} (-1)^t \binom{2m-1}{t} \binom{m+r-t-1}{r-t}^2.$$

This completes the proof of Theorem. □

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